TECHNICAL NOTE

FIRST INTEGRALS OF A SPECIAL SYSTEM OF ODES

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Abstract In this paper we suggest a method to calculate the first integrals of a special system of the first order of differential equations. Then we use the method for finding the solutions of some differential equations such as, the differential equation of RLC circuit.

Keywords Distribution, First Integral, RLC Circuit, Heat Capacity

چکیده در این مقاله روشی برای محاسبهٔ انتگرال های اول یک دستگاه معادلات دیفرانسیل مرتبهٔ اول بیان شده است. سپس این روش را برای یافتن جواب های برخی از معادلات دیفرانسیلَ، مانند معادلهٔ دیفرانسیل مدار RLC به کار می‌بریم.

1. INTRODUCTION

Let us consider a system of ordinary differential equations:

$$\begin{cases} \dfrac{dx}{dt} = A_1 x + A_2 y + A_3 \\ \dfrac{dy}{dt} = B_1 x + B_2 y + B_3 \end{cases}$$

First of all we will show that this system always have first integrals, next by a method which will be explained, we obtain the first integrals, in the sequel. The system which we are exposed to, is a special case and not a general one, because the important point is as we will see, it needs tedious computations to approach to the first integrals of this simple and special system, and it needs mathematical software for our purpos.

2. EXISTENCE OF A FIRST INTEGRAL

Before starting the main part, it is necessary to change the form of the system to make the calculation more direct, on the condition that the solutions will not vary.

2.1. Frobenius Theorem The Frobenius theorem is one of the most important theorems in theory of differential equations which, its result guarantees that the system would have the first integrals, before we needs a lot of details.

First of all to make the calculations easier we translate the two independent variables of equations by $x - \frac{A_3}{A_1}$ in the first one and $y - \frac{B_3}{B_2}$ in the second one. Thus we have the new following ODEs system which has the same solutions as the first system:

$$\begin{cases} \dfrac{dx}{dt} = A_1 x + A_2 y \\ \dfrac{dy}{dt} = B_1 x + B_2 y \end{cases}. \quad (1)$$

The solutions of this system is a 3-dimensional submanifold of 5-dimensional jet space, which is denoted by

$$J = \left\{ \left(x, y, t, \frac{dx}{dt}, \frac{dy}{dt} \right) : x, y, t \in R \right\}.$$

2.1.1. Definition Suppose M is an n-dimensional manifold and $p \in M$. A choice of k-dimensional linear subspace $D_p \subset T_pM$ is called a k-dimensional tangent distribution or a k-dimensional distribution. D_p is called smooth if $D = \cup_{P \in M} D_p \subset TM$ is a smooth subbundle of TM.

2.1.2. Lemma Let M be a smooth n-manifold, and let $D \subset TM$ be a k-dimensional distribution. Then D is smooth if and only if each point $p \in M$ has a neighborhood U on which there are smooth 1-forms $\omega^1, \ldots, \omega^{n-k}$ such that for each $q \in U$,

$$D_q = \mathrm{Ker}\omega^1\Big|_q \cap \cdots \cap \mathrm{Ker}\omega^{n-k}\Big|_q.$$

For details of proof see [3].

More precisely, if we denote the annihilator of D by $\mathrm{Ann}(D) = \{\omega \in \Omega^1(M) : \omega = 0 \text{ on } D\}$, then for any ω^i defined in lemma 2.1.2 we have $\omega^i \in \mathrm{Ann}(D)$.

If D is a distribution generates by $\{V_1, \ldots, V_k\}$, then D could be discussed by $\omega^1, \ldots, \omega^{n-k}$ too. We will show such a distribution as $D = F(V_1, \ldots, V_k) = F(\omega^1, \ldots, \omega^{n-k})$.

2.1.3. Definition Suppose $D = F(V_1, \ldots, V_k)$ is a k-dimensional distribution. The distribution $D^{(1)}$ which is generated by the vector fields $\{V_1, \ldots, V_k\}$ and by all possible sorts of commutators $[V_i, V_j]$ (i < j; i, j = 1,...,k), is called the first derivative of D, i.e.,

$$D^{(1)} = F(V_1, \ldots, V_k, [V_1, V_2], \ldots, [V_1, V_k], \ldots, [V_{k-1}, V_k]).$$

2.1.4. Lemma Let D = $F(V_1,\ldots,V_k)$ is a distribution such that D = $F(\omega^1,\ldots,\omega^{n-k})$. Then $D^{(1)}$ = D if and only if for i = 1,...,n-k

$$d\omega^i \wedge \omega^1 \wedge \ldots \wedge \omega^{n-k} = 0. \quad (2)$$

See [1] for a proof.

2.1.5. Definition A smooth distribution D on a smooth manifold M is called Completely Integrable Distribution or a CID distribution if all points of M contain in an integral manifold of D. (A submanifold $N \subset M$ is called an integral manifold of D if $T_pN \subset D$).

Now we are ready to have the Frobenius theorem.

2.1.6. Theorem (Frobenius theorem) Let D be a smooth distribution on a smooth manifold M. D is CID if and only if $D^{(1)} = D$.

See [3,4] for two different kinds of proofs.

Now by using the Frobenius theorem it will be shown that any first order ODE has a first integral. This is the result of the Frobenius theorem and its following corollary, before we have a necessary definition.

2.1.7. Definition Let D be a smooth distribution on smooth manifold M, a smooth function $\phi \in C^\infty(M)$ is called a first integral for D if $d\phi \in \mathrm{Ann}(D)$. Here $C^\infty(M)$ means the set of all smooth real valued functions on manifold M.

Another definition for a CID distribution D, is that D is CID if and only if there exist n-k first functional independent integrals $\phi_1, \ldots, \phi_{n-k}$ such that $D = F(d\phi_1, \ldots, d\phi_{n-k})$.

So if a distribution is CID it means that it has first integral(s).

2.1.8. Corollary Suppose D is a smooth distribution such that $D = F(\omega)$. Then D is CID if and only if

$$\omega \wedge d\omega = 0. \quad (3)$$

Now consider a first order ODE

$$y' = \frac{dy}{dx} = f(x, y), \quad (4)$$

It is clear that the Equation 4 is obtained by taking the 1-form $\omega = dy - f(x,y)dx$, to zero, so ω satisfies the Equation 3, thus the distribution corresponds to the Equation 4 has a first integral.

3. SYMMETRIES OF DISTRIBUTIONS

Symmetry of a distribution is the transformation of the manifold M that maps distribution into itself.

In other words, a diffeomorphism F: M→ M is a symmetry of a distribution D if

$$F_*(D_p) = D_{F(p)}, \tag{5}$$

for all $p \in M$. Here F_* is the push forward map of F.

Assume that D = $F(\omega^1,\ldots,\omega^{n-k})$, then the Equation 5 means that the differential 1-forms $F^*(\omega^1),\ldots,F^*(\omega^{n-k})$ determine the same distribution D and therefore can be expressed in terms of basic forms $\omega^1,\ldots,\omega^{n-k}$. That is

$$\begin{gathered} F^*(\omega^1) \wedge \omega^1 \wedge \ldots \wedge \omega^{n-k} = 0, \\ \vdots \\ F^*(\omega^{n-k}) \wedge \omega^1 \wedge \ldots \wedge \omega^{n-k} = 0. \end{gathered}$$

Where F^* is the pull back map of F.

3.1. Infinitesimal Symmetry Suppose V is an smooth vector field on manifold M, and θ_t be its flow, then we know $\theta_t : M \to M$ induces a diffeomorphism on M.

3.1.1. Definition θ_t which has defined above is called an infinitesimal symmetry or a symmetry of a distribution D if θ_t along the vector field V consist of symmetries of D, i.e, $(\theta_t)_*(D_p) = D(\theta_t(p))$, for all points $p \in M$ and t.

Denote Sym (D) the set of all infinitesimal symmetries of distribution D.

3.1.2. Theorem Let D be a distribution and $\Gamma(D)$ denotes the set of all vector fields on D, and then the following statements are equivalent.

- $V \in \mathrm{Sym}(D)$.
- $\forall\ W \in \Gamma(D) \Rightarrow [V,W] \in \Gamma(D)$.
- $\forall \omega \in \mathrm{Ann}(D) \Rightarrow L_V(\omega) \in \mathrm{Ann}(D)$.

In this theorem $L_V(\omega)$, denotes the Lie derivative of ω respect to V. For the proof of the theorem see [3], and for more details of Lie derivative see [4].

3.1.3. Corollary Sym (D) has a real Lie algebra structure with respect to the commutator of the vector fields.

3.2. Distribution with a Commutative Symmetry Algebra Let ∑ be a commutative symmetry Lie algebra which its dimension is equal to the dimension of D = $F(\omega^1,\ldots,\omega^n)$ is equal to k. Let $V_1,\ldots V_k$ be a basis of ∑ and let D is a CID distribution.

Then form the matrix

$$Z = \begin{pmatrix} \omega^1(V_1) & \cdots & \omega^1(V_k) \\ \vdots & \vdots & \vdots \\ \omega^k(V_1) & \cdots & \omega^k(V_k) \end{pmatrix}. \tag{6}$$

Because of independence of ω^is, then Z^{-1} exists.

Now we are going to construct a new basis $\overline{\omega}^1,\ldots,\overline{\omega}^k$ for Ann (D).

These basis are constructed by the following relation:

$$\begin{pmatrix} \overline{\omega}^1 \\ \vdots \\ \overline{\omega}^k \end{pmatrix} = Z^{-1} \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^k \end{pmatrix}. \tag{7}$$

Then it is possible to see that $\overline{\omega}^i$ s are closed, so the functions $\phi_i(p) = \int_\alpha \overline{\omega}^i$ are called the first integrals of D. Here α is a path from the fix point p_0 to a point $p \in M$.

For example when D = F(ω) be a CID distribution and V be a symmetry of D, then the 1-form $\overline{\omega} = \frac{1}{\omega(V)}\omega$, is closed and the function

$\phi = \int_{\alpha} \frac{\omega}{\omega(V)}$, is the first integral of D.

For another example, consider an ODE in the form of Equation 4. We know that the corresponding 1-form is $\omega = dy - f(x,y)dx$, suppose that $V = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$ is a symmetry of a distribution, that is the symmetry of the equation. (For scrutinizing the symmetries of ODEs see [5,6]). Then, Z = b(x,y)-f(x,y)a(x,y) and the differential 1-form $\overline{\omega} = \frac{dy - f(x,y)dx}{b(x,y) - f(x,y)a(x,y)}$, is closed, and the function $\phi = \int_{\alpha} \overline{\omega}$ is a first integral of the equation. The function Z^{-1} is called an integrating factor for the equation.

One can use a differential equation to describe the behavior of fluids channeled by a funnel and use the first integral method for its solution. Consider a conical funnel of angle θ at its apex with the hole of radius r_0. According to the generally accepted law of hydraulics, the velocity of outflow of a fluid from a hole under hydrostatic pressure is given by the formula $v = \eta\sqrt{2gh}$ cm/s with an empirical coefficient η (for water η = 3/5). Here "h" is the heat of the fluid over the hole and g ≈ 981 cm/s is the gravitational constant. To describe the outflow from the funnel, it suffices to determine the height h = h(t) of the fluid in the funnel. To determine the unknown function h(t), let us write down the balance of the fluid. The volume of the fluid that flows from the hole in dt seconds is equal to $dv = \pi r_0^2 v dt \equiv \pi r_0^2 \sqrt{vghdt}$ cm^2. On the other hand, the decrease of the fluid volume, due to the negative increment (-dh) of the height of the fluid in the funnel, is given by dv = $-\pi r^2(t)dh$, where r(t) is the radius of the funnel at the height h = h(t) an hence r(t) = h tan(θ/2). It follows that the balance equation is

$$\pi \tan^2(\theta/2)\, h^2 dh + \pi\eta r_0^2 \sqrt{2ghdt} = 0.$$

Thus, we arrive at the differential equation

$$h^{3/2}\frac{dh}{dt} + \frac{\eta\sqrt{2gr_0{}^2}}{\tan^2(\theta/2)} = 0,$$

Where it is in the form of Equation 4, consequently it has a first integral. Because "t" does not exist in the equation explicitly, the vector field $V = \frac{\partial}{\partial t}$, is its symmetry and $\phi = \int_{\alpha}\left(dt + \frac{h^{3/2}}{\lambda}dh\right)$, is the first integral of the differential equation of outflow from a funnel, where $\lambda = \eta\sqrt{2gr_0{}^2} / \tan^2(\theta/2)$ is a constant. It is possible to see that

$$\phi = t + \frac{2}{5\lambda}h^{5/2},$$

by taking ϕ to zero we obtain

$$h = \left(-\frac{5}{2}\lambda t\right)^{2/5},$$

Which is the solution of the differential equation of outflow from a funnel.

3.3. First Integral of the System (1) Now we are ready to obtain first integrals of the system (1). This system corresponds to the following differential 1-forms:

$$\omega^1 = dx - (A_1x + A_2y)dt,$$

$$\omega^2 = dy - (B_1x + B_2y)dt. \tag{8}$$

The corresponding distribution is $D = F(\omega^1, \omega^2)$.

Now if $V \in \Gamma(D)$ has the form $V = a(x,y,t)\frac{\partial}{\partial x} + b(x,y,t)\frac{\partial}{\partial y} + c(x,y,t)\frac{\partial}{\partial t}$, then we should have $\omega^1(V) = \omega^2(V) = 0$,

Consequently

$$a(x,y,t) = (A_1x + A_2y)c(x,y,t),$$

$$b(x,y,t) = (B_1x + B_2y)c(x,y,t),$$

by substituting a(x,y,t) and b(x,y,t) in V we have

$$D = F\left(V = (A_1x + A_2y)\frac{\partial}{\partial x} + (B_1x + B_2y)\frac{\partial}{\partial y} + \frac{\partial}{\partial t} \right).$$

Let

$W = \tilde{a}(x,y,t)\frac{\partial}{\partial x} + \tilde{b}(x,y,t)\frac{\partial}{\partial y} + \tilde{c}(x,y,t)\frac{\partial}{\partial t}$, be a symmetry of the distribution, i.e., $W \in \mathrm{Sym}(D)$, the second part of theorem 3.1.3 implies that

$$[V,W] \in \Gamma(D). \tag{9}$$

Now we consider those symmetries of D which contain translations, scales and rotations.

Now substituting

$$\tilde{a}(x,y,t) = a_1x + a_2y + a_3t + a_4,$$

$$\tilde{b}(x,y,t) = b_1x + b_2y + b_3t + b_4,$$

$$\tilde{c}(x,y,t) = c_1x + c_2y + c_3t + c_4,$$

in the relation (9), implies that

$$\tilde{a}A_1 + \tilde{b}A_2 - (A_1x + A_2y)a_1 - (B_1x + B_2y)a_2 - a_3 = \lambda(x,y,t)(A_1x + A_2y),$$

$$\tilde{a}B_1 + \tilde{b}B_2 - (A_1x + A_2y)b_1 - (B_1x + B_2y)b_2 - b_3 = \lambda(x,y,t)(B_1x + B_2y),$$

$$(A_1x + A_2y)c_1 + (B_1x + B_2y)c_2 + c_3 = -\lambda(x,y,t).$$

Here $\lambda(x,y,t)$ is an arbitrary function of x, y and t.

One of the solutions of the above equations system respect to a_i s, b_i s and c_i s is

$$a_1 = \frac{1}{B_1}(b_1A_1 + b_2B_1 - b_1B_2)$$

$$a_2 = \frac{b_1A_2}{B_1},$$

and

$$a_3 = a_4 = b_3 = b_4 = c_1 = c_2 = c_3 = 0.$$

Consequently we obtain one of the special form of W. That is

$$W = \left[\frac{1}{B_1}\left(b_1A_1 + b_2B_1 - b_1B_2 \right)x + \frac{b_1A_2}{B_1}y \right]\frac{\partial}{\partial x} + (b_1x + b_2y)\frac{\partial}{\partial y}.$$

Let us suppose that $b_1 = 1$ and $b_2 = 0$ thus,
$W_1 = \frac{1}{B_1}\left[(A_1 - B_2)x + A_2y \right]\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$,

and

If $b_1 = 0$ and $b_2 = 1$ we have,

$$W_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

it is possible to see that these two vector fields make a set of two independent symmetries for system (1). Next we construct matrix Z,

$$Z = \begin{pmatrix} \frac{1}{B_1}\left[(A_1 - B_2)x + A_2y \right] & x \\ x & y \end{pmatrix}.$$

By using relation (7) we calculate the new basis $\overline{\omega}^1$ and $\overline{\omega}^2$, as

$$\begin{pmatrix} \overline{\omega}^1 \\ \overline{\omega}^2 \end{pmatrix} = \frac{B_1}{K}\begin{pmatrix} y & -x \\ -x & \frac{1}{B_1}\left[(A_1 - B_2)x + A_2y \right] \end{pmatrix}\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

Where

K is $\left(A_1 - B_2 \right)xy + A_2y^2 - B_1x^2$,

So we have

$$\overline{\omega}^1 = \frac{B_1}{K}\left\{ ydx - xdy + \left[B_1x^2 - A_2y^2 + (B_2 - A_1)xy \right]dt \right\},$$

$\overline{\omega}^2 =$

$$\frac{1}{K}\Big\{-B_1 x dx + \Big[(A_1 - B_2)x + A_2 y\Big]dy + \Big[(2A_1B_1 + B_1B$$

$$+ A_2B_1)x^2 + (A_1B_2 + B_2{}^2 - A_2B_1)xy - A_2B_2 y^2\Big]dt\Big\}.$$

($\overline{\omega}^1$ and $\overline{\omega}^2$ are closed.), which make a new basis for the annihilator of the distribution corresponds to the system (1), so we can find the first integrals of the system (1) with the following integrations:

$$\phi(x, y, t) = \int_\alpha \overline{\omega}^1,$$

$$\eta(x, y, t) = \int_\alpha \overline{\omega}^2 .$$

Because of the two 1-forms are closed, these integrations are independent with respect to path α.

Consequently if

$$E = A_1{}^2 + B_2{}^2 + 4A_2B_1 - 2A_1B_2,$$

$$F = \frac{2A_2 y + (B_2 - A_1)x}{x\sqrt{E}},$$

$$G = \frac{2B_1 x + (B_2 - A_1)y}{y\sqrt{E}},$$

We have

$\varphi(x, y, t) =$

$$\frac{2xB_1}{\sqrt{E}}\arctan(F) + \frac{2B_1}{\sqrt{E}}\Bigg\{\arctan(-G)\left(\frac{B_2 - A_1}{2B_1 x} + \frac{1}{y}\right) +$$

$$\frac{\sqrt{E}}{4B_1 x}\mathrm{Ln}(G^2 + 1) - \frac{2B_1}{\sqrt{E}}\arctan\Bigg[\frac{2(E + (A_1 - B_2)^2)y}{4B_1 x\sqrt{E}} -$$

$$\frac{(B_2 - A_1)}{\sqrt{E}}\Bigg]\Bigg\} + \Big[B_1 x^2 - A_2 y^2 + (B_2 - A_1)xy\Big]t.$$

The function η has a very complicated form and it does not need to be calculated here. By integrating the restriction of 1-form $\overline{\omega}^1$ on the set H = $\{(x, y, t) : \phi(x, y, t) = 0\}$, we will obtain the general solution of the system (1).

4. PRACTICAL APPLICATIONS

Now let us have a practical example. An RLC circuit (also known as a resonant circuit or a tuned circuit) is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel. Every RLC circuit consists of two components: a power source and resonator. They are two types of power source-Thevenin and Norton. Likewise, there are two types of resonator-series LC and parallel LC. As a result there are four configuration of RLC circuit:

- Series LC with Thevenin power of source
- Series LC with Norton power source
- Parallel LC with Thevenin power of source
- Parallel LC with Norton power of source.

There are two fundamental parameters that describe the behavior of RLC circuit: the resonant frequency and the damping factor. In addition other parameters derived from these first are discussed below.

The undamped resonance or natural of an RLC circuit (in radiance per second) is given by $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\Delta\omega = \frac{R}{L}$.

Consider a Series RLC with Thevenin power source. In this circuit, the there components are all in series with the voltage source. Let's have some notations for this circuit:

V – The voltage of the power source,
i – The current in the circuit,
R – The resistance of the resistor,
L – The inductance of the inductor,
C – The capacitance of the capacitor,
q – The charge across the capacitor.

Given the parameter V, R, L, and C, the solution for the current q using Kirchhoff's voltage law (KVL) gives $V_R + V_L + V_C = V$.

For a time-changing voltage v (t), this becomes

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(s)ds = V(t)\,.$$

Using the relation between charge and current: $i(t) = \frac{dq}{dt}$, the above expression can be expressed in terms of charge across the capacitor:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q(t) = V(t).$$

Dividing by L gives the following second order differential equation:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q(t) = \frac{1}{L}V(t).$$

For our purpose it is necessary to assume that the voltage of the circuit is constant, thus we have

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q(t) = \frac{1}{L}V\,. \tag{10}$$

By substituting q(t) = y(t) in the Equation (10) we obtain the following system of ODEs:

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{R}{L}x(t) - \dfrac{1}{LC}y(t) - \dfrac{1}{L}V \\ \dfrac{dy(t)}{dt} = x(t) \end{cases}$$

Similarly by a suitable translation which we have done for obtaining the system (1), we can eliminate the term $-\frac{1}{L}V$ in the first equation, so we have:

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{R}{L}x(t) - \dfrac{1}{LC}y(t) \\ \dfrac{dy(t)}{dt} = x(t) \end{cases} \tag{11}$$

This system has the following solutions

$$x(t) =$$

$$-\frac{1}{2L}\left\{\left(R - \sqrt{R^2 - 4LC}\right)C_1\exp\left[-\frac{\left(R - \sqrt{R^2 - 4LC}\right)t}{2L}\right]\right.$$

$$\left. + \left(R + \sqrt{R^2 - 4LC}\right)C_2\exp\left[-\frac{\left(R + \sqrt{R^2 - 4LC}\right)t}{2L}\right]\right\},$$

$$y(t) = C_1\exp\left[-\frac{\left(R + \sqrt{R^2 - 4LC}\right)t}{2L}\right] +$$

$$C_2\exp\left[-\frac{\left(R + \sqrt{R^2 - 4LC}\right)t}{2L}\right].$$

As for the system (1) we can construct the first integrals of the system (11), the solutions could have obtained from the first integrals.

Let us have another applied example; Heating a Single-Story House. A single-story house is being heated with a forced-air central heating system. Imagine for simplicity that the house is composed of two main compartments: the lower living area and the upper attic area. Only the living area is heated directly by the furnace, which generates 75000 Btu/hr. Heat transfer takes place between the living and attic areas of the house as indicated by such as vertical arrows. There is also heat loss through the walls of the house to the out side, as well as through the roof over the attic. We assume that, initially the temperature inside the house and the attic is the same as that of the outside: a cold 35˚F. At time t = 0 the furnace is turned on and begins to heat the house. We are interested in knowing when the temperature in the living area reaches a comfortable 68˚F, assuming the outside temperature remains at a constant 35˚F. Let us construct a model of the heat-transfer behavior.

We define x(t) and y(t) as the temperatures of the living and attic areas, respectively, at time t. The change in temperature in the living area depends on the addition of heat from the furnace and the loss of heat to the outside and to the attic. The rate at which the furnace affects the temperature is the number of Btu per hour times the heat capacity of the living area. The heat capacity itself is a function of such variables as the size of the living area and the thermal characteristics of the objects inside the living area. Let us assume that the heat capacity of the living area is 0.2˚F per thousand Btu. Then the furnace can provide 75(0.2) = 15˚F each hour to the living area.

According to Newton's law of cooling (see [1]), the rate of temperature change of a region is proportional to the difference between the temperature of the region and the temperature of an adjacent region. For the living area the heat loss through the outside walls account for a change of $k_1(35-x)$, and the heat loss to the attic is $k_2(y-x)$. The proportionality constants k_1 and k_2 are assumed to be positive and depend on the insulation and materials of the walls and ceiling. Thus for the living area the rate of change in temperature is given by $dx/dt = 15 + k_1(35-x) + k_2(y-x)$. In a similar way we can derive the rate of change in the temperature for the attic area. Together these two rates from the system

$$\begin{cases} \dfrac{dx}{dt} = 15 + k_1(35 - x) + k_2(y - x) \\ \dfrac{dy}{dt} = k_2(x - y) + k_3(35 - y) \end{cases} \tag{12}$$

Where the constant k_3 is the system depends on the roofing materials.

In specific situation the proportionality constant k_1, k_2 and k_3 are often specified as the reciprocals of the time constant for the heat transfer to take place between the two adjacent regions involved. For instance, if the time constant for the heat transfer between the living area and outside in 4 hr, then $k_1 = 1/4$. The time constant $1/k_1$ between the living area and the outside, for example represents the time it takes for the temperature difference 35-x(t) to change form 35-x(0) to $\dfrac{35 - x(0)}{e} \approx 0.368[35 - x(0)]$.

A typical value for the time constant of building 1s 2-4 hr, but it can be shorter if there are open windows or doors and longer if the building is well insulated.

Now return to first integral again. First we should change the system in the form of system (1) by a suitable change of variable. If we construct the first integral of the system (12), $\phi(x, y, t)$, and take it to zero we have

$$y(t) = C_1 \exp\{-\frac{1}{2}\Big(2k_2 + k_3 - k_1 -$$

$$\sqrt{4k_2{}^2 + k_3{}^2 - 2k_1k_3 + k_1{}^2}\Big)t\} +$$

$$C_2\exp\{-\frac{1}{2}\Big(2k_2 + k_3 - k_1 -$$

$$\sqrt{4k_2{}^2 + k_3{}^2 - 2k_1k_3 + k_1{}^2}\Big)t\},$$

Where C_1 and C_2 are arbitrary constants, x(t) can be obtained from on of the equation of system (12).

5. FIRST INTEGRALS OF A SYSTEM OF N-ODES

In this part we have a general and more comprehensive system of ODEs in the form of

$$\begin{cases} \dfrac{dy_1}{dx} = f_1(x, y_1, \ldots, y_{n-1}) \\ \quad\vdots \\ \dfrac{dy_{n-1}}{dx} = f_{n-1}(x, y_1, \ldots, y_{n-1}) \end{cases} \tag{13}$$

A general solution of this system has the form

$$y_i(x) = \phi_i\Big(x, C_1, \ldots, C_{n-1}\Big),$$

Where i = 1,...,n-1, whence, upon solving with respect to the constants of integration C_i,

$$\phi_i\Big(x, y_1, \ldots, y_{n-1}\Big) = C_i, \tag{14}$$

for i = 1,...,n-1. The relation (14) provides the general integral for the system (13). The left hand side of each relation (14) reduces to a constant when $y_1,\ldots,y_{n-1}$ are replaced by the coordinates $y_1(x),\ldots,y_{n-1}(x)$ of any solutions of the system (13). For this reason every single relation in (14) is known as a first integral of the system (13). According to the systems (11) and (12) and in the general form system (1), it is enough to find one of the two first integrals for finding one of the solutions, another one is obtaining from the equations of the system.

6. ACKNOWLEDGMENT

When we do not have any model for finding solution(s) of an ODE, first integral(s) of that ODE can give(s) us the solution(s). But some times, system such as ODEs similar to system (1), first integrals have such complicated forms, which is not appropriate to find the solutions from first integrals. It is necessary to say that the model of first integrals is related to the form of symmetries of ODE. May be in some special cases such as, differential equation of outflow from a funnel, the first integral has more simple form.

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