

A BOUNDARY ELEMENTS AND PARTICULAR INTEGRALS IMPLEMENTATION FOR THERMOELASTIC STRESS ANALYSIS

M. Hemayati

*Surface Effect Research Institute, Shahid Charmin Boulevard
Shiraz, Iran*

G. Karami

*Department of Mechanical Engineering, School of Engineering, Shiraz University
Shiraz, Iran, karami@succ.shirazu.ac.ir*

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Abstract A formulation and an implementation of two-dimensional Boundary Element Method (BEM) analysis for steady state, uncoupled thermoelastic problems is presented. This approach differs from other treatments of thermal loads in BEM analysis in which the domain integrals due to the thermal gradients are to be incorporated in the analysis via particular-integrals. Thus unlike Finite Elements or Field Boundary Elements algorithms the domain discretization becomes unnecessary. The algorithm and the formulation are implemented in a general purpose, multi-region two-dimensional analysis. Isoparametric quadratic elements are employed to represent the geometry and the field variables. Examples are presented to demonstrate the accuracy and versatility of the method.

Key Words Thermoelasticity, Boundary-Element-Method, Particular Integrals

چکیده یک روش جدید عناصر مرزی برای مسائل دو بعدی ترموالاستیسیته در این مقاله ارائه گردیده است. روش ارائه شده برای حل مسائل با توزیع دمای یکنواخت و گذرا در دامنه کاربرد داشته و تفاوت آن با روشهای قبلی در آنستکه انتگرالهای روی دامنه، ناشی از نیروهای حرارتی به کمک انتگرالهای مخصوص در سیستم عناصر مرزی در نظر گرفته می شوند. بدین ترتیب بر خلاف روش عناصر محدود و یا روشهای متعارف عناصر مرزی، نیازی به المان بندی دامنه نخواهد بود. این روش برای سیستمهای شامل چند ناحیه نیز بکار می رود و در آن از المانهای درجه دوم و آیزوپارامتریک برای نشان دادن توابع و هندسه جسم استفاده شده است. بمنظور نشان دادن توانایی و دقت روش، مثالهایی ارائه شده است.

1. INTRODUCTION

Usually the thermoelastic problems may be solved using small modifications to the pure elastic formulations by treatment of the temperature gradients as a kind of body forces. In BEM analysis this will include an extra domain integral to the resulted boundary-only integrals of elastic formulation. Hence the domain of the problem should be discretized for the sole implementation of thermal forces. This obviously would loose the benefits of boundary-only BEM analysis. To avoid the domain discretization several transforming schemes are proposed and implemented. Other methods would include

the particular integrals that would replace the domain integrals by approximate equivalent values. Among the transforming schemes is the Galerkin vector method, which was introduced by Cruse [1] and applied to a limited range of body force problems. Danson [2] have also used this concept for the transformation of body force domain integrals to surface integrals. Rizzo and Shippy [3], Karami and Fenner [4], Karami and Kuhn [5] and Karami [6] have employed an efficient method based on replacing the body force and temperature changes by a scalar potential function in the so-called pseudo-body force approach. Nardini and Brebbia [7], Wroble et al. [8], Nowak and

Brebbia [9] and Neves and Brebbia [10] have used Dual Reciprocity and Multiple Reciprocity Methods (DRM, and MRM). Other solutions include the use of Papkovitch and Neuber stress functions (see for example Kuhn [11]). Also, Sharp and Crouch [12] has developed a formulation, which conceptually can be implemented without volume integration.

The use of particular integrals in BEM was tentatively discussed by Watson [13] and Banerjee and Butterfield [14], but has received little attention thereafter. In 1986, Ahmad and Banerjee [15] successfully employed the concept in a two-dimensional free vibration analysis. The axisymmetric free vibration formulations were developed by Wang and Banerjee [16], and Banerjee et al. [17] extended the theory to acoustic eigenfrequency analysis. Furthermore, particular integral formulations have been presented for gravitational and centrifugal body forces in axisymmetric, two- and three-dimensional stress analysis [17,18].

The particular integral formulation presented in this paper is developed for two-dimensional uncoupled thermoelastic stress analysis, using quadratic isoparametric boundary elements to model the geometry and field variables of the surface based on the previous work by Karami [6] for two dimensional elastic and thermoelastic problems. A global shape function is used to represent the temperature distribution in the region. Using this global shape function, the particular integrals are developed for the region. At last, the particular integrals are used together with the (boundary only) displacement integral equation to produce a solution for the thermoelastic analysis. Sample problems involving different types of temperature gradients are solved to prove the accuracy and versatility of the method. The uncoupled thermoelastic BEM formulation presented is applicable to both steady state and transient temperature distributions, with heat source and initial temperature gradients without any need for volume integration.

2. THE GOVERNING EQUATIONS

In the theory of thermoelasticity the total strain can be divided into elastic strain and thermal strain as

follow,

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^T \quad (1)$$

in which for an isotropic material one can express the thermal strain in terms of temperature difference, T , as $\varepsilon_{ij}^T = \delta_{ij} \alpha T$, where α is the thermal coefficient of expansion. The elastic stress strain equations or Hooke's law may be written as,

$$\sigma_{ij} = 2\mu \varepsilon_{ij}^e + \delta_{ij} \frac{2\mu\nu}{1-2\nu} \varepsilon_{kk}^e - \delta_{ij} 2\mu \frac{\nu}{1-2\nu} \alpha T \quad (2.a)$$

$$\sigma_{ij} = 2\mu \varepsilon_{ij}^e + \delta_{ij} \frac{2\mu\nu}{1-2\nu} \varepsilon_{kk}^e \quad i, j = 1, 2 \quad (2.b)$$

In the above equations and subsequently, part (a) and (b) of an equation apply to plane stress and plane strain, respectively. From Equations 2 and (1), the stress may be written in terms of total strain as,

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \delta_{ij} \frac{2\mu\nu}{1-2\nu} \varepsilon_{kk} - \delta_{ij} 2\mu \frac{1+\nu}{1-2\nu} \alpha T \quad (3.a)$$

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \delta_{ij} \frac{2\mu\nu}{1-2\nu} \varepsilon_{kk} - \delta_{ij} \frac{2\mu}{1-2\nu} \alpha T \quad (3.b)$$

Using the above constitutive relations together with the total strain-displacement relations and the equilibrium equation, one can write the Navier Equation in two dimensions as,

$$u_{i,jj} + \frac{1}{1-2\nu} u_{j,ji} = -\frac{f_i}{\mu} + 2 \frac{1+\nu}{1-2\nu} \alpha T_{,i} \quad (4.a)$$

$$u_{i,jj} + \frac{1}{1-2\nu} u_{j,ji} = -\frac{f_i}{\mu} + \frac{2}{1-2\nu} \alpha T_{,i} \quad (4.b)$$

in which, the value of Poisson's ratio should take its effective value [6].

According to Goodier's body force analogy, the thermal loading can be treated as a body force and incorporated in the boundary integral equation through a volume integral. Therefore, the displacement integral equation satisfying Equation 4 can be written as [6,20],

$$C_{ij}(\xi)u_i(\xi) = \int_s [U_{ij}(x, \xi)t_i(x) - T_{ij}(x, \xi)u_i(x)]dS(x) + \int_v (3\lambda + 2\mu)\alpha T(x)H_j(x, \xi)dV(x) \quad i, j=1,2 \quad (5)$$

where $u_i(x)$ is the real displacement; $t_i(x) = \sigma_{ij} n_j$ is the real traction; δ_{ij} is the kronecker delta; $C_{ij}(\xi) = \delta_{ij}$ for interior points is dependent on surface geometry at ξ for boundary points and U_{ij} and T_{ij} are second order kernels for displacements and tractions, respectively [6,20]. The first integral is a boundary integral whereas the second integral is a domain integral. Note that through an application of the divergence theorem, the gradient operator has been removed from the temperature variable $T(x)$. A similar equation can be written for stress. In the above format, the domain discretization is necessary in order to evaluate the domain integrals.

3. PARTICULAR INTEGRALS APPROACH

If there is no external forces f_i , Equation 4 may be simplified as,

$$(\lambda + \mu)u_{i,jj} + \mu u_{i,jj} = (3\lambda + 2\mu)\alpha T_i \quad i,j=1,2 \quad (6)$$

in which λ and μ are Lamé's constants and T is the change in temperature.

In operator notation, the thermoelastic, inhomogeneous differential Equation 6 may be written as,

$$L(u_i) = \beta T_i \quad (7)$$

in which $L(u_i)$ is a self-adjoint homogeneous differential operator showing the Left-hand side of Equation 6 and βT_i is the known inhomogeneous quantity with, $\beta = \alpha(3\lambda + 2\mu)$.

The solution of the inhomogeneous Equation 7 consists of two parts as follow,

$$u_i = u_i^c + u_i^p$$

where u_i^c is a complimentary function satisfying the homogeneous equation,

$$L(u_i^c) = 0 \quad (8)$$

A particular integral u_i^p , which satisfies the inhomogeneous equation,

$$L(u_i^p) = \beta T_i \quad (9)$$

is not unique. By adding u_i^c to u_i^p and applying boundary conditions, a unique solution to the boundary value problem produces. The complementary functions thus for displacement at point ξ is,

$$C_{ij}(\xi)u_i^c(\xi) = \int_s [U_{ij}(x, \xi)t_i^c(x) - T_{ij}(x, \xi)u_i^c(x)]dS(x) \quad (10)$$

where the t_i^c and u_i^c are the complementary functions for traction and displacement, respectively.

4. PARTICULAR INTEGRALS

In according with linear quasi-static thermoelastic theory, the particular integral for displacement can be expressed as a gradient of a thermoelastic displacement potential $h(x)$,

$$u_i^p(x) = kh_{,i}(x) \quad (11)$$

in which, $k = \frac{\alpha(3\lambda + 2\mu)}{(\lambda + 2\mu)}$. After substituting

Equation 11 into Equation 6 and simplifying, yields,

$$h_{,ij}(x) = T(x) \quad (12)$$

Now, assume that the function $h(x)$ be represented by an infinite series. An expression relating $h(x)$ to a set of fictitious scalar densities $\phi(\xi_n)$ via a global shape function $C(x, \xi_n)$, can be written as,

$$h(x) = \sum_{n=1}^{\infty} C(x, \xi_n)\phi(\xi_n) \quad (13)$$

in which $C(x, \xi_n)$ is a suitable function of spatial coordinates x and ξ_n . The best results were obtained with the following expression for $C(x, \xi_n)$,

$$C(x, \xi_n) = A_0^2 [\rho^2 - b_n \rho^3]$$

where, A_0 is a characteristic length, all distances are non-dimensionalized by it; ρ is the Euclidean distance between the field point x and the source

point ξ_n , and b_n is a suitably chosen constant. For the present discussion, assume $b_n = 1$.

The particular integral for displacement is found, using Equations 13 and 11,

$$u_i^p(x) = \sum_{n=1}^{\infty} D_i(x, \xi_n) \phi(\xi_n) \quad (14)$$

where:

$$D_i(x, \xi_n) = kC_{,i}(x, \xi_n) = kA_0(2 - 3b_n\rho)y_i$$

$$y_i = [x_i - (\xi_n)_i]$$

$i = 1, 2$ for two dimensions.

Applying the Laplacian operator to Equation 13, the temperature distribution will be,

$$T(x) = \sum_{n=1}^{\infty} K(x, \xi_n) \phi(\xi_n) \quad (15)$$

in which,

$$K(x, \xi_n) = C_{,ii}(x, \xi_n) = [2d - 3(1+d)b_n\rho]$$

$d = 2$ for two dimensional (plain strain) analysis.

Now by substitution of Equation 14 into the strain – displacement relation, a particular integral for strain can be found,

$$\epsilon_{kl}^p(x) = \sum_{n=1}^{\infty} E_{kl}(x, \xi_n) \phi(\xi_n) \quad (16)$$

in which,

$$E_{kl}(x, \xi_n) = kC_{,kl}(x, \xi_n) = k \left[2\delta_{kl} - 3b_n(\delta_{kl}\rho + \frac{y_k y_l}{\rho}) \right]$$

and the particular integral for stress can be found by introducing above equation into the stress-strain law for thermoelasticity,

$$\sigma_{ij}^p(x) = \sum_{n=1}^{\infty} S_{ij}(x, \xi_n) \phi(\xi_n) \quad (17)$$

where,

$$S_{ij}(x, \xi_n) = D_{ijkl}^e E_{kl}(x, \xi_n) - \delta_{ij} \beta K(x, \xi_n), \quad \text{and}$$

$$D_{ijkl}^e = \lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl} \quad \beta = \alpha(3\lambda + 2\mu)$$

At last, by multiplying the above equation by an

appropriate normal, a particular integral for traction will be obtained,

$$t_i^p(x) = \sum_{n=1}^{\infty} H_i(x, \xi_n) \phi(\xi_n) \quad (18)$$

where, $H_i(x, \xi_n) = S_{ij}(x, \xi_n) n_j(x)$ and $n_j(x) =$ unit normal at x in the j th direction. In the case of plane stress, the modified material constants $\bar{\alpha}$ and $\bar{\lambda}$ must be used instead of α and λ in the above equations which are valid for plane strain condition, where,

$$\bar{\alpha} = \frac{\alpha(3\lambda + 2\mu)}{2(2\lambda + \mu)} \quad \bar{\lambda} = \frac{2\mu\lambda}{\lambda + 2\mu}$$

5. NUMERICAL APPROACH

The functions $u_i^p(x)$ and $t_i^p(x)$ must be evaluated at each boundary node before a solution to the governing equation can be achieved. For this purpose, particular integrals for displacement, traction and temperature distribution may be written as infinite series for N finite terms as follow,

$$u_i^p(x) = \sum_{n=1}^{\infty} D_i(x, \xi_n) \phi(\xi_n) \quad (19a)$$

$$t_i^p(x) = \sum_{n=1}^{\infty} H_i(x, \xi_n) \phi(\xi_n) \quad (19b)$$

$$T(x) = \sum_{n=1}^{\infty} K(x, \xi_n) \phi(\xi_n) \quad (19c)$$

To evaluate $u_i^p(x)$ and $t_i^p(x)$ in the first two equations, we need N fictitious nodal quantities $\phi(\xi_n)$. For this reason, we have written N temperature Equations 19c at each ξ_n node. In the matrix form,

$$\{T\} = [K] \{\Phi\} \Rightarrow \{\Phi\} = [K]^{-1} \{T\} \quad (20)$$

in which $[K]$ is an $N \times N$ matrix. Since the increment of temperature distribution is known, the fictitious nodal values $\{\phi(\xi_n)\}$ is determined from above equation and using them in Equations 19a and 19b, allows calculation of $u_i^p(x)$ and $t_i^p(x)$ at all boundary points. In this way the boundary value

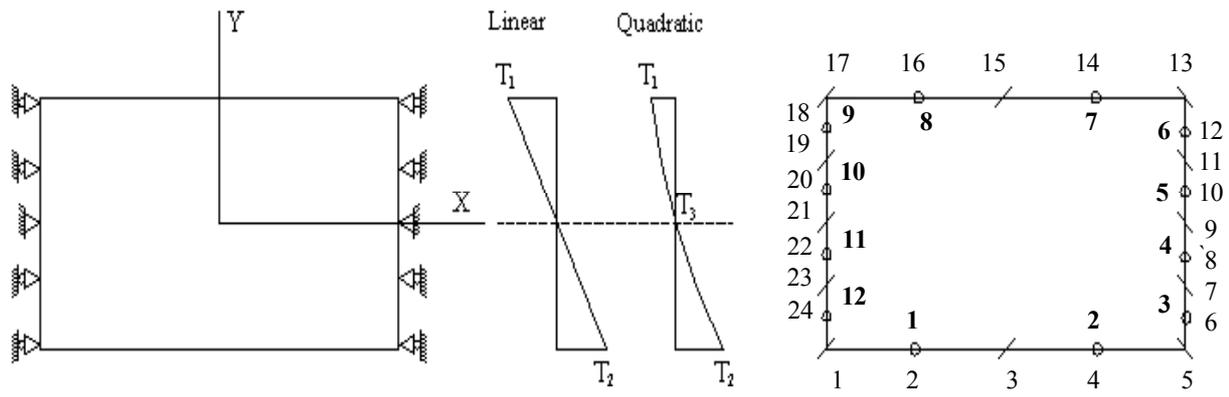


Figure 1. Beam subjected to linear and quadratic temperature change, geometry and temp. distribution along the width; Boundary element discretization

problem can be solved in the following manner.

6. METHOD OF SOLUTION

The boundary integral equation for complementary displacement is discretized and integrated in the usual manner for a system of boundary nodes. The resulting equation is then expressed in matrix form as,

$$[U]\{t^c\} - [T]\{u^c\} = \{0\} \tag{21}$$

As stated before,

$$\{u_i\} = \{u_i^c\} + \{u_i^p\} \quad \text{and} \quad \{t_i\} = \{t_i^c\} + \{t_i^p\}$$

Substituting $\{u_i^c(x)\}$ and $\{t_i^c(x)\}$ from above equations in Equation 21 leads to,

$$[U]\{t\} - [T]\{u\} = [U]\{t^p\} - [T]\{u^p\}$$

where the particular integral terms on the right-hand side of this equation are known temperature dependent quantities. After assembling the unknown boundary quantities and corresponding coefficients on the left-hand side and the known boundary conditions on the right, the final system can be written as,

$$[A^b]\{x\} = \{b^b\} + \{b^p\}$$

in which $[A^b]$ is a block-banded matrix, vector $\{x\}$ represents the unknown boundary conditions and

vector $\{b^p\}$ is the contribution of the particular integral. This system of equations is solved for the unknown vector $\{x\}$ by standard numerical techniques.

7. NUMERICAL EXAMPLES

In order to investigate the applicability, accuracy and generality of the particular integrals method in BEM analysis of thermoelastic problems, three examples are solved, and the results are compared with those of analytical solutions.

Example 1. Beam Subjected to Linear and Quadratic Temperature Change

A beam fixed at both ends is assumed to be subjected to two different cases of temperature change. Plane stress case was assumed and the material of the beam is taken to be, $E = 10^6 \text{ N/m}^2$, $\nu = 0.3$, $\alpha = 10^{-7} \text{ deg}^{-1}$ Figure 1 illustrates the beam geometry and the temperature variation along the depth of the beam. A BEM discretization of the beam is also shown. The numerical solution for the normal stresses, σ_x along the y-axis in the center of the beam were compared with analytical solutions. For linear temperature change with $T_1 = -1000^\circ$ and $T_2 = 1000^\circ$, the exact values of the normal stress are given analytically by [21],

$$\sigma_x = -E\alpha \frac{(T_1 - T_2)y}{h}$$

where h is the width of the beam and y is the coordinate shown in Figure 1.

TABLE 1. Normal Stress σ_x (N/m²) Along The Centerline In Y-Direction For Linear And Quadratic Temperature Distribution In A Beam.

Y-axes	Linear Temperature Distribution		Quadratic Temperature Distribution	
	Exact	BEM (P.I.)	Exact	BEM (P.I.)
- 0.500 h	- 100.000	-101.2172	- 4.000	-3.9908
- 0.375 h	- 75.000	-74.2744	-2.8125	-2.6733
- 0.250 h	- 50.000	-51.3937	-1.7500	-1.8046
- 0.125 h	- 25.000	-25.1422	-0.8125	-0.8844
0.000 h	0.000	0.00011	0.0000	0.0092
0.125 h	25.000	25.1421	0.6875	0.7440
0.250 h	50.000	51.3937	1.2500	1.2790
0.375 h	75.000	74.2743	1.6875	1.6031
0.500 h	100.000	101.2172	2.0000	1.9935

For the second case a quadratic temperature variation of the form, $T = 2(T_1+T_2-2T_3)y^2/h^2 + (T_1-T_2)y/h + T_3$, with $T_1 = -20^\circ$, $T_2 = 40^\circ$ and $T_3 = 0^\circ$ is implemented. The exact values for the normal stress under such a temperature distribution are given by [21],

$$\sigma_x = - E \alpha [2(T_1 + T_2 - 2T_3) \frac{y^2}{h^2} + (T_1 - T_2) \frac{y}{h}]$$

The BEM and numerical results can be found in Table 1. As can be seen the accuracy of the numerical formulation is very well satisfied for the two cases of linear and quadratic temperature variation. However, the errors in the case of quadratic temperature variation are slightly higher than the other case, as can be expected.

Example 2. Internal Pressure Cylinder Under Thermal Loading In this example a cylinder under combined pressure as well as a temperature gradient is considered. The temperature distribution is assumed to have the form of, $T = Ar^2$, where A is a constant. The material and geometrical properties are as follows, $E=10^7$ N/m²; $\alpha =10^{-5}$ deg⁻¹; $a = 10$ cm; $b=20$ cm; $P_i=1000$ Pa; $A=2.5$. The answer to this problem is the superposition of the analytical solution from the two cases of loading, i.e., the internal pressure and thermal gradient loading [21]. Figure 2 shows the geometry with temperature distribution and boundary element discretization.

The results are given in Table 2 for radial displacements, radial and tangential stresses. The accuracy of the results is well satisfied

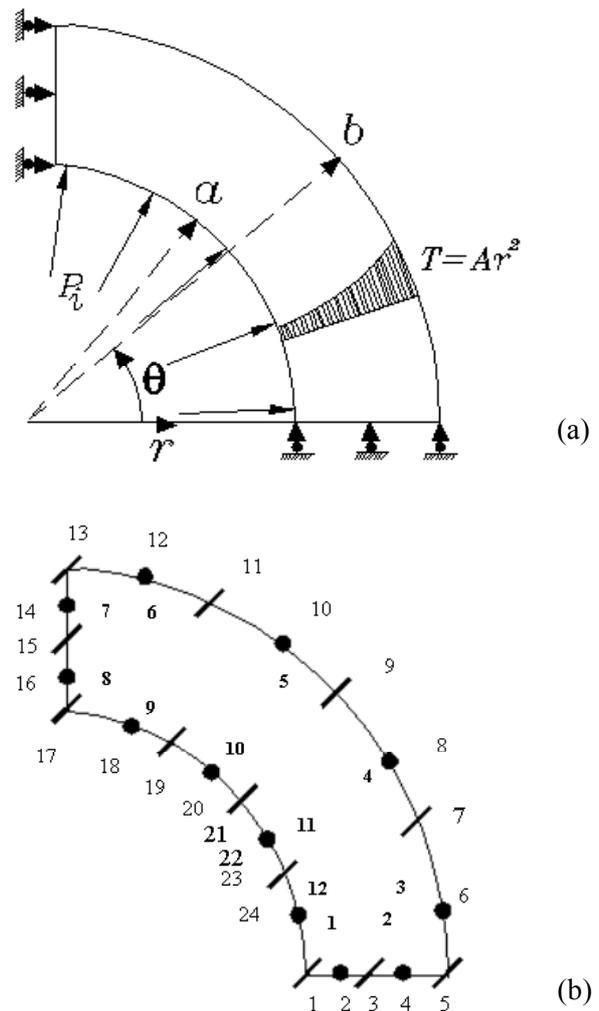


Figure 2. A pressurized cylinder under a quadratic temperature distribution, (a) Geometry and temperature distribution, (b) BEM discretization. $E = 10^7$ N/m².

TABLE 2. Radial Displacements (cm) And Stresses (N/m²) In Axisymmetric Thermoelastic Response For A Pressurized Cylinder Under A Radial Temperature Distribution.

NODE	Radial Displacement (u_r)		Radial stress (σ_r)		Hoop stress (σ_θ)	
	Exact	BEM	Exact	BEM	Exact	BEM
1	0.08322	0.08309	-1000.0	-1030.2	27536.12	27554.02
2	0.09051	0.09035	7415.8	7677.17	26833.67	26907.55
3	0.10645	0.10636	8422.72	8413.04	1373.926	1379.19
4	0.13076	0.13053	5737.16	5773.81	-24961.29	-25151.86
5	0.16383	0.16357	0.0	-4.13	-52633.03	-52655.22

Example 3. Thermal Analysis of Rotating Disc

Let's consider a thin disc of uniform thickness with a central hole, rotating with a constant angular velocity ω rad/sec and in addition is subjected to a thermal loading according to, $T=Ar^2 + B$, where A and B are constants. The analytic solutions for resulting stresses and radial displacement are the superposition of two different cases thermal and inertial loading due to rotation [21].

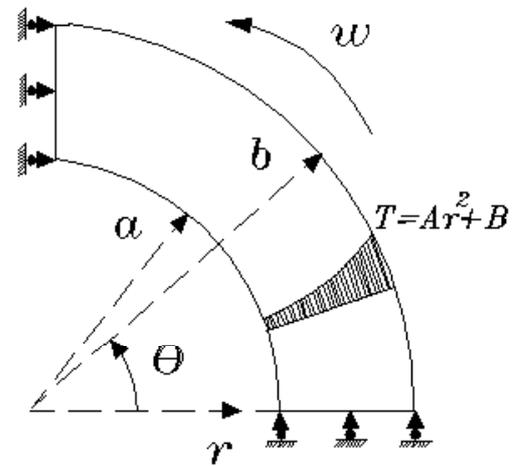
The appropriate boundary conditions of traction-free edges on a disc with a concentric hole are $\sigma_r = 0$, at $r = a$ and $r = b$, the inner and outer radii respectively.

For a plane stress case, the disc has a uniform thickness with inner and outer radii of 0.1m and 0.2 m, respectively. The mesh contains 12 three-node continuous elements and a total of 24 nodes. Because of symmetry, one-fourth of the geometry was modeled as shown in Figure 3. The data used was, $E=10^5 \text{ Nm}^{-2}$; $\nu = 0.3$; $\alpha = 0.001 \text{ deg}^{-1}$; $\rho = 2.4 \text{ kg/m}^3$; $\omega = 100 \text{ rad/sec}$; $A = 2500$; $B = 500$.

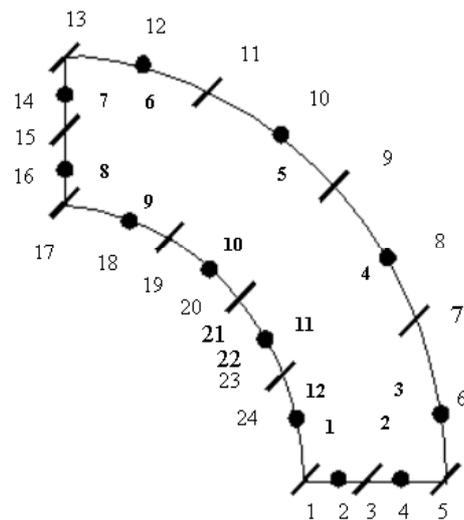
Table 3 contains a comparison between analytical and BEM numerical results using particular integrals for u_r and also σ_r and σ_θ along the radius of the disc. Good agreement is seen for both displacements and stresses due to thermoelastic behavior.

8. ACKNOWLEDGMENT

The first author would like to appreciate the



(a)



(b)

Figure 3. A rotating disc with temp. distribution $T=Ar^2+B$, (a) Geometry and temperature distribution, (b) BEM model. $E = 10^7 \text{ N/m}^2$.

TABLE 3. Radial Displacements (cm) And Stresses (N/m²) In Thermoelastic Response For A Rotating Disc Under A Radial Temperature Distribution.

NODE	Radial Displacement (u_r)		Radial stress (σ_r)		Hoop stress (σ_θ)	
	Exact	BEM	Exact	BEM	Exact	BEM
1	0.0562	0.0571	0.0	4.12	4584.0	4535.74
2	0.0691	0.0702	557.08	532.96	2454.4	2466.00
3	0.0826	0.0839	617.22	620.49	560.12	580.76
4	0.0976	0.0982	400.85	413.73	-1350.3	-1383.40
5	0.1114	0.1131	0.0	-2.5	-3383.98	-3382.04

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