ANALYTIC APPROACH TO INVESTIGATION OF FLUCTUATION AND FREQUENCY OF THE OSCILLATORS WITH ODD AND EVEN NONLINEARITIES

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Abstract In this paper we examine fluctuation and frequency of the governing equation of oscillator with odd and even nonlinearities without damping and we present a new efficient modification of the He's homotopy perturbation method for this equation. We applied standard and modified homotopy perturbation method and compare them with the numerical solution (NS), also we applied He's Energy balance method (EBM) for study frequency of this equation. By compare modified homotopy perturbation method with numerical solution we find that this modified homotopy perturbation method works very well for the wide range of time and boundary conditions for nonlinear oscillator, and comparison of the result obtained using this method for frequency with those obtained by Energy balance method reveals that the former is very effective and convenient. The new modified method accelerates the rapid convergence of the solution, reduces the error solution and increases the validity range for fluctuation and frequency.

Keywords Homotopy Perturbation Method (HPM), Nonlinear Undamped Oscillator, Energy Balance Method (EBM), Modified Homotopy Perturbation Method (MHPM)

چکیده در این مقاله ما نوسان و بسامد معادله حاکم بر نوسان گر با درجه غیر خطی زوج و فرد بدون میرایی را مورد آزمایش قرار داده و روش جدید هموتوپی پرتوربیشن بهبود یافته آقای هی را ارائه می دهیم.ما روش های استاندارد وبهبود یافته HPM را بکار برده وآنها را با راه حل عددی مقایسه می کنیم.همچنین برای بدست آوردن فرکانس این معادله روش انرژی بالانس هی مورد استفاده قرار می گیرد.با مقایسه HPM بهبودیافته با راه حل عددی در می یابیم که این روش بهبودیافته HPM برای بازه بزرگی از مسائل زمانی و شرایط مرزی نوسان گر غیر خطی بسیار خوب کار کرده و با مقایسه نتایج بدست آمده از این روش برای فرکانس و روش انرژی بالانس نشان می دهد که روش اول (هموتوپی پرتوربیشن بهبود یافته) بسیار موثر و مناسب است. روش بهبود یافته جدید رسیدن به همگرایی سریع راه حل را تسریع کرده و خطا را کاهش داده و بازه معتبر برای نوسان و بسامد را افزایش می دهد.

1. INTRODUCTION

Nonlinear oscillations systems are such phenomena that mostly occur nonlinearly. These systems are important in engineering because many practical engineering components consist of vibrating systems that can be modeled using oscillator systems such as elastic beams supported by two springs or mass-on-moving belt or nonlinear pendulum and vibration of a milling machine [1,2].

The development of numerical techniques for solving nonlinear algebraic equations is a subject of considerable interest. There are many papers that deal with nonlinear algebraic equations. The application of homotopy perturbation method in linear and nonlinear problems has been devoted by scientists and engineers [3-21], because this method is to continuously deform a simple problem which is easy to solve into the under study problem which is difficult to solve. This method, homotopy perturbation method (HPM), proposed first by He [3,4], for solving differential and integral equations, linear and nonlinear has been the subject of extensive analytical and numerical studies. The method is a coupling of the traditional perturbation method and homotopy in topology. This method, which does not require a small parameter in an equation, has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences. This HPM has already been applied successfully to solve Laplace equation, nonlinear dispersive K(mp) equations, heat radiation equations, nonlinear integral equations, nonlinear heat conduction and convection equations, nonlinear oscillators, nonlinear Schrödinger equations, nonlinear wave equations, nonlinear chemistry problems, and to other fields [5-21]. This HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions. Thus He's HPM is a universal one which can solve various kinds of nonlinear equations.

Recently, some modifications of this method have published to facilitate and accurate the calculations and accelerate the rapid convergence of the series solution and reduce the size of work [22-27] and some new methods were found to overcome the shortcomings, such as parameter-expansion method [28-33]. It is the purpose of the present paper to examine fluctuation and frequency of the oscillator's governing equation with strong (odd and even) nonlinearities and introduce a new reliable modification of the HPM. The new modification demonstrates an accurate solution if compared with standard HPM and Energy balance method [34,35], and therefore it has been shown that to be computationally efficient in applied fields. In addition the new modified HPM may give the exact solution for nonlinear equations by using two iterations only. The obtained results suggest that this newly improvement technique introduces a powerful improvement for solving nonlinear problems.

In this paper, we consider the following oscillator equation that is governing equation for many mechanical systems with odd and even nonlinearities without damping.

$$\ddot{x} + \mu x + \beta x|x| + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{1}$$

Where the following equation presented for the relation between the deflection of this spring and the force acting upon it:

$$F = k_1 x + k_2 x^2 + k_3 x^3,$$

or

$$F = m(\mu x + \beta x^2 + \varepsilon x^3), \tag{2}$$

There are many mechanical systems that model by mass and spring, which some of them shown in Figure 1.

Also some of two degree of freedom mechanical systems simplified to Equation 1 shown in Figure 2.

2. ANALYSIS OF THE METHODS

2.1. Analysis of the Homotopy Perturbation Method

The Homotopy perturbation method is a combination of the classical perturbation technique and Homotopy technique. To explain the basic idea of the HPM for solving nonlinear differential equations we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3}$$

Subject to boundary condition

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{4}$$

Where A is a general differential operator, B a

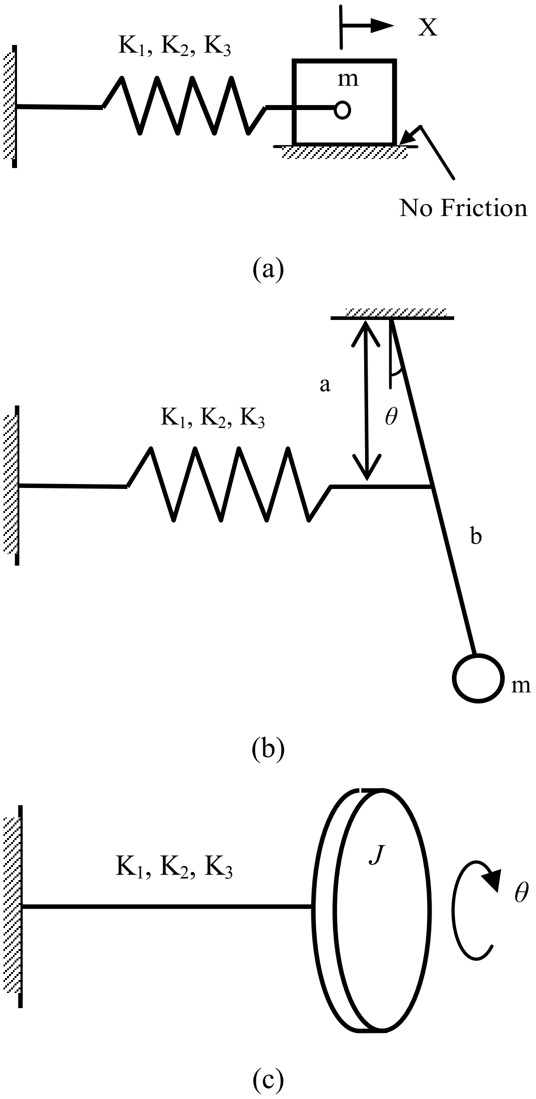


Figure 1. The mass-nonlinear spring systems, (a) $m\ddot{x} + k_1x + k_2x|x| + k_3x^3 = 0$, (b) $J\ddot{\theta} + mgb\theta + k_1a^2\theta + k_2a^3\theta|\theta| + k_3a^4\theta^3 = 0$ and (c) $J\ddot{\theta} + k_1\theta + k_2\theta|\theta| + k_3\theta^3 = 0$.

boundary operator, f(r) is a known analytical function, Γ is the boundary of domain Ω and ∂u/∂n denotes differentiation along the normal drawn outwards from Ω. The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N. Equation 3 therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \tag{5}$$

In case that the nonlinear Equation 3 has no "small parameter", we can construct the following Homotopy:

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p(N(v) - f(r)) = 0, \tag{6}$$

Where,

$$v(r,p) : \Omega \times [0,1] \to R, \tag{7}$$

In Equation 6, $P \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Equation 6 can be written as a power series in p, as following:

$$v = v_0 + pv_1 + p^2v_2 + ..., \tag{8}$$

and the best approximation for solution is:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ..., \tag{9}$$

When, Equation 6 correspond to Equations 3 and 9 becomes the approximate solution of Equation 3. Some interesting results have been attained using this method. Convergence and stability of this method is shown in [36].

2.2. The New Modified HPM The present new modified HPM that is used to solve the nonlinear undamped oscillator is similar to standard HPM. In this way, the homotopy parameter p is used to expand the square of the unknown angular frequency ω as follows:

$$\mu = \omega^2 - p\alpha_1 - p^2\alpha_2 - ..., \tag{10}$$

or

$$\omega^2 = \mu + p\alpha_1 + p^2\alpha_2 + ..., \tag{11}$$

Where μ is coefficient of u(r) in Equation 3, that the right hand of Equation 10 replace to it. Also α(i=1,2,...) are arbitrary parameters that to be determined.

The only different between present HPM and standard HPM is expansion of angular frequency ω, and we can approximate frequency by obtain α in every section.

$$\omega^2 = \mu + \alpha_1 + \alpha_2 + ..., \tag{12}$$

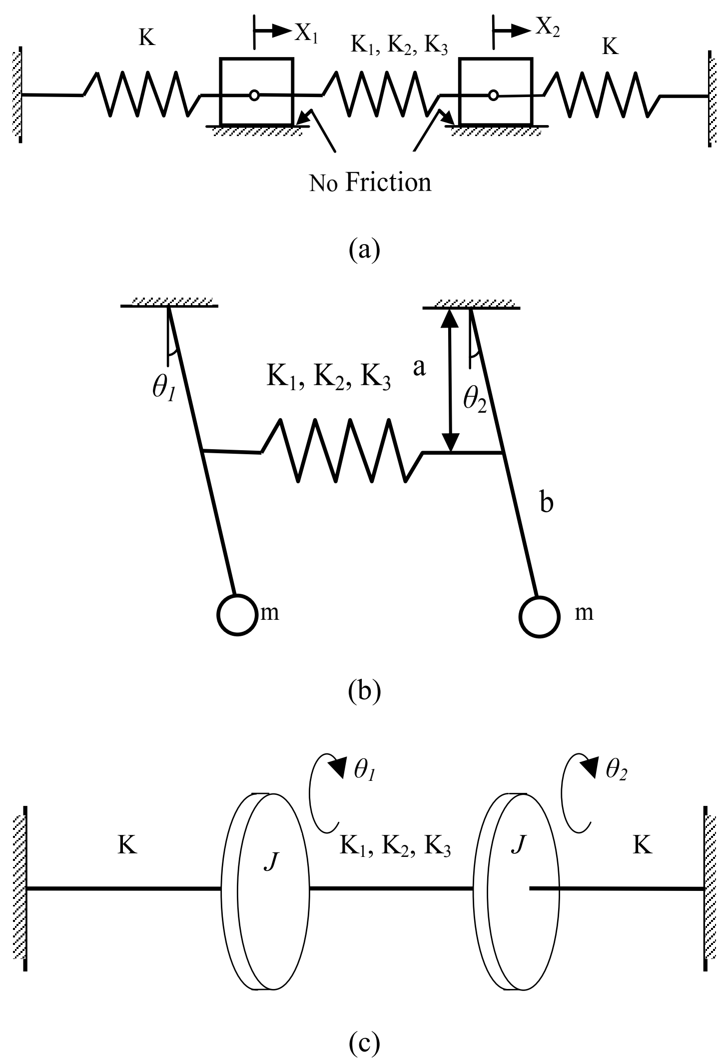


Figure 2. The 2DOF mass-nonlinear spring systems, (a)

$$\begin{cases} m\ddot{x}_1 + kx_1 - k_1(x_2 - x_1) - k_2(x_2 - x_1)\left|(x_2 - x_1)\right| - k_3(x_2 - x_1)^3 = 0 \\ m\ddot{x}_2 + kx_2 + k_1(x_2 - x_1) + k_2(x_2 - x_1)\left|(x_2 - x_1)\right| + k_3(x_2 - x_1)^3 = 0 \end{cases}$$

using the new variables u and v be defined as follows:
$u = x_2 - x_1,\ v = x_2 + x_1$ this equation can be put into a different form $\begin{cases} m\ddot{v} + kv = 0 \\ m\ddot{u} + ku + 2k_1u + 2k_2u\left|u\right| + 2k_3u^3 = 0 \end{cases}$

$$\begin{cases} J\ddot{\theta}_1 + mgb\theta_1 - k_1a^2(\theta_2 - \theta_1) - k_2a^3(\theta_2 - \theta_1)\left|(\theta_2 - \theta_1)\right| - k_3a^4(\theta_2 - \theta_1)^3 = 0 \\ J\ddot{\theta}_2 + mgb\theta_2 + k_1a^2(\theta_2 - \theta_1) + k_2a^3(\theta_2 - \theta_1)\left|(\theta_2 - \theta_1)\right| + k_3a^4(\theta_2 - \theta_1)^3 = 0 \end{cases}, \text{(b)}$$

using the new variables u and v be defined as follows:
$u = \theta_2 - \theta_1,\ v = \theta_2 + \theta_1$ this equation can be put into a different form $\begin{cases} J\ddot{v} + mgbv = 0 \\ J\ddot{u} + mgbu + 2k_1a^2u + 2k_2a^3u\left|u\right| + 2k_3a^4u^3 = 0 \end{cases}$

$$\text{and (c)} \begin{cases} J\ddot{\theta}_1 + k\theta_1 - k_1(\theta_2 - \theta_1) - k_2(\theta_2 - \theta_1)\left|(\theta_2 - \theta_1)\right| - k_3(\theta_2 - \theta_1)^3 = 0 \\ J\ddot{\theta}_2 + k\theta_2 + k_1(\theta_2 - \theta_1) + k_2(\theta_2 - \theta_1)\left|(\theta_2 - \theta_1)\right| + k_3(\theta_2 - \theta_1)^3 = 0 \end{cases}$$

using the new variables u and v be defined as follows:
$u = \theta_2 - \theta_1,\ v = \theta_2 + \theta_1$ this equation can be put into a different form $\begin{cases} J\ddot{v} + kv = 0 \\ J\ddot{u} + ku + 2k_1u + 2k_2u\left|u\right| + 2k_3u^3 = 0. \end{cases}$

2.3. Energy Balance Method In this method according to basic idea of the energy balance method, if θ = 0, it shows the whole energy is in form of kinetic energy and if θ = π/2, it shows the whole energy is in form of potential energy, in θ = π/4 there is a balance between the potential energy and kinetic energy so we can benefit from this point.

Then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method.

In the present paper, we consider a general nonlinear oscillator in the form [37]:

$$u'' + f(u(t)) = 0 \tag{13}$$

In which u and t are generalized displacement and time variables, respectively.

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + F(u) \right) dt \tag{14}$$

Where $T = 2\pi / w$ is period of the nonlinear oscillator, $F(u) = \int f(u)du$.

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + F(u) = F(A) \tag{15}$$

or:

$$R(t) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \tag{16}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation, A. So let us consider such initial conditions:

$$u(0) = A, \quad u'(0) = 0 \tag{17}$$

Assume that its initial approximate guess can be expressed as:

$$u(t) = A\cos(\omega t) \tag{18}$$

Substituting (17) into u term of (15), yield:

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A\cos\omega t) - F(A) = 0 \tag{19}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Equation 17 is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{\frac{2(F(A) - F(A\cos\omega t))}{A^2 \sin^2 \omega t}} \tag{20}$$

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{\frac{2(F(A) - F(A\cos\omega t))}{A^2 \sin^2 \omega t}}} \tag{21}$$

3. APPLICATIONS

3.1. Solution using Homotopy Perturbation Method In this section, we will apply the HPM to nonlinear ordinary differential Equation 1. According to the HPM, we can construct a homotopy of Equation 1 as follows:

$$H(x,p) = (1-p)(\ddot{x} + \mu x) + p(\ddot{x} + \mu x + \beta x|x| + \varepsilon x^3) \tag{22}$$

Assume that the solution of Equation 1 can be written as a power series in p:

$$x = x_0 + px_1 + p^2 x_2 + \ldots \tag{23}$$

Substituting Equation 23 into Equation 22 we have:

$$\begin{aligned} H(x,p) = {} & (1-p)(\ddot{x}_0 + p\ddot{x}_1 + p^2\ddot{x}_2 + \ldots + \mu(x_0 + px_1 \\ & + p^2 x_2 + \ldots)) + p[(\ddot{x}_0 + p\ddot{x}_1 + p^2\ddot{x}_2 + \ldots + \mu(x_0 + \\ & px_1 + p^2 x_2 + \ldots) + \beta(x_0 + px_1 + p^2 x_2 + \ldots) \\ & \left|(x_0 + px_1 + p^2 x_2 + \ldots)\right| + \varepsilon(x_0 + px_1 + p^2 x_2 + \ldots)^3] \end{aligned} \tag{24}$$

Equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0: \quad \ddot{x}_0 + \mu x_0 = 0, \; x_0(0) = A, \; \dot{x}_0(0) = 0 \tag{25}$$

$$p^1: \quad \ddot{x}_1 + \mu x_1 + \beta x_0 |x_0| + \varepsilon x_0^3 = 0, \quad x_1(0) = 0, \; \dot{x}_1(0) = 0 \tag{26}$$

$$p^2: \ddot{x}_2 + \mu x_2 + 2\beta x_1 |x_0| + 3\varepsilon x_0^2 x_1 = 0, \; x_2(0) = 0, \; \dot{x}_2(0) = 0 \tag{27}$$

$$P^3: \ddot{x}_3 + \mu x_3 + \beta(2x_2|x_0| + x_1|x_1|) + 3\varepsilon(x_0 x_1^2 + x_0^2 x_2) = 0, \; x_3(0) = 0, \; \dot{x}_3(0) = 0 \tag{28}$$

The solution of Equation 25 is

$$x_0(t) = A\cos(\sqrt{\mu}\, t) \tag{29}$$

Substitution of this result into Equation 26 gives:

$$\ddot{x}_1 + \mu x_1 + \beta A\cos(\sqrt{\mu}\, t)\left|A\cos(\sqrt{\mu}\, t)\right| + \varepsilon(A\cos(\sqrt{\mu}\, t))^3 = 0, \tag{30}$$

It is possible to do the following Fourier series expansion:

$$f(t) = \beta A\cos(\sqrt{\mu}\, t)\left|A\cos(\sqrt{\mu}\, t)\right| + \varepsilon(A\cos(\sqrt{\mu}\, t))^3 = \sum_{n=0}^{\infty} a_{2n+1}\cos((2n+1)\sqrt{\mu}\, t) = a_1\cos(\sqrt{\mu}\, t) + a_3\cos(3\sqrt{\mu}\, t) + \ldots \tag{31}$$

Where:

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2}\begin{pmatrix}[\beta A\cos(\theta)|A\cos(\theta)| \\ +\varepsilon(A\cos(\theta))^3](\cos(2n+1)\theta)\end{pmatrix} d\theta \tag{32}$$

and

$$a_1 = \frac{3}{4}A^3\varepsilon + \frac{8}{3\pi}A^2\beta, \; a_3 = \frac{1}{4}A^3\varepsilon + \frac{8}{15\pi}A^2\beta, \; a_5 = \frac{1}{4}A^3\varepsilon + \frac{8}{15\pi}A^2\beta, \; \ldots, \tag{33}$$

$f(t)$ has an infinite number of harmonics and it is difficult to solve the new differential equation; however we can truncate the series expansion at Equation 31 and write an approximate equation $f(t)$ in the form:

$$f^{(N)}(t) = \sum_{n=0}^{N} a_{2n+1}\cos((2n+1)\omega t), \tag{34}$$

Equation 34 has only a finite number of harmonics. It is possible to make this approximation because the absolute value of the coefficient b_{2n+1} decreases when n increases as we can easily verify from Equations 31 and 32. Comparing Equations 31 and 34, it follows that

$$f(t) = \lim_{N\to\infty} f^{(N)}(t), \tag{35}$$

In the simplest case we consider N = 1 (n = 0,1) in Equation 34, and we obtain

$$f^{(2)}(t) = a_1\cos(\omega t) + a_3\cos(3\omega t), \tag{36}$$

Substituting Equation 31 into Equation 30,
When N = 1 (n = 0,1) we have:

$$\ddot{x}_1 + \mu x_1 + a_1\cos(\omega t) + a_3\cos(3\omega t) = 0, \; x_1(0) = 0, \; \dot{x}_1(0) = 0, \tag{37}$$

By solving Equation 37 we obtain:

$$x_1 = -\frac{a_5 + 3a_3 - 6a_1}{24\mu}(\cos(\sqrt{\mu}\, t) + \frac{a_5}{24\mu}(\cos(5\sqrt{\mu}\, t) + \frac{a_3}{8\mu}(\cos(3\sqrt{\mu}\, t) - \frac{a_1}{2\mu^{3/2}}(\mu t\sin(\sqrt{\mu}\, t) + \frac{\sqrt{\mu}}{2}\cos(\sqrt{\mu}\, t)), \tag{38}$$

Substituting Equations 29 and 38 into Equation 27, we have

$$\ddot{x}_2 + \mu x_2 + (2\beta\left|A\cos(\sqrt{\mu}\, t)\right| + 3\varepsilon A\cos(\sqrt{\mu}\, t)^2)(-\frac{a_5 + 3a_3 - 6a_1}{24\mu}(\cos(\sqrt{\mu}\, t) + \frac{a_5}{24\mu}(\cos(5\sqrt{\mu}\, t) + \frac{a_3}{8\mu}(\cos(3\sqrt{\mu}\, t) - \frac{a_1}{2\mu^{3/2}}(\mu t\sin(\sqrt{\mu}\, t) + \frac{\sqrt{\mu}}{2}\cos(\sqrt{\mu}\, t))) = 0, \tag{39}$$

The same procedure as was used for calculating x_1 we obtain the following expression for x_2:

$$x_2 = -\frac{b_5 + 3b_3 - 6b_1}{24\mu}$$
$$(\cos(\sqrt{\mu}\,t) + \frac{b_5}{24\mu}(\cos(5\sqrt{\mu}\,t)$$
$$+\frac{b_3}{8\mu}(\cos(3\sqrt{\mu}\,t) - \frac{b_1}{2\mu^{3/2}} \tag{40}$$
$$(\mu t\sin(\sqrt{\mu}\,t) + \frac{\sqrt{\mu}}{2}\cos(\sqrt{\mu}\,t)),$$

Where

$$b_1 = -\frac{A}{3360\,\pi\mu}(4480\sqrt{\mu}\,a_1\beta t + 5040\sqrt{\mu}\,a_1 A\varepsilon t +$$
$$315\pi a_5 A\varepsilon + 630\,\pi\, a_3 A\varepsilon + 768\, a_5\beta + 1792 a_3\beta),$$

$$b_3 = -\frac{A}{30240\pi\mu}(24192\sqrt{\mu}a_1\beta t + 15120\sqrt{\mu}a_1 A\varepsilon t +$$
$$2835\pi a_3 A\varepsilon + 256 a_5\beta + 11520 a_3\beta)$$

$$b_5 = -\frac{A}{332640\,\pi\mu}(63360\sqrt{\mu}\,a_1\beta t + 166320\sqrt{\mu}\,a_1 A\varepsilon t +$$
$$20790\pi\, a_5 A\varepsilon + 31185\pi\, a_3 A\varepsilon + 58112 a_5\beta + 59136 a_3\beta) \tag{41}$$

Having x_i, i = 1,2,...,n, the solutions are as follows:

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \ldots + x_n(t)$$

3.2. The New Modified HPM

To illustrate the new modified HPM, we expand the solution x(t) and the square of the unknown angular frequency ω as follows:

$$\mu = \omega^2 - p\alpha_1 - p^2\alpha_2 - \ldots, \tag{42}$$

$$x = x_0 + px_1 + p^2 x_2 + \ldots \tag{43}$$

Where α_i (i = 1, 2...) are to be determined.

Substituting Equations 42 and 43 into Equation 1 we have:

$$H(x,p) = (1-p)(\ddot{x}_0 + p\ddot{x}_1 + p^2\ddot{x}_2 + \ldots +$$
$$\mu(x_0 + px_1 + p^2 x_2 + \ldots)) + p[(\ddot{x}_0 + p\ddot{x}_1 + p^2\ddot{x}_2 + \ldots +$$
$$(\omega^2 - p\alpha_1 - p^2\alpha_2 - \ldots)(x_0 + px_1 + p^2 x_2 + \ldots) +$$
$$\beta(x_0 + px_1 + p^2 x_2 + \ldots)\left|(x_0 + px_1 + p^2 x_2 + \ldots)\right| +$$
$$\varepsilon(x_0 + px_1 + p^2 x_2 + \ldots)^3] \tag{44}$$

Equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0 : \ddot{x}_0 + \omega^2 x_0 = 0,\ x_0(0) = A,\ \dot{x}_0(0) = 0. \tag{45}$$

$$p^1 : \ddot{x}_1 + \omega^2 x_1 + \beta x_0\left|x_0\right| + \varepsilon x_0^3 - \alpha_1 x_0 = 0,$$
$$x_1(0) = 0,\ \dot{x}_1(0) = 0. \tag{46}$$

$$p^2 : \ddot{x}_2 + \omega^2 x_2 + 2\beta x_1\left|x_0\right| + 3\varepsilon x_0^2 x_1 -$$
$$\alpha_2 x_0 - \alpha_1 x_1 = 0,\ \dot{x}_2(0) = 0,\ x_2(0) = 0. \tag{47}$$

$$p^3 : \ddot{x}_3 + \omega^2 x_3 + \beta(2x_2\left|x_0\right| + x_1\left|x_1\right|) + 3\varepsilon(x_0 x_1^2 +$$
$$x_0^2 x_2) - \alpha_3 x_0 - \alpha_2 x_1 - \alpha_1 x_2 = 0,$$
$$\dot{x}_3(0) = 0,\ x_3(0) = 0. \tag{48}$$

The solution of Equation 45 is:

$$x_0(t) = A\cos(\omega t) \tag{49}$$

Substitution of this result into Equation 46 gives:

$$\ddot{x}_1 + \omega^2 x_1 + \beta A\cos(\omega t)\left|A\cos(\omega t)\right| +$$
$$\varepsilon(A\cos(\omega t))^3 - \alpha_1 A\cos(\omega t) = 0, \tag{50}$$

It is possible to do the following Fourier series expansion:

$$\beta A\cos(\omega t)\left|A\cos(\omega t)\right| + \varepsilon(A\cos(\omega t))^3 = \sum_{n=0}^{\infty} a_{2n+1}$$
$$\cos((2n+1)\omega t) = a_1\cos(\omega t) + a_3\cos(3\omega t) + \ldots \tag{51}$$

Where

$$a_{2n+1} = \frac{4}{\pi} \int_0^{\pi/2} \Big([\beta A\cos(\theta)|A\cos(\theta)| + \varepsilon (A\cos(\theta))^3](\cos(2n+1)\theta) \Big) d\theta \tag{52}$$

and

$$\begin{aligned} a_1 &= \frac{3}{4}A^3\varepsilon + \frac{8}{3\pi}A^2\beta, \\ a_3 &= \frac{1}{4}A^3\varepsilon + \frac{8}{15\pi}A^2\beta, \\ a_5 &= A^3\varepsilon + \frac{8}{15\pi}A^2\beta, \ \ldots \end{aligned} \tag{53}$$

Substituting Equation 51 into 50, we have:

$$\ddot{x}_1 + \omega^2 x_1 + \sum_{n=0}^{\infty} a_{2n+1} \cos((2n+1)\omega t) - \alpha_1 A \cos(\omega t) = 0,$$

or

$$\ddot{x}_1 + \omega^2 x_1 + \sum_{n=1}^{\infty} a_{2n+1} \cos((2n+1)\omega t) + (a_1 - \alpha_1 A)\cos(\omega t) = 0, \tag{54}$$

No secular terms in $x_1(t)$ requires eliminating contributions proportional to cos(ωt) in the Equation 54 and we obtain

$$\alpha_1 = \frac{a_1}{A}, \tag{55}$$

Taking into account Equations 55 and 54, we rewrite Equation 54 in the form:

$$\ddot{x}_1 + \omega^2 x_1 = -\sum_{n=1}^{\infty} a_{2n+1} \cos((2n+1)\omega t), \tag{56}$$

With initial conditions $x_1(0) = 0$ and $\dot{x}_1(0) = 0$. The periodic solution to Equation 56 can be written as:

$$x_1(t) = \sum_{n=0}^{\infty} b_{2n+1} \cos((2n+1)\omega t) = b_1 \cos(\omega t) + b_3 \cos(3\omega t) + \ldots \tag{57}$$

Substituting Equation 57 into Equation 56 we obtain:

$$\omega^2 \sum_{n=0}^{\infty} b_{2n+1}(1 - (2n+1)^2)\cos((2n+1)\omega t) = -\sum_{n=1}^{\infty} a_{2n+1} \cos((2n+1)\omega t), \tag{58}$$

We can write the following expression for the coefficients b_{2n+1}:

$$b_{2n+1} = \frac{a_{2n+1}}{((2n+1)^2 - 1)\omega^2} = \frac{a_{2n+1}}{4n(n+1)\omega^2}, \quad \text{for } n \geq 1 \tag{59}$$

Taking into account that $x_1(0) = 0$, Equation 31 gives

$$b_1 = -\sum_{n=1}^{\infty} b_{2n+1} \tag{60}$$

$x_1(t)$ has an infinite number of harmonics and it is difficult to solve the new differential equation; however we can truncate the series expansion at Equation 57 and write an approximate equation $x_1^{(N)}(t)$ in the form

$$x_1^{(N)}(t) = b_1^{(N)} \cos(\omega t) + \sum_{n=1}^{N} b_{2n+1} \cos((2n+1)\omega t) \tag{61}$$

Where

$$b_1^{(N)} = -\sum_{n=1}^{N} b_{2n+1} \tag{62}$$

Equation 61 has only a finite number of harmonics. It is possible to make this approximation because the absolute value of the coefficient b_{2n+1} decreases when n increases as we can easily verify from Equations 51 and 59. Comparing Equations 57 and 61, and Equations 60 and 61, it follows that:

$$x_1(t) = \lim_{n\to\infty} x_1^{(N)}(t), \quad b_1 = \lim_{n\to\infty} b_1^{(N)} \tag{63}$$

In the simplest case we consider N = 1 (n = 0,1) in Equations 61 and 62, and we obtain:

$$x_1^{(1)}(t) = b_3(-\cos(\omega t) + \cos(3\omega t)), \tag{64}$$

From Equation 59 the following expression for the coefficient b_3 is obtained:

$$b_3 = \frac{a_3}{8\omega^2}, \tag{65}$$

and from Equation 42 and 55, writing p = 1, we can find that the first-order approximate frequency

$$\omega_1(A) = \sqrt{\mu + \alpha_1} = \sqrt{\mu + \frac{a_1}{A}} \tag{66}$$

Substituting Equations 49, 64 and 66 into Equation 3.26 gives the following equation for $x_2(t)$:

$$\begin{aligned}&\ddot{x}_2 + \omega^2 x_2 + \\ &2\beta b_3(-\cos(\omega t) + \cos(3\omega t))\left|A\cos(\omega t)\right| + \\ &3\varepsilon A^2 \cos^2(\omega t) b_3(-\cos(\omega t) + \cos(3\omega t)) - \\ &\alpha_2 A\cos(\omega t) - \alpha_1 b_3(-\cos(\omega t) + \cos(3\omega t)) = 0,\end{aligned} \tag{67}$$

It is possible to do the following Fourier series expansion:

$$\begin{aligned}&2\beta b_3(-\cos(\omega t) + \cos(3\omega t))\left|A\cos(\omega t)\right| + \\ &3\varepsilon A^2 \cos^2(\omega t) b_3(-\cos(\omega t) + \cos(3\omega t)) - \\ &\alpha_1 b_3(\cos(3\omega t)) = \sum_{n=0}^{\infty} c_{2n+1}\cos((2n+1)\omega t) = \\ &c_1\cos(\omega t) + c_3\cos(3\omega t) + \ldots\end{aligned} \tag{68}$$

Where

$$c_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} \begin{pmatrix} [2\beta b_3(-\cos(\theta) + \cos(3\theta)) \\ \left|A\cos(\omega t)\right| - \alpha_1 b_3(\cos(3\theta)) \\ + 3\varepsilon A^2\cos^2(\theta) b_3(-\cos(\theta) \\ + \cos(3\theta))]\cos((2n+1)\theta) \end{pmatrix} d\theta \tag{69}$$

and

$$\begin{aligned}&c_1 = \frac{3}{4}A^2\varepsilon(b_3 + 3b_1) + \frac{16}{3\pi}A\beta(\frac{b_3}{5} + b_1), \\ &c_3 = \frac{3}{8}A^2\varepsilon(b_3 + \frac{b_1}{2}) + \frac{4}{\pi}A\beta(\frac{36}{35}b_3 + \frac{4}{15}b_1) - \alpha_1 b_3, \\ &c_5 = \frac{3}{4}A^2\varepsilon b_3 + \frac{16}{3\pi}A\beta(\frac{5b_3}{21} - \frac{b_1}{35}), \ \ldots\end{aligned} \tag{70}$$

Substituting Equation 31 into Equation 30, we have:

$$\begin{aligned}&\ddot{x}_2 + \omega^2 x_2 + \sum_{n=0}^{\infty} c_{2n+1}\cos((2n+1)\omega t) \\ &- \alpha_2 A\cos(\omega t) + \alpha_1 b_3\cos(\omega t) = 0,\end{aligned}$$

or

$$\begin{aligned}&\ddot{x}_2 + \omega^2 x_2 + \sum_{n=1}^{\infty} c_{2n+1}\cos((2n+1)\omega t) \\ &+ (\alpha_1 b_3 - \alpha_2 A + c_1)\cos(\omega t) = 0,\end{aligned} \tag{71}$$

No secular terms in $x_2(t)$ requires eliminating contributions proportional to cos(ωt) in the Equation 71 and we obtain

$$\alpha_2 = \frac{c_1 + \alpha_1 b_3}{A}, \tag{72}$$

Taking into account Equation 72 and 71, we rewrite Equation 71 in the form

$$\ddot{x}_2 + \omega^2 x_2 = -\sum_{n=1}^{\infty} c_{2n+1}\cos((2n+1)\omega t), \tag{73}$$

With initial conditions $x_2(0) = 0$ and $\dot{x}_2(0) = 0$. The periodic solution to Equation 73 can be written as:

$$x_2(t) = \sum_{n=0}^{\infty} d_{2n+1} \cos((2n+1)\omega t) = d_1 \cos(\omega t) + d_3 \cos(3\omega t) + \ldots \quad (74)$$

Substituting Equation 74 into Equation 73 we obtain:

$$\omega^2 \sum_{n=0}^{\infty} d_{2n+1}(1-(2n+1)^2)\cos((2n+1)\omega t) = -\sum_{n=1}^{\infty} c_{2n+1}\cos((2n+1)\omega t), \quad (75)$$

We can write the following expression for the coefficients b_{2n+1}:

$$d_{2n+1} = \frac{c_{2n+1}}{((2n+1)^2-1)\omega^2} = \frac{c_{2n+1}}{4n(n+1)\omega^2},$$

for $n \geq 1$ (76)

Taking into account that $x_2(0) = 0$, Equation 74 gives

$$d_1 = -\sum_{n=1}^{\infty} d_{2n+1} \quad (77)$$

The same procedure as was used for approximate x_1 we obtain the following expression for x_2:

$$x_2^{(2)}(t) = -(d_3+d_5)\cos(\omega t) + d_3\cos(3\omega t) + d_5\cos(5\omega t), \quad (78)$$

and from Equations 42 and 72, writing p = 1, we can find that the first-order approximate

$$\omega_2(A) = \sqrt{\mu+\alpha_1+\alpha_2} = \sqrt{\mu + \frac{a_1}{A} + \frac{c_1}{A} + \frac{a_1 b_3}{A^2}} \quad (79)$$

3.3. Energy Balance Method for Equation 1,

$$f(x) = \mu x + \beta x^2 + \varepsilon x^3,$$

and

$$F(x) = \mu\frac{x^2}{2} + \beta\frac{x^3}{3} + \varepsilon\frac{x^4}{4}$$

Its variational and Hamiltonian formulations can be readily obtained as follows:

$$J(x) = \int_0^t \left(-\frac{1}{2}x'^2 + \mu\frac{x^2}{2} + \beta\frac{x^3}{3} + \varepsilon\frac{x^4}{4} \right) dt \quad (80)$$

$$H = \frac{1}{2}x'^2 + \mu\frac{x^2}{2} + \beta\frac{x^3}{3} + \varepsilon\frac{x^4}{4} = \mu\frac{A^2}{2} + \beta\frac{A^3}{3} + \varepsilon\frac{A^4}{4} \quad (81)$$

$$R(t) = \frac{1}{2}x'^2 + \mu\frac{x^2}{2} + \beta\frac{x^3}{3} + \varepsilon\frac{x^4}{4} - \mu\frac{A^2}{2} - \beta\frac{A^3}{3} - \varepsilon\frac{A^4}{4} = 0 \quad (82)$$

Substituting (17) into (82), we obtain:

$$R(t) = \frac{A^2\omega^2}{2}\sin(\omega t)^2 + \mu\frac{A^2\cos(\omega t)^2}{2} + \beta\frac{A^3\cos(\omega t)^3}{3} + \varepsilon\frac{A^4\cos(\omega t)^4}{4} - \mu\frac{A^2}{2} - \beta\frac{A^3}{3} - \varepsilon\frac{A^4}{4} = 0 \quad (83)$$

We obtain the following result:

$$\omega = \left(-\mu\cos(\omega t)^2 - \beta\frac{2A\cos(\omega t)^3}{3} - \varepsilon\frac{A^2\cos(\omega t)^4}{2} + \mu + \beta\frac{2A}{3} + \varepsilon\frac{A^2}{2}\right)^{1/2} \Big/ \sin(\omega t) \quad (84)$$

with $T = 2\pi/\omega$, yields:

$$T = 2\pi\sin(\omega t) \Big/ \left(-\mu\cos(\omega t)^2 - \beta\frac{2A\cos(\omega t)^3}{3} - \varepsilon\frac{A^2\cos(\omega t)^4}{2} + \mu + \beta\frac{2A}{3} + \varepsilon\frac{A^2}{2}\right)^{1/2} \quad (85)$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \sqrt{-\beta\frac{\sqrt{2}A}{3} + \varepsilon\frac{3A^2}{4} + \mu + \beta\frac{4A}{3}} \qquad (86)$$

with $T = 2\pi/\omega$, yields:

$$T = \frac{2\pi}{\sqrt{-\beta\frac{\sqrt{2}A}{3} + \varepsilon\frac{3A^2}{4} + \mu + \beta\frac{4A}{3}}} \qquad (87)$$

4. CONCLUSIONS

In the present work, we have applied He's Homotopy Perturbation Method (HPM), modification He's Homotopy Perturbation Method (MHPM) and He's Energy balance method (EBM) to investigation of fluctuation and frequency of the oscillator's governing equation with strong nonlinearities. This equation is solved by the numerical method using the software MAPLE 11, whose results of the different methods of HPM and MHPM are compared in Figures 3-5, and the result for A = 5, A = 10 have been shown in Figure 4.

Observe that HPM is just valid for short region, but the new modification HPM solution exactly the same with the numerical solution (NS) for this strongly nonlinear problem. These approximate analytical solutions are in an excellent agreement with the corresponding numerical solutions.

Figure 5 shows the comparison between numerical solution and new modification HPM for different value of A, μ, β and ε.

The results of the different methods of MHPM and EBM for frequency are compared in Figures 6-12.

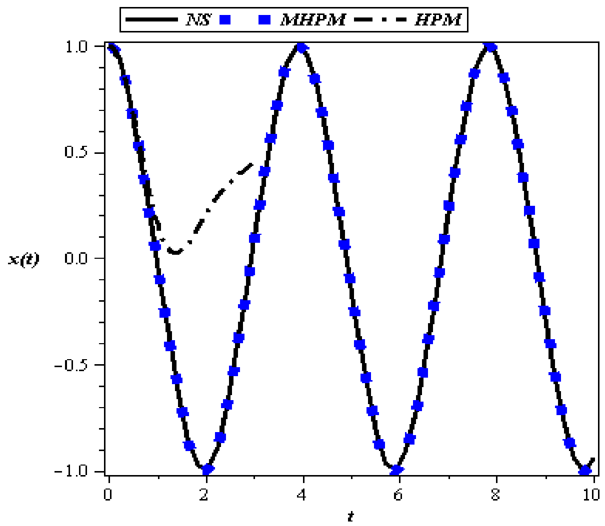
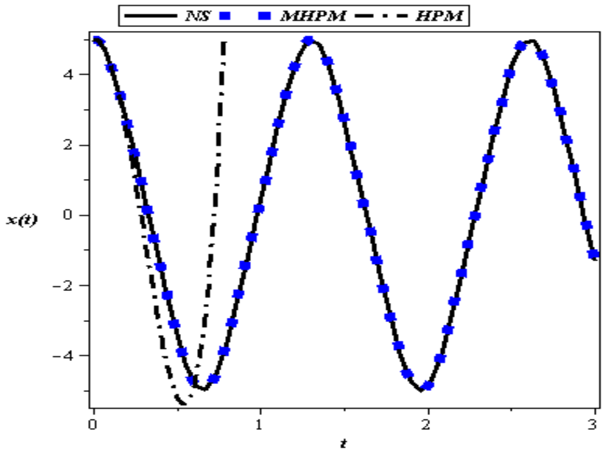
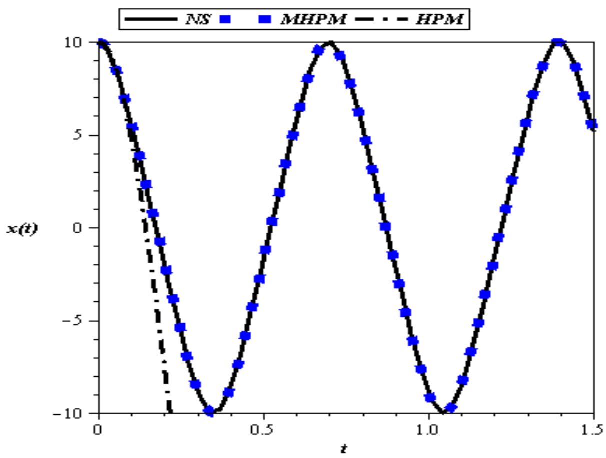


Figure 3. The comparison between standard HPM, modified HPM and numerical solutions for A = 1, μ = 1, β = 1, $\varepsilon = 1$.

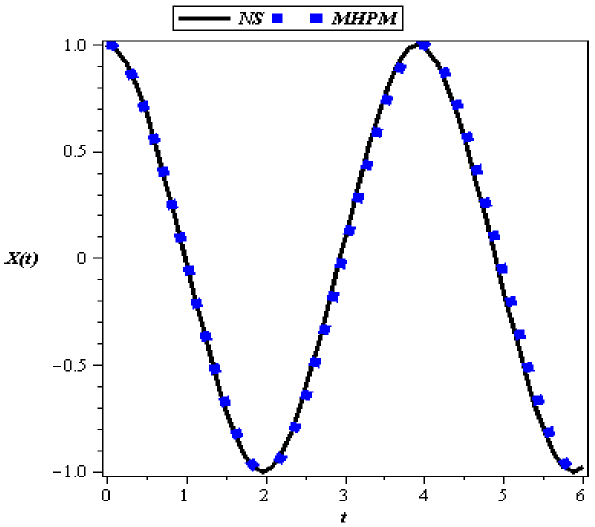


(a)

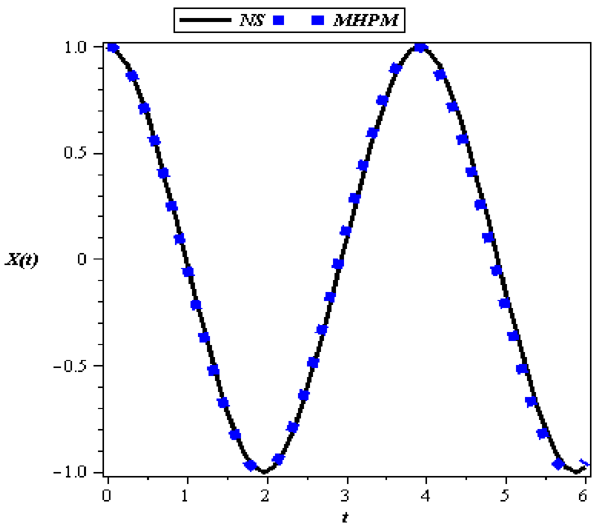


(b)

Figure 4. The comparison between standard HPM, modified HPM and numerical solutions for (a) A = 5, μ = 1, β = 1, $\varepsilon = 1$ and (b) A = 10, μ = 1, β = 1, $\varepsilon = 1$.



(a)



(b)

Figure 5. The comparison between standard HPM, modified HPM and numerical solution for (a) $A = 1$ $\mu = 0.5$, $\beta = 1$, $\varepsilon = 1.5$, (b) $A = 1$, $\mu = 1$, $\beta = 1.5$, $\varepsilon = 0.5$.

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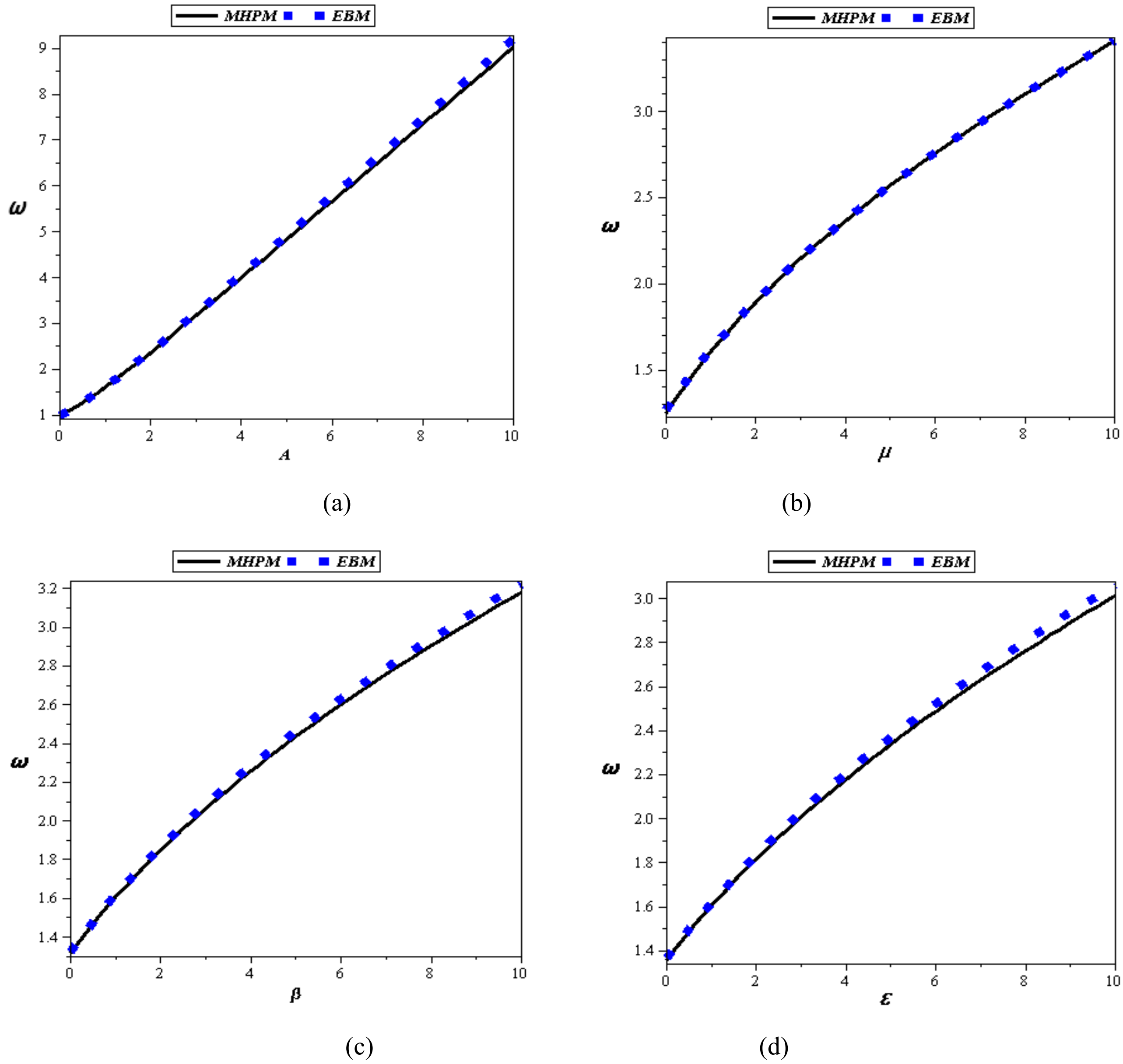

Figure 6. The comparison between modified HPM and EBM for (a) μ = 1, β = 1, ε = 1, (b) A = 1, β = 1, ε = 1 (c) A= 1, μ = 1, ε = 1 (d) A = 1, β = 1, μ = 1.

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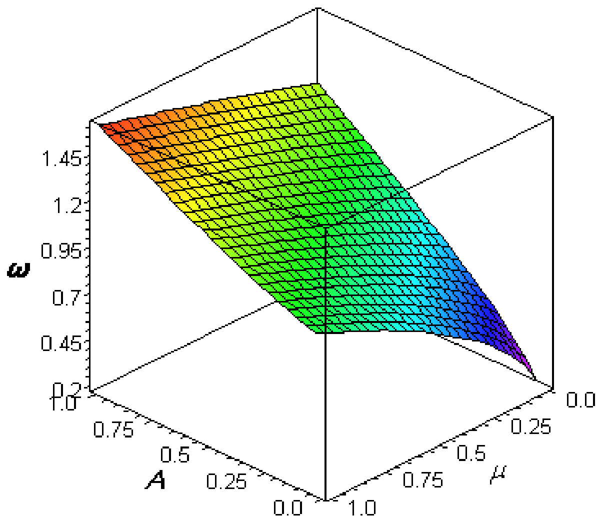

Figure 7. The MHPM results for ω for β = 1, ε = 1 .

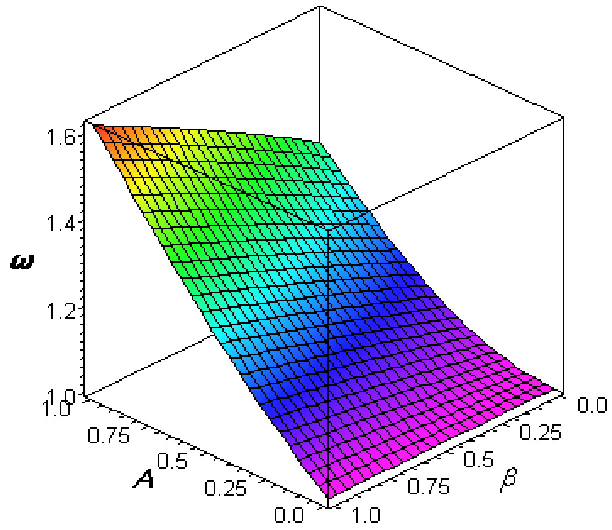

Figure 9. The MHPM results for ω for μ= 1, ε = 1 .

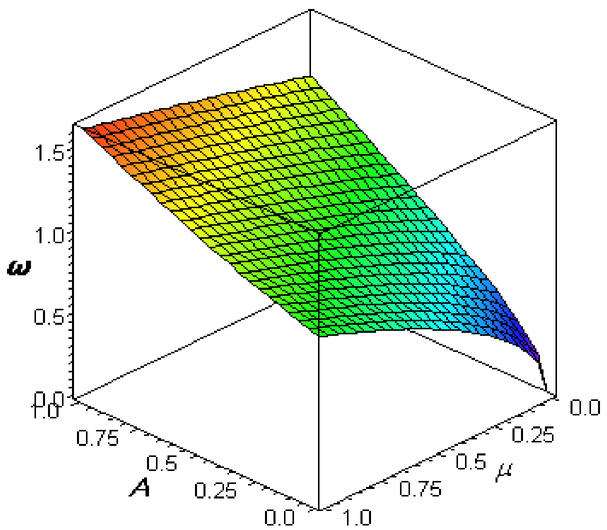

Figure 8. The EBM results for ω for β = 1, ε = 1.

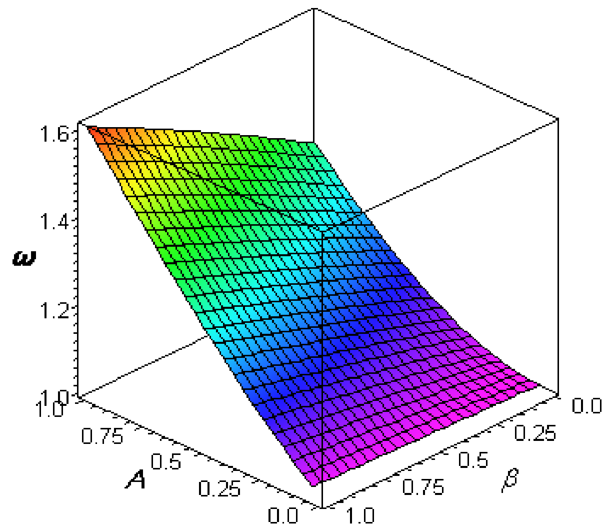

Figure 10. The EBM results for ω for μ = 1, ε = 1 .

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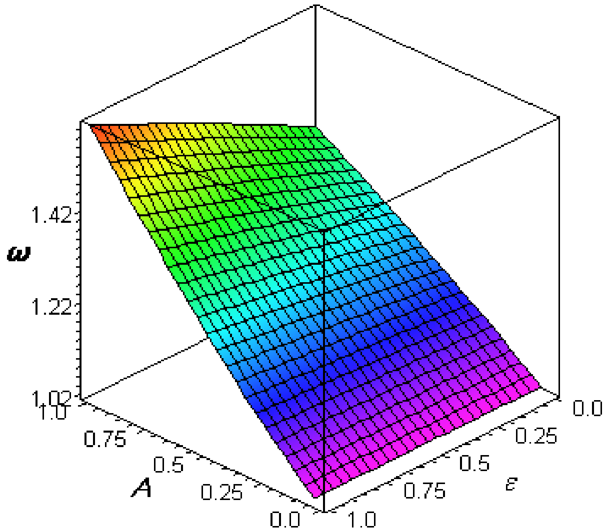

Figure 11. The MHPM results for ω for μ=1, $\beta = 1$.

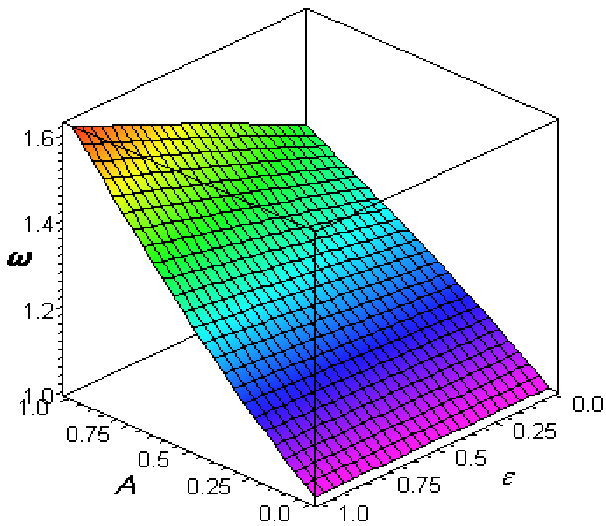

Figure 12. The EBM results for ω for μ= 1, $\beta = 1$.

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