

# A GENERAL BOUNDARY ELEMENT FORMULATION FOR THE ANALYSIS OF VISCOELASTIC PROBLEMS

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(Received: April 24, 2009 – Accepted in Revised Form: May 20, 2010)

**Abstract** The analysis of viscoelastic materials is one of the most important subjects in engineering structures. Several works have been so far made for the integral equation methods to viscoelastic problems. From the basic assumptions of viscoelastic constitutive equations and weighted residual techniques, a simple but effective Boundary Element (BE) formulation is developed for the Kelvin viscoelastic solid models. This formulation needs only Kelvin's fundamental solution of isotropic elastostatics with material constants prescribed as explicit functions of time. It is able to solve the quasistatic problems with any load time-dependence and boundary conditions. A system of time-dependent equations is derived by imposing the convenient approximations and adopting the kinematical relations for strain rates. This approach avoids the use of relaxation functions and mathematical transformations. The main feature of the proposed formulation is the absence of domain discretizations, which simplifies the treatment of problems involving infinite domains. A computer code has been developed in the programming environment of MATLAB software. At the end of this paper, two numerical examples have been provided to validate this formulation.

**Keywords** Viscoelastic Solids, Boundary Element Approach, Kelvin Solid Model

**چکیده** تحلیل مسائل ویسکوالاستیک یکی از مهمترین موضوعات در سازه‌های مهندسی می‌باشد. تلاش‌هایی تاکنون به منظور حل مسائل ویسکوالاستیک با رهیافت‌های مختلف معادلات انتگرالی انجام گرفته است. در این مقاله، یک فرمول‌بندی المان‌های مرزی ساده ولی بسیار مؤثر با استفاده از فرضیه‌های بنیادین معادلات مشکله ویسکوالاستیک و اصول پسماند وزنی برای مدل‌های ویسکوالاستیک جامد کلوینی توسعه یافته است. این فرمول‌بندی جدید تنها به جواب اساسی کلوین مورد استفاده در مسائل الاستواستاتیک ایزوتروپیک نیاز دارد که ثوابت مادی در آن به صورت توابع صریحی از زمان توصیف شده‌اند. قابلیت اعمال هر نوع بارهای وابسته به زمان و شرایط مرزی در این رهیافت برای حل مسائل شبه استاتیک وجود دارد. مجموعه‌ای از معادلات وابسته به زمان با اعمال فرضیات مناسب و تعمیم روابط سینماتیکی برای نرخ کرنش‌ها استخراج شده است. در این رهیافت از توابع وارهیدگی یا خرزشی و تبدیلات ریاضی پیچیده استفاده نمی‌شود. مهمترین ویژگی فرمول‌بندی ارائه شده در عدم نیاز به گستره‌سازی‌های دامنه‌ای است که آن را برای استفاده در مسائل با دامنه بینهایت نیز مطلوب ساخته است. یک کد کامپیوتری در محیط برنامه‌نویسی نرم افزار ماتلب (MATLAB) توسعه یافته است. دو مثال عددی کاربردی به منظور معتبرسازی این فرمول‌بندی در پایان مقاله ارائه شده‌اند.

## 1. INTRODUCTION

Because of the complexity of viscoelastic models which include time as an independent variable, the available exact analytical solutions have been obtained for only a few simplified problems. In the literature of computational solid mechanics, a large number of the computational methods to solve the various mechanical problems can be found. The

rigorous predictions of viscoelastic behavior usually rely on numerical approaches such as the Finite Difference Method (FDM), Finite Element Method (FEM), and Boundary Element Method (BEM). During recent years, the BEM has found considerable applications in the solution of engineering problems, such as contact mechanics, viscoelasticity, elastoplasticity, elastodynamics, fracture mechanics and geomechanics. The more

importantly, the problems that once seemed completely inflexible for an analytic study can be now analyzed by this technique rather easily. BEM has the advantage of requiring only boundary data as input and, no division of the domain under consideration into elements. In addition, the BEM is ideally suited for the analysis of infinite domains problems involving because only their surface has to be discretized and conditions at infinity are automatically satisfied by the Kelvin's fundamental solution.

The viscoelastic problems can be effectively treated by the BEM formulation. Numerous researchers developed the boundary element formulations to model the viscoelastic behavior. The first classical procedure for solving the viscoelastic problems is the use of mathematical transformations, in which, based on corresponding theorem [1-5], viscoelastic solutions are usually obtained by inverting the transformation of their associated elastic solutions. The viscoelastic equations can be transformed into equivalent elastic ones by means of Laplace transforms. After solving the transformed problem, a numerical inversion can be performed recovering the desired time domain behavior. Due to the complexity of the viscoelastic behavior, however, such treatment is feasible only in cases with simple geometry or idealized boundary conditions. In addition, this method is too complicated to use unless the relaxation or creep moduli can be accurately interpreted by a simple model. This procedure presents some difficulties when the viscous parameters are time varying, or when complicated time dependent boundary conditions are imposed. Another procedure is also available based on incremental scheme [6-10]. This kind of procedure uses the relaxation functions to transform the convolutional aspect of the viscous behavior in discrete contributions added to the elastic response.

The other available formulation follows the same scheme applied to viscoplastic ones [11-13]. These techniques are based on the quasistatic incremental schemes where the time behavior of the solution is recovered by the stress decay, therefore, imposing the external loads with arbitrary time dependence presents some difficulties. In the following, several important works are investigated.

The usual approach, originally adopted by

Rizzo and Shippy [1], has been to formulate a BEM solution for the Laplace transforms of all variables satisfying an associated elastic problem, and then the solution in time domain is obtained by numerical inversion. Also, incremental BEM solutions in the time domain were first formulated by Shinokawa, et al [6]. Kusama and Mitsui [14] analyzed the linear viscoelastic problems using the boundary element method together with the numerical inversion of the Laplace transform. Their method reduced the amount of manual and computing time as compared with the finite element method.

The weighted residual technique, the indirect BEM, the truncated indirect BEM, and the direct BEM can be used to analyze nonlinear soil-structure interaction in time domain. They illustrated and compared by using 1-D dynamic problem of the spherical cavity in an infinite space by Wolf and Dorbe [15]. For realistic time steps, all formulations led to accurate results, but the weighted residual technique and the truncated indirect BEM were much more efficient than the direct BEM in time domain. Sim and Kwak [16] formulated isotropic, linear viscoelastic problems in time domain by BEM. The viscoelastic fundamental solutions were represented in terms of the constant coefficients of relaxation functions. From the reciprocal work theorem, an alternative form of boundary integral equations was derived by integration by parts. This form required the regularity of field variables to be one order less than that in the usual formulation. A time marching process was incorporated in the numerical method.

Shinokawa and Mitsui [17] presented a combined BEM/FEM method to analysis of viscoelastic problems using the time marching method without the provision of cells. They represented some numerical examples with the tunnel and trench excavation problems in the viscoelastic analysis and confirmed the applicability of BEM to geotechnical analysis. Carini and De Donato [18] obtained the expressions of the fundamental solutions due to unit force, displacement and strain discontinuities for the general viscoelastic model. They derived full set of fundamental solutions for the linear viscoelastic problems from the relevant elastic fundamental solutions using the well-known correspondence principle. Fundamental solutions

were given for 3D-continua, 2D-plane strain and the 2D-plane stress problems using the wide range of linear viscoelastic constitutive law models. An elaborate scheme, generating the time domain fundamental solution in a more general case, was developed by Lee, et al [19]. They proposed the method based on the Laplace transform and the correspondence principle. The relaxation function was expanded in a sum of exponentials, and the transformed fundamental solutions were inverted numerically into real time space.

Pan, et al [20] presented a BEM formulation for 3D linear and viscoelastic bodies subjected to the body force of gravity. First, Laplace transformation was used to suppress the time variable, and then the solutions of displacements and stresses were found in transformed domain. The time domain solutions were then found using an accurate and efficient numerical inversion method which required only real calculations for all quantities. The Green's functions in Laplace domain were obtained through the correspondence principle.

Schanz [21] applied a quadrature rule for the convolution integrals, and/or the convolution quadrature method (CQM). With this numerical quadrature formula, he determined the integration weights from the Laplace transformed fundamental solution and a linear multi-step method. Finally, he obtained a BE formulation in time domain using all the advantages of the Laplace domain formulation. Even materials with complex Poisson's ratio, leading to time-dependent integral free terms in the boundary integral equation, can be treated by this formulation. Mesquita and Coda [22 – 23] implemented Kelvin and Boltzmann viscoelastic models in a 2-D BE atmosphere. Their general methodology was based on differential constitutive relations for viscoelasticity. First part of their work described a methodology using the internal cells. This methodology makes it possible to consider viscous parameters, which are not proportional to elastic tensor. They obtained a simple time-marching process from the kinematical relations. In second part, the important algebraic operations were introduced into the formulation allowing analysis of the viscoelastic problems without internal cells.

Wang and Birgisson [24] presented a time domain BEM to model the quasistatic linear

viscoelastic behavior of asphalt pavements. In this viscoelastic analysis, the fundamental solution was derived in terms of the elemental displacement discontinuities (DDs) and a boundary integral equation was formulated in time domain. The unknown DDs were assumed to vary quadratically in the spatial domain and to vary linearly in time domain. The equations were then solved incrementally through the whole time history using an explicit time-marching approach. The spatial and time integrals were evaluated analytically, which gives highly accurate results and fast convergence of the numerical scheme. Recently, the researchers developed an alternative FEM formulation to analyze the viscoelastic problems based on the constitutive equations of the Kelvin and SLS models [25, 26]. Ashrafi and Farid [27] presented a general BEM formulation from the basic assumptions of SLS viscoelastic constitutive equations, in which it needs only the Kelvin's fundamental solution of isotropic elastostatic problems. These formulations are based on the differential constitutive relation of the SLS viscoelastic model. They produced the time differential systems of equations, which can be solved by an appropriate time marching process.

In the present work, a direct boundary integral formulation is proposed which needs only the Kelvin's fundamental solution of isotropic elastostatics with material constants prescribing as explicit functions of time. In addition, this formulation is based on the differential constitutive relation of Kelvin viscoelastic solid model. The resulting algorithm is able to solve the general quasistatic viscoelastic problems of solid structures with any load time-dependence and boundary conditions. The proposed BE formulation is completely developed to model the viscoelastic behavior of engineering structures. It is necessary to consider the domain integrals by BE formulation due to viscous effects. Avoiding the internal elements is the main objective of this paper, resulting discretizations only at boundary of the problems. Therefore, it reduces the number of variables to be computed, and makes easily treatment of infinite and semi-infinite viscoelastic problems. Quasistatic response of a compressible cylinder under the internal pressure as well as, problem of a pressurized crack in a viscoelastic infinite plane as two numerical examples are

provided to validate the proposed formulation.

## 2. VISCOELASTIC CONSTITUTIVE RELATIONS

Mathematically, the usefulness of constitutive equations is to describe the relationships among the kinematical, mechanical, and thermal field equations and to permit the formulations of well-posed problems in the continuum mechanics. Physically, constitutive equations define various idealized materials which serve as models for the behavior of real materials. All actual materials store and dissipate energy in varying degrees during a loading/unloading cycle [28]. A number of important engineering materials, such as solid polymers, simultaneously store and dissipate mechanical energy when subjected to the applied forces [29]. This form of response as a combination of both liquid-like and solid-like features is termed as *viscoelasticity*. Thus, the valid constitutive equations for the viscoelastic behavior incorporate elastic deformation and viscous flow as special cases, and at same time provide for the response patterns that characterize the combined behavior. Intrinsically, such constitutive equations will involve not only stress and strain, but time-rates of both stress and strain as well [28].

The behavior of viscoelastic materials in the uniaxial stress closely resembles that the models built from discrete elastic and viscous elements. Extensive experimental evidence has shown that practically all engineering materials behave elastically in dilatation; therefore without serious loss of generality, we may assume the fundamental constitutive equations for the linear viscoelastic behavior in differential operator form to be [28]:

$$\{P\} S_{ij} = 2 \{Q\} \eta_{ij} \quad (1)$$

$$\sigma_{ii} = 3K \varepsilon_{ii} \quad (2)$$

for isotropic media.  $S_{ij}$  and  $\eta_{ij}$  are the elements of shear stress and strain tensor, respectively. The coefficients  $\{P\}$  and  $\{Q\}$  are differential time operators of the form:

$$\{P\} = \sum_{i=0}^N p_i \frac{\partial^i}{\partial t^i} \quad (3)$$

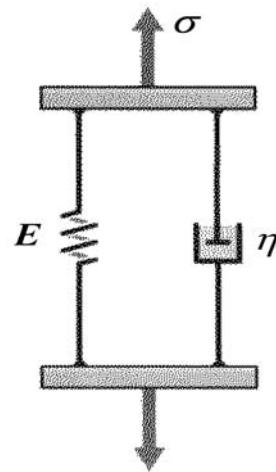
$$\{Q\} = \sum_{i=0}^M q_i \frac{\partial^i}{\partial t^i} \quad (4)$$

in which the coefficients  $p_i$  and  $q_i$  (not necessarily constants) represent the viscoelastic properties. Also,  $K$  is the bulk modulus. Note further that this pair of equations specifies separately the deviatoric and volumetric responses.

Based upon the two fundamental elements of elastic spring and viscous dashpot to model viscoelastic behavior, it is easy to construct the viscoelastic models by suitable combinations of this pair of elements. One especially simple combination that immediately comes to mind is the Kelvin or Voigt viscoelastic model. This model is a simple representation of this kind of behavior. The Kelvin solids consist of a spring and dashpot in parallel, as shown in Figure 1. For which, Equations 1 and 2 reduce to the following single equation [28]:

$$\sigma_{ij} = (E + \eta \partial_t) \varepsilon_{ij} \quad (5)$$

where  $E$  and  $\eta$  represent the material constants (Figure 1), and the partial derivative with respect to time is denoted by  $\partial_t$ . In general, from Figure 1, the following relations can be expressed:



**Figure 1.** Sketch of the Kelvin model representing a viscoelastic solid.

$$\sigma_{ij} = \sigma_{ij}^e + \sigma_{ij}^v \quad (6)$$

$$\varepsilon_{ij} = \varepsilon_{ij}^e = \dot{\varepsilon}_{ij} \quad (7)$$

where the Cartesian coordinates are represented by subscripts  $i$  and  $j$ , while the superscript  $v$  and  $e$  represent viscous and elastic parts, respectively.

The stress-strain equations for the linear elastic solids assuming infinitesimal strains can be written as [28, 30]:

$$\sigma_{ij}^e = C_{ijkl} \varepsilon_{kl} \quad (8)$$

where  $C_{ijkl}$  are the elastic constants representing the properties of the body, and can be defined as [30]:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{km} + \mu (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) \quad (9)$$

where  $\lambda$  and  $\mu$  are Lame's constants, given by:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (10)$$

$$\mu = G = \frac{E}{2(1+\nu)} \quad (11)$$

in which  $E$  and  $\nu$  are Young's modulus and Poisson ratio, respectively. In addition,  $\delta_{ij}$  is the Kronecker delta.

Similarly, for the viscous stress components, following relations can be given:

$$\sigma_{ij}^v = K_{ijmn} \dot{\varepsilon}_{mn} = K_{ijmn} \dot{\varepsilon}_{mn} \quad (12)$$

where the constant  $K_{ijmn}$  represent viscous properties of the body, and can be defined as [30]:

$$K_{ijmn} = (\beta_\lambda \lambda) \delta_{ij} \delta_{mn} + (\beta_\mu \mu) (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \quad (13)$$

in which  $\beta_\lambda$  and  $\beta_\mu$  are the hydrostatic and deviatoric viscosity coefficients, respectively

By substituting Equations 8 and 12 into Equation 6, we have:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} + K_{ijkl} \dot{\varepsilon}_{kl} \quad (14)$$

In this numerical formulation, in order to obtain only boundary integral equations, a simplification for the viscosity coefficients  $\beta_\lambda$  and  $\beta_\mu$  is assumed, i.e.  $\beta_\lambda = \beta_\mu = \beta$ . Therefore, Equation 14 changes into:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} + \beta C_{ijkl} \dot{\varepsilon}_{kl} \quad (15)$$

### 3. BOUNDARY ELEMENT FORMULATION

The boundary element method is based on boundary integral equations. There are several methods of deriving the boundary element formulations: the reciprocal theorem, the weighted residual concept and the variational approach. Here, the boundary integral equations, by extending to the Kelvin solid model, will be derived using the weighted residual concept.

The viscous effects should be included into the equilibrium equation of the body by relating numerically strain time rates with velocity in a way that the viscous characteristics of the body satisfy the boundary conditions together with the elastic ones. In order to do these requirements, the equilibrium equations for a general viscoelastic body can be written as:

$$\sigma_{ij,i} + B_j = \rho \ddot{u}_j + c \dot{u}_j \quad (16)$$

Or

$$\sigma_{ij,i}^e + \sigma_{ij,i}^v + B_j = \rho \ddot{u}_j + c \dot{u}_j \quad (17)$$

where  $B_j$  is the body force acting in  $j$  direction.

Since, in this work, the dynamic effects will not be considered, Equation 17 should be rewritten as:

$$\sigma_{ij,i}^e + \sigma_{ij,i}^v + B_j = 0 \quad (18)$$

**3.1. Displacement Integral Equations** Here, the viscoelastic integral equations for boundary and interior points are obtained using the weighted residual technique for the equilibrium Equations:

$$\sigma_{ij,j} + B_i = 0 \quad (19)$$

In BEM, the Kelvin fundamental solution of an elastic infinite body is adopted as a proper function for weighting the differential equilibrium relation [31-32]. Therefore, Equation 19 can be weighted over the considered domain  $D$  as:

$$0 = \int_D \psi_{ki} \langle \sigma_{ij,j} + B_i \rangle dv \quad (20)$$

where  $\psi_{ki}$  is the Kelvin fundamental solution. It represents the effect of a unit concentrated load applied at a point located in an infinite domain. Integrating Equation 20 by parts and then applying the divergence theorem yields:

$$0 = \int_{\partial D} \psi_{ki} \sigma_{ij} n_j ds - \int_D \psi_{ki,j} \sigma_{ij} dv + \int_D \psi_{ki} B_i dv \quad (21)$$

where  $\partial D$  is the boundary of the body and  $n_j$  is the outward normal vector component.

By knowing that,

$$\sigma_{ij} n_j = t_i^n \quad (22)$$

and

$$\begin{aligned} \varepsilon_{kij}^\psi \sigma_{ij} &= 0.5(\psi_{ki,j} + \psi_{kj,i})\sigma_{ij} \\ &= 0.5(\psi_{kj,i} + \psi_{ki,j})\sigma_{ij} = \psi_{ki,j}\sigma_{ij} \end{aligned} \quad (23)$$

where  $\varepsilon_{kij}^\psi$  is the strain fundamental term, Equation 21 changes into:

$$0 = \int_{\partial D} \psi_{ki} t_i^n ds - \int_D \varepsilon_{kij}^\psi \sigma_{ij} dv + \int_D \psi_{ki} B_i dv \quad (24)$$

The next equation is the starting point to derive the viscoelastic integral equations. By imposing the viscoelastic relations, i.e. Equation 15, into Equation 24, we have:

$$0 = \int_{\partial D} \psi_{ki} t_i^n ds - \int_D \varepsilon_{kij}^\psi (C_{ijmn} \varepsilon_{mn}) dv - \int_D \varepsilon_{kij}^\psi (\beta C_{ijmn} \dot{\varepsilon}_{mn}) dv + \int_D \psi_{ki} B_i dv \quad (25)$$

Also, by knowing that,

$$\begin{aligned} \varepsilon_{kij}^\psi C_{ijmn} \varepsilon_{mn} &= \sigma_{kmn}^\psi \varepsilon_{mn} \\ &= \sigma_{kmn}^\psi u_{m,n} = \sigma_{kij}^\psi u_{i,j} \end{aligned} \quad (26)$$

And

$$\begin{aligned} \varepsilon_{kij}^\psi \beta C_{ijmn} \dot{\varepsilon}_{mn} &= \beta \sigma_{kmn}^\psi \dot{\varepsilon}_{mn} \\ &= \beta \sigma_{kmn}^\psi \dot{u}_{m,n} = \beta \sigma_{kij}^\psi \dot{u}_{i,j} \end{aligned} \quad (27)$$

Equation 25 changes into:

$$\begin{aligned} 0 &= \int_{\partial D} \psi_{ki} t_i^n ds - \int_D \sigma_{kij}^\psi u_{i,j} dv \\ &\quad - \int_D \beta \sigma_{kij}^\psi \dot{u}_{i,j} dv + \int_D \psi_{ki} B_i dv \end{aligned} \quad (28)$$

By using integrating by parts of the second and third terms of Equation 28, we have:

$$\begin{aligned} 0 &= \int_{\partial D} \psi_{ki} t_i^n ds - \int_{\partial D} \sigma_{kij}^\psi n_j u_i ds \\ &\quad + \int_D \sigma_{kij,j}^\psi u_i dv - \beta \int_{\partial D} \sigma_{kij}^\psi n_j \dot{u}_i ds \\ &\quad + \beta \int_D \sigma_{kij,j}^\psi \dot{u}_i dv + \int_D \psi_{ki} B_i dv \end{aligned} \quad (29)$$

Equation 29 can be rewritten by using the fundamental equilibrium equation, i.e.

$$\sigma_{kij,j}^\psi = -\delta(x, y) \delta_{ki} \quad (30)$$

where  $\delta(x, y)$  is the Dirac's delta distribution, in which  $y$  is a field point and  $x$  is the source point. By knowing that,

$$\sigma_{kij}^\psi n_j = t_{ki}^{n\psi} \quad (31)$$

Equation 29 changes into:

$$\begin{aligned} \mathcal{E}_{ki} u_i(x) + \beta \mathcal{E}_{ki} \dot{u}_i(x) &= \int_{\partial D} \psi_{ki} t_i^n ds \\ &\quad - \int_{\partial D} t_{ki}^{n\psi} u_i ds - \beta \int_{\partial D} t_{ki}^{n\psi} \dot{u}_i ds + \int_D \psi_{ki} B_i dv \end{aligned} \quad (32)$$

where the free term  $\mathcal{E}_{ji}$  is exactly what was obtained in the elastostatic formulations and can be found in the BE handbooks [31-32]. Equation 32 is

the integral equation for the general Kelvin–Voigt viscoelastic solids.

The body force domain integral can be easily transformed into its boundary integral equation, which results an equation written exclusively for boundary values. The simple and robust method, which is called the radial integration method, was used for transforming the domain integrals into the equivalent boundary integrals [33]. Any 2-D or 3-D domain integral can be evaluated in a unified way without need to discretize the domain into internal elements. By assuming that the body force  $B_i$  is constant, the domain explicit integral equation is converted to the boundary integral equation. Therefore, we have:

$$\begin{aligned} \int_D \psi_{ki} B_i dv &= B_i \int_D \psi_{ki} dv = B_i \int_\theta \int_r \psi_{ki} r dr d\theta \\ &= B_i \int_{\partial D} \int_r \psi_{ki} r dr \frac{1}{r} \frac{\partial r}{\partial n} ds = B_i \int_{\partial D} B_{ki}^\psi ds \end{aligned} \quad (33)$$

in which,  $B_{ki}^\psi$ , for the Kelvin fundamental solution, is given by:

$$\begin{aligned} B_{ki}^\psi &= \frac{r}{16\pi G(1-\nu)} \\ &\times \left\langle (4\nu - 3) \left( \ln r - \frac{1}{2} \right) \delta_{ki} + r_{,k} r_{,i} \right\rangle \frac{\partial r}{\partial n} \end{aligned} \quad (34)$$

By applying above changes into Equation 32, results:

$$\begin{aligned} \mathcal{C}_{ki} u_i(x) + \beta \mathcal{C}_{ki} \dot{u}_i(x) &= \int_{\partial D} \psi_{ki} ds \\ &- \int_{\partial D} t_{ki}^{n\psi} u_i ds - \beta \int_{\partial D} t_{ki}^{n\psi} \dot{u}_i ds + B_i \int_{\partial D} B_{ki}^\psi ds \end{aligned} \quad (35)$$

Finally, it is worth to note that the necessary kernels  $\psi_{ki}$  and  $t_{ki}^{n\psi}$ , for the 2-D elastostatic problems can be obtained in BE handbooks with the either plane strain or plane stress conditions [31-32]. In a similar way to displacement integral equations, the boundary stress integral equations can be derived to calculate the stress fields on the boundary.

**3.2. Stress Integral Equations** To derive the stress integral equation for interior points, one starts by deriving the strain integral equation. At interior points, the displacement integral equation is given by:

$$\begin{aligned} u_k(x) + \beta \dot{u}_k(x) &= \int_{\partial D} \psi_{ki} t_i^n ds - \int_{\partial D} t_{ki}^{n\psi} u_i ds \\ &- \beta \int_{\partial D} t_{ki}^{n\psi} \dot{u}_i ds + B_i \int_{\partial D} B_{ki}^\psi ds \end{aligned} \quad (36)$$

Denoting the displacements  $u_i$  and the displacement rates  $\dot{u}_i$  in tensor notation, the both small strains and small strain rates are related to displacements and displacement rates via [34]:

$$\varepsilon_{kl} = \frac{1}{2} \{ u_{k,l} + u_{l,k} \} \quad (37)$$

$$\dot{\varepsilon}_{kl} = \frac{1}{2} \{ \dot{u}_{k,l} + \dot{u}_{l,k} \} \quad (38)$$

By substituting the above relations in Equation 36 and by considering that the derivatives are done with respect to the source point location, we find:

$$\begin{aligned} \varepsilon_{kl}(x) + \beta \dot{\varepsilon}_{kl}(x) &= \int_{\partial D} \varepsilon_{kil}^{\psi} t_i^n ds - \int_{\partial D} \tilde{t}_{kil}^{n\psi} u_i ds \\ &- \beta \int_{\partial D} \tilde{t}_{kil}^{n\psi} \dot{u}_i ds + B_i \int_{\partial D} \tilde{B}_{kil}^{\psi} ds \end{aligned} \quad (39)$$

The total stress is obtained using the constitutive Equation 15 in Equation 39, resulting in:

$$\begin{aligned} \sigma_{mn}^e(x) + \sigma_{mn}^v(x) &= \int_{\partial D} \bar{\sigma}_{min}^{\psi} t_i^n ds - \int_{\partial D} \bar{t}_{min}^{n\psi} u_i ds \\ &- \beta \int_{\partial D} \bar{t}_{min}^{n\psi} \dot{u}_i ds + B_i \int_{\partial D} \bar{B}_{min}^{\psi} ds \end{aligned} \quad (40)$$

Now, the total stress state is obtained using the general constitutive Equation 6 and Equation 40:

$$\begin{aligned} \sigma_{mn}(x) &= \int_{\partial D} \bar{\sigma}_{min}^{\psi} t_i^n ds - \int_{\partial D} \bar{t}_{min}^{n\psi} u_i ds \\ &- \beta \int_{\partial D} \bar{t}_{min}^{n\psi} \dot{u}_i ds + B_i \int_{\partial D} \bar{B}_{min}^{\psi} ds \end{aligned} \quad (41)$$

where  $\bar{\sigma}_{ijk}^{\psi}$ ,  $\bar{t}_{ijk}^{n\psi}$ ,  $\bar{B}_{ijk}^{\psi}$  and  $\varepsilon_{ijk}^{\psi}$  are the new kernels based on the Kelvin's fundamental solution which

can be defined by following relations (by using [31]):

$$\bar{\sigma}_{ijk}^{\psi} = -\frac{1}{8\pi(1-\nu)} \frac{1}{r^2} \times [(1-2\nu)(\delta_{ij} r_{,k} + \delta_{jk} r_{,i} - \delta_{ik} r_{,j}) + 3r_{,i} r_{,j} r_{,k}] \quad (42)$$

$$\begin{aligned} \bar{t}_{ijk}^{n\psi} &= -\frac{E}{8\pi(1-\nu^2)} \frac{1}{r^3} \left\langle 3 \frac{\partial r}{\partial n} \left[ (1-2\nu)\delta_{ij} r_{,k} \right. \right. \\ &\quad \left. \left. + \nu(\delta_{ik} r_{,j} + \delta_{kj} r_{,i}) - 5r_{,i} r_{,j} r_{,k} \right] \right. \\ &\quad \left. + 3\nu(n_i r_{,j} r_{,k} + n_j r_{,i} r_{,k}) \right. \\ &\quad \left. + (1-2\nu)[3n_k r_{,i} r_{,j} + n_j \delta_{ik} + n_i \delta_{jk}] \right. \\ &\quad \left. - (1-4\nu)n_k \delta_{ij} \right\rangle \end{aligned} \quad (43)$$

$$\bar{B}_{ijk}^{\psi} = \bar{\sigma}_{ijk}^{\psi} \frac{\partial r}{\partial n} \quad (44)$$

and

$$\begin{aligned} \varepsilon_{ijk}^{\psi}(x, y) &= -\frac{1+\nu}{4\pi(1-\nu)E} \frac{1}{r} \\ &\times [(3-4\nu)(\delta_{ij} r_{,k}) - \delta_{jk} r_{,i} - \delta_{ik} r_{,j} + 2r_{,k} r_{,i} r_{,j}] \end{aligned} \quad (45)$$

In order to determine the elastic and viscous stress states of total stress fields, from Equation 41, Equation 8 can be written in following form:

$$\dot{\sigma}_{ij}^e = C_{ijkl} \dot{\varepsilon}_{kl} = \frac{1}{\beta} [\beta C_{ijkl} \dot{\varepsilon}_{kl}] = \frac{1}{\beta} \sigma_{ij}^v \quad (46)$$

By substituting the above relation into the equilibrium equation, a time-dependent equation can be derived as:

$$-\sigma_{ij} + \sigma_{ij}^e + \beta \dot{\sigma}_{ij}^e = 0 \quad (47)$$

Finally, we can solve numerically Equation 47 by an adaptive linear approximation for the elastic stress field.

#### 4. NUMERICAL DISCRETIZATION

It is only possible to solve boundary element formulation (i.e. Equations 35 and 41) analytically

for very simple problems. The first step in the discretization is to divide the boundary  $\partial D$  into  $N_e$  elements, so that Equations 35 and 41 become:

$$\begin{aligned} \mathcal{E}_{ki} u_i(x) + \beta \mathcal{E}_{ki} \dot{u}_i(x) &= \sum_{r=1}^{N_e} \int_{\partial D_r} \psi_{ki} t_i^n ds \\ &- \sum_{r=1}^{N_e} \int_{\partial D_r} t_{ki}^{n\psi} u_i ds - \sum_{r=1}^{N_e} \beta \int_{\partial D_r} t_{ki}^{n\psi} \dot{u}_i ds \\ &+ \sum_{r=1}^{N_e} B_i \int_{\partial D_r} B_{ki}^{\psi} ds \end{aligned} \quad (48)$$

and

$$\begin{aligned} \sigma_{mn}(x) &= \sum_{r=1}^{N_e} \int_{\partial D_r} \bar{\sigma}_{min}^{\psi} t_i^n ds - \sum_{r=1}^{N_e} \int_{\partial D_r} \bar{t}_{min}^{n\psi} u_i ds \\ &- \sum_{r=1}^{N_e} \beta \int_{\partial D_r} \bar{t}_{min}^{n\psi} \dot{u}_i ds + \sum_{r=1}^{N_e} B_i \int_{\partial D_r} \bar{B}_{min}^{\psi} ds \end{aligned} \quad (49)$$

where  $\partial D = \sum \partial D_n$ .

#### 5. ISOPARAMETRIC ELEMENTS

One of the most significant improvements in the BEM was the introduction of the parametric representation of both geometry and unknown functions similar to isoparametric formulation in the FEM. In this type of formulation, the boundary parameter  $y_i$ , the unknown displacement and velocity fields  $u_i$  and  $\dot{u}_i$ , and also the traction fields  $t_i$  are approximated by introducing interpolation functions in following forms, respectively:

$$\begin{aligned} y_i(\xi) &= \sum_{\alpha=1}^m N_{\alpha}(\xi) y_i^{\alpha} \\ u_i(\xi) &= \sum_{\alpha=1}^m N_{\alpha}(\xi) u_i^{\alpha} \\ \dot{u}_i(\xi) &= \sum_{\alpha=1}^m N_{\alpha}(\xi) \dot{u}_i^{\alpha} \\ t_i^n(\xi) &= \sum_{\alpha=1}^m N_{\alpha}(\xi) t_i^{n\alpha} \end{aligned} \quad (50)$$

where,  $N_\alpha$  called shape functions, are polynomials of degree  $m-1$ . The quantities  $y_i^\alpha$ ,  $u_i^\alpha$ ,  $\dot{u}_i^\alpha$  and  $t_i^{n\alpha}$  are the values of the functions at node  $\alpha$ . These shape functions are defined in terms of non-dimensional coordinates  $\xi$  ( $-1 \leq \xi \leq 1$ ). Here, for discretization of the boundary, the linear elements ( $m = 2$ ) have been used; therefore for these elements, we have:

$$\begin{aligned} N_1 &= \frac{1}{2}(1 - \xi) \\ N_2 &= \frac{1}{2}(1 + \xi) \end{aligned} \quad (51)$$

A discretized boundary element formulation can be obtained by substituting the relations of Equation 50 into the chief integral Equations 48 and 49, to obtain:

$$\begin{aligned} \mathcal{E}_{ki} u_i(x) + \beta \mathcal{E}_{ki} \dot{u}_i(x) &= \sum_{r=1}^{N_e} \sum_{\alpha=1}^m \int_{\partial D_r} \psi_{ki} N_\alpha t_i^{nr\alpha} ds \\ &\quad - \sum_{r=1}^{N_e} \sum_{\alpha=1}^m \int_{\partial D_r} t_{ki}^{n\psi} N_\alpha u_i^{r\alpha} ds \\ &\quad - \sum_{r=1}^{N_e} \sum_{\alpha=1}^m \beta \int_{\partial D_r} t_{ki}^{n\psi} N_\alpha \dot{u}_i^{r\alpha} ds \\ &\quad + \sum_{r=1}^{N_e} B_i \int_{\partial D_r} B_{ki}^\psi ds \end{aligned} \quad (52)$$

and

$$\begin{aligned} \sigma_{mn}(x) &= \sum_{r=1}^{N_e} \sum_{\alpha=1}^m \int_{\partial D_r} \bar{\sigma}_{\min}^\psi N_\alpha t_i^{nr\alpha} ds \\ &\quad - \sum_{r=1}^{N_e} \sum_{\alpha=1}^m \int_{\partial D_r} \bar{t}_{\min}^{n\psi} N_\alpha u_i^{r\alpha} ds \\ &\quad - \sum_{r=1}^{N_e} \sum_{\alpha=1}^m \beta \int_{\partial D_r} \bar{t}_{\min}^{n\psi} N_\alpha \dot{u}_i^{r\alpha} ds \\ &\quad + \sum_{r=1}^{N_e} B_i \int_{\partial D_r} \bar{B}_{\min}^\psi ds \end{aligned} \quad (53)$$

Due to the similar shape functions used for approximation of the geometry and functions, the formulation is referred to as *isoparametric*. After choosing the same number for the source points and nodes and then calculating all integrals, the

discretized boundary element equations may now be written in the matrix form as:

$$\mathbf{H}\mathbf{u}(t) + \beta \mathbf{H}\dot{\mathbf{u}}(t) = \mathbf{G}\mathbf{t}(t) + \mathbf{D}\mathbf{B}(t) \quad (54)$$

and

$$\mathbf{\sigma}(t) = \bar{\mathbf{G}}\mathbf{t}(t) + \bar{\mathbf{D}}\mathbf{B}(t) - \bar{\mathbf{H}}\mathbf{u}(t) - \beta \bar{\mathbf{H}}\dot{\mathbf{u}}(t) \quad (55)$$

where  $t$  represents the time.

## 6. NUMERICAL ALGORITHM

To solve the time-dependent differential matrix Equations 54 and 55, it can be necessary to approximate velocity in time domain by a proper time marching treatment. This is carried out by choosing a linear behavior along the time, as follows:

$$\dot{\mathbf{u}}_{s+1} = \frac{\mathbf{u}_{s+1} - \mathbf{u}_s}{\Delta t} \quad (56)$$

The following linear time marching process, by substituting Equation 56 into Equation 54, has been derived:

$$\ddot{\mathbf{H}}\mathbf{u}_{s+1} = \mathbf{G}\mathbf{t}_{s+1} + \mathbf{F}_s \quad (57)$$

in which

$$\ddot{\mathbf{H}} = \left[ 1 + \frac{\beta}{\Delta t} \right] \mathbf{H} \quad (58)$$

and

$$\mathbf{F}_s = \frac{\beta}{\Delta t} \mathbf{H}\mathbf{u}_s + \mathbf{D}\mathbf{B}_{s+1} \quad (59)$$

As previous values are known, now it is necessary to solve the matrix system of Equation 57 for the current time ( $t_{n+1}$ ). In addition, the boundary conditions in time domain are prescribed by interchanging the columns of  $\ddot{\mathbf{H}}$  and  $\mathbf{G}$

matrices.  $\mathbf{u}_{n+1}$  and  $\mathbf{t}_{n+1}$  can be obtained from Equation 57.

For calculating the total stress state, Equation 55 for the current time ( $t_{n+1}$ ) can be computed as:

$$\boldsymbol{\sigma}_{s+1} = \bar{\mathbf{G}} \mathbf{t}_{s+1} + \bar{\mathbf{D}} \mathbf{B}_{s+1} - \bar{\mathbf{H}} \mathbf{u}_{s+1} - \beta \bar{\mathbf{H}} \dot{\mathbf{u}}_{s+1} \quad (60)$$

in which  $\dot{\mathbf{u}}_{s+1}$  is derived from Equation 56. By assuming the linear behavior along the time domain for the elastic stress rates as:

$$\dot{\boldsymbol{\sigma}}_{s+1}^e = \frac{\boldsymbol{\sigma}_{s+1}^e - \boldsymbol{\sigma}_s^e}{\Delta t} \quad (61)$$

and by substituting it into Equation 47, we have:

$$\boldsymbol{\sigma}_{s+1}^e = \frac{1}{\left(1 + \frac{\beta}{\Delta t}\right)} \left( \boldsymbol{\sigma}_{s+1} + \frac{\beta}{\Delta t} \boldsymbol{\sigma}_s^e \right) \quad (62)$$

Elastic stress state can be obtained from Equation 62, in which  $\boldsymbol{\sigma}_s^e$  is known and  $\boldsymbol{\sigma}_{s+1}$  is derived by Equation 60. The presented algorithm has been cast into a unique program and has been solved using the commercial software MATLAB. A computer code "VBE\_KEL" was developed into MATLAB.

## 7. NUMERICAL EXAMPLES AND RESULTS

For validating accuracy of presented formulation, it has been used to solve the two numerical viscoelastic problems shown below whose results can be compared with the analytical solutions.

**7.1. Example One; Pressurization of a Viscoelastic Compressible Cylindrical Tank**  
The problem of the pressurization of a viscoelastic cylinder is of technical important [2-4]. The considerations here are for sufficiently long cylinders such that plane strain conditions can be assumed.

A thick-walled cylindrical tank, as shown in Figure 2, under radial internal pressure  $P_i$  is

analyzed. In the plane of the cross section, we use a polar coordinate system  $r, \theta$  and normal to them  $z$  axis. Continuity of the deformation in plane strain demands that  $\varepsilon_z = 0$ . Theory of elasticity yields the following formulas for the radial stress  $\sigma_r$ , the tangential stress  $\sigma_\theta$  and the radial displacement  $u$  subjected to internal pressure  $P_i$ :

$$\sigma_r = A - \frac{B}{r^2} \quad (63)$$

$$\sigma_\theta = A + \frac{B}{r^2} \quad (64)$$

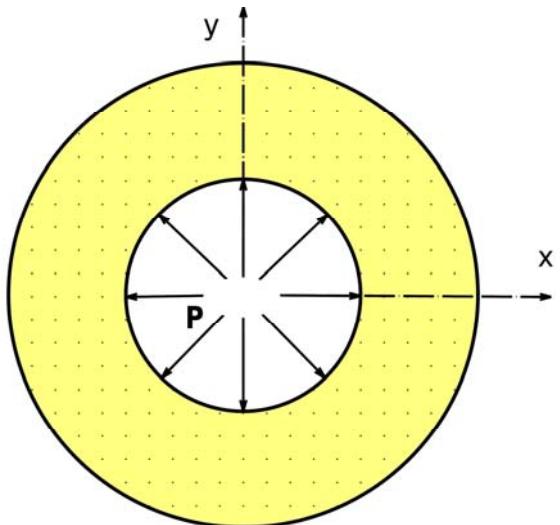


Figure 2. The cylindrical tank under the radial internal pressure ( $P$ ).

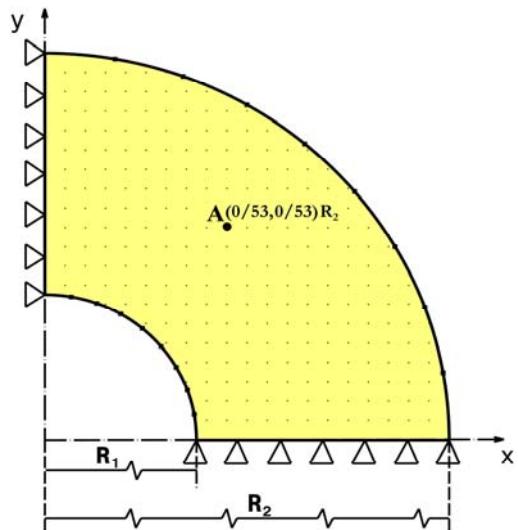


Figure 3. BE Discretization of the model under the radial internal pressure.

and

$$u(r) = \frac{1+\nu}{E} \left( A[1-2\nu]r + \frac{B}{r} \right) \quad (65)$$

in which constants  $A$  and  $B$  are determined from boundary conditions. For this problem, the boundary conditions are:

$$\begin{aligned} r = R_1 &\Rightarrow \sigma_r = -p_i \\ r = R_2 &\Rightarrow \sigma_r = 0 \end{aligned} \quad (66)$$

Upon inserting above conditions:

$$A = \frac{p_i R_1^2}{R_2^2 - R_1^2} \quad B = \frac{p_i R_1^2 R_2^2}{R_2^2 - R_1^2} \quad (67)$$

where  $R_1$  and  $R_2$  are the inside and outside radii of tank, respectively.

As shown by Equations 63 and 64, the stresses in elastic body are independent of the material constants. Therefore, in accordance with the elastic-viscoelastic correspondence principle, the stresses will be the same when the cylinder is made of a viscoelastic material. Since the displacement  $u$  is a function of material constants, it will be time dependent. The time dependence may be found by employing the corresponding principle [2-4]. With considering cylinder as a Kelvin viscoelastic solid and applying correspondence principle, the actual radial displacement is obtained:

$$u(r, t) = \left\{ \frac{3r}{6K + q_0} \left[ 1 - \exp \left( -\frac{(6K + q_0)t}{q_1} \right) \right] + \frac{R_2^2}{q_0 r} \left[ 1 - \exp \left( -\frac{q_0 t}{q_1} \right) \right] \right\} \times A \quad (68)$$

in which  $K$  is the bulk modulus, and  $q_0$  and  $q_1$  are the coefficients of the Kelvin solids. The analytical and numerical results for displacements have been calculated for the following parameters:

$$\begin{aligned} q_0 &= 9 \times 10^5 \\ K &= 2 \times 10^5 \text{ (MPa)} \\ q_1 &= 2 \times 10^6 \text{ (MPa)(Days)} \end{aligned} \quad (69)$$

TABLE 1. Parameters of the First Modeling.

Mechanical Properties	$E = 20$ (MPa)	$\nu = 0.4$	$\beta = 14$ (days)
Geometry	$R_1 = 10$ (in) $= 250$ (mm)	$R_2 = 2(R_1)$	---
Time Increment	$\Delta t = 1$ (day)	---	---
Pressurization	$P = 2000$ (kPa)	---	---

Due to double symmetry of the problem, only a quarter of problem is modeled, as shown in Figure 3. The geometry and mechanical properties of the model are shown in Table 1.

The numerical solution is obtained by using 30 elements and using 20 steps in time-marching process to arrive at  $t = 120$  days. The inner and outer wall radial displacement obtained using this numerical formulation is compared with the analytical ones as shown in Figure 4 and Figure 5, respectively. The radial displacement at interior point (A:  $[0.53 R_2, 0.53 R_2]$ ) obtained by this numerical formulation has also been compared with the analytical one as shown in Figure 6. It can be observed that the agreement of the obtained results of this formulation with those of analytic ones is very good.

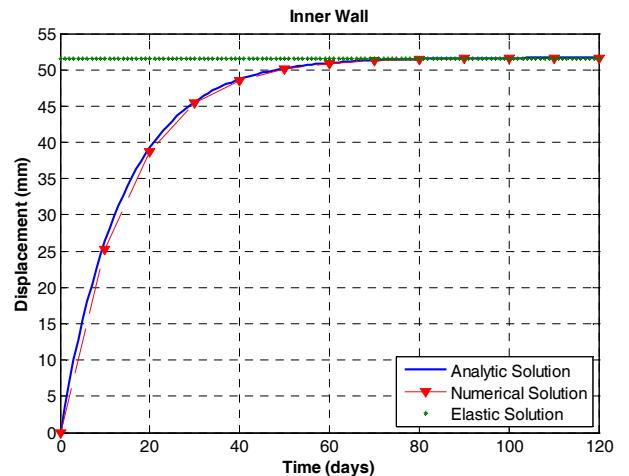
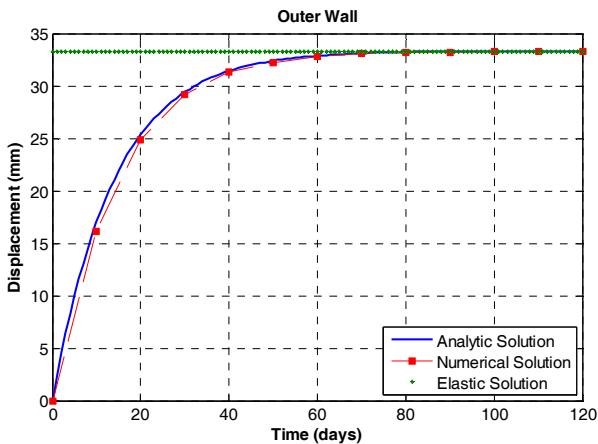
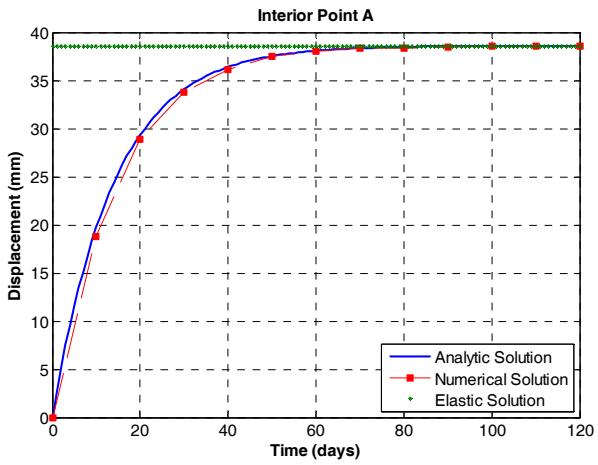


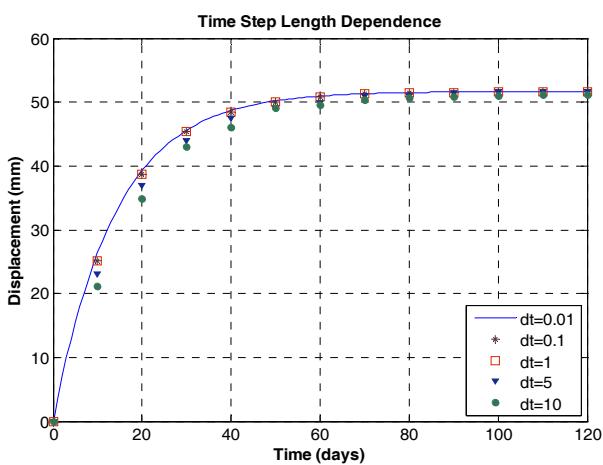
Figure 4. The radial inner wall displacement as compare to the analytical solution.



**Figure 5.** The radial outer wall displacement as compared to the analytical solution.



**Figure 6.** The radial displacement at interior point (*A*) as compared to the analytical solution.



**Figure 7.** Dependence effect to the time step length of the radial inner wall displacement as compared to the corresponding analytical solution.

Time step dependence for the proposed formulation is also shown in Figure 7 for the inner wall radial displacement. As it can be observed, the obtained results are rather accurate even for the large time steps. However, the final elastostatic solution is achieved for any chosen time step.

It is worth noting that this formulation can easily be used for plane stress conditions by changing in material constants. A hole or a crack in an infinite viscoelastic structure can be modeled as a Kelvin solid and is then analyzed by this formulation for plane stress conditions.

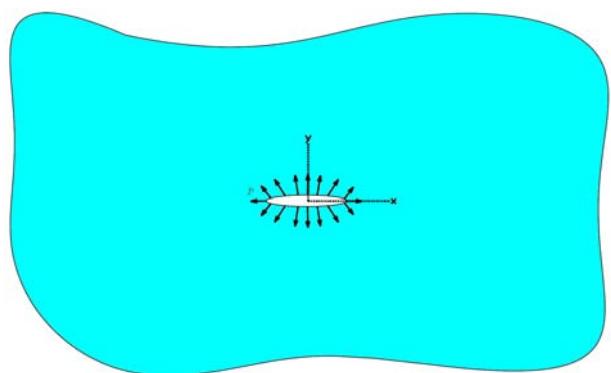
## 7.2. Example Two; A Pressurized Crack in an Infinite Viscoelastic Plane Structure

As another example, we consider the problem of a pressurized crack in a viscoelastic plane structure made of the Kelvin model. For simplicity, the crack is assumed to be aligned along the *x* axis with the center located at the origin of the coordinate system as shown in Figure 8. The length of the crack is  $2a$  and the crack surfaces have been subjected to the uniform normal pressure  $P$ .

The analytical solution for the crack opening displacement for the elastic structure cases is given as [35]:

$$\Delta u(x) = \left( \frac{2(1-\nu)}{G} \right) \sqrt{a^2 - x^2} P \quad (70)$$

where  $G$  and  $\nu$  are the elastic constants and also  $\Delta u$  denotes the crack opening displacement.



**Figure 8.** Pressurized crack in an infinite viscoelastic plane structure.

Since the crack opening displacement is a function of the material constants, it will be time dependent. The time dependence may be found by employing the corresponding principle [2-4]. The viscoelastic solution, using the correspondence principle and the analytical Laplace transform inversion, can be obtained as:

$$\Delta u(x,t) = 4\sqrt{a^2 - x^2} \times (c_1 - c_2 \exp[-c_3 t] - c_4 \exp[-c_5 t]) P H(t) \quad (71)$$

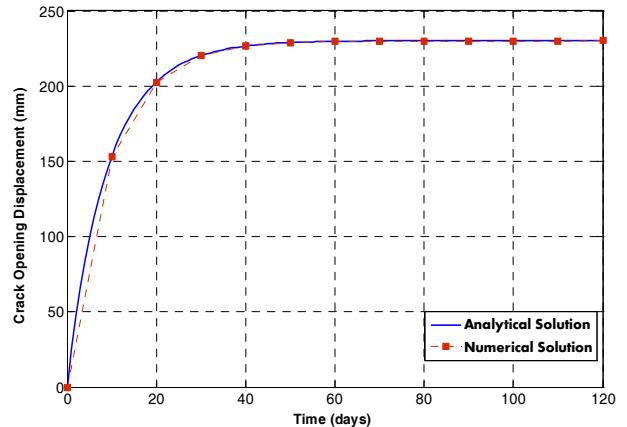
where  $H(t)$  is the heaviside step function, and the constants  $c_i$  are defined as:

$$\begin{aligned} c_1 &= \frac{9K + 2q_0}{3q_0^2 + 18Kq_0} \\ c_2 &= \frac{1}{36K + 6q_0} \\ c_3 &= \frac{6K + q_0}{q_1} \\ c_4 &= \frac{1}{2q_0} \\ c_5 &= \frac{q_0}{q_1} \end{aligned} \quad (72)$$

in which  $K$  is the bulk modulus, and also  $q_0$  and  $q_1$  are the coefficients of the Kelvin solids. The analytical and numerical results for the crack opening displacement have been calculated for the following parameters:

$$\begin{aligned} q_0 &= 9 \times 10^4 \\ K &= 2 \times 10^4 \text{ (MPa)} \\ q_1 &= 1 \times 10^6 \text{ (MPa)(Days)} \end{aligned} \quad (73)$$

Figure 9 shows the evolution of the crack displacement at the center of the crack ( $x = 0$ ) along time. The numerical solution has been obtained by using 28 elements to approximate the crack and using 20 steps in the time-marching process to arrive at  $t = 120$  days. The good agreement between the numerical and analytical solutions is observed in Figure 9.



**Figure 9.** Evolution of the crack opening displacement along the time domain.

## 5. CONCLUDING REMARKS

In this paper, a new formulation to perform simplified viscoelastic analysis was presented by the BEM. Using a weighted residual procedure and a proper kinematical relation between strain and material velocities of boundary points, it is possible to write boundary integral representation for displacement and velocity. The resulting algorithm was able to solve the quasistatic viscoelastic problems with any time-dependence load and boundary conditions. Only the Kelvin's fundamental solution of isotropic elastostatics was needed for this formulation. The main advantage of the presented approach was that the integral representation including only boundary values. It has been imposed a spatial approximation for boundary values achieving a system of time differential equations. This system was easily solved by choosing the linear time approximation for velocity. A computer code was developed in programming environment of MATLAB software, and validation of the proposed formulation was provided by solving two numerical examples.

One particular field where BEM applications have been limited is the viscoelastic fracture mechanics [36]. Introducing the fundamental solutions for specific cracked geometries such as cracks along the interfaces of bi-material continua can be considered as the continuous of the present work. Moreover, the material nonlinearity and anisotropy as well as temperature variations [37]

which highly influence the viscoelastic properties can be taken into account in the BEM viscoelastic analysis.

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