QUEUE WITH HETEROGENEOUS SERVER UNDER RESEQUENCING CONSTRAINT

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Abstract In this paper, a markovian queue with two types of servers and buffer space is considered. The resequencing constraint is imposed according to which the customers leave the system in the same order in which they entered it. For finite buffer queue, the steady state queue size distribution has been obtained.

Key Words Markovian Queue, Buffer Space, Two Servers, Hetergeneous, Resquencing

چکیده دراین مقاله، یک سیستم صف مارکوی با دو خدمت دهنده و یک محل میانی برای تشکیل صف جدید مورد بررسی قرار می گیرد. تشکیل سیستم صف جدید درمحل میانی براساس همان ترتیب ورودی آنها است. این مسئله با فضای میانی نامتناهی قبلاً حل شده است ولیکن دراین مقاله حجم فضای متناهی محدود تلقی شده است.

INTRODUCTION

There may be real world queueing situations wherein customers are compelled to leave the system in the same chronological order in which they arrive. Washburn [1] was a pioneer in this regard and has studied M/M/s queue. An extension of this model to finite waiting space was studied by Sharma et al. [2]. Jain et al. [3] obtained an expression for expected waiting time of customers for a discouragement queueing model having two types of customers and nopassing. The resequencing buffer is an important issue in manufacturing system, distributed computing system communication network, etc., and requires a solution for minimizing waiting time and maximizing the efficiency and reliability of the system. In such systems, the job to be completed requires services of more than one servers in a predetermined sequence.

Two servers queueing models have been studied extensively by several reserchers. Two servers queue sequencing with and without an intermediate waiting space was studied by Avi-Itzhak and Yadin [4] and Clark [5]. A queueing model with resequencing constraints was studied by Lien [6]. He obtained an expression for the average resequencing delay. An extension of this model, by including allocation of customers with thresh-hold type policy, was studied by Hiadis and Lien [7]. A resequencing system with disordering due to infinite server was studied by Baccelli et al. [8], Harrus and Plateau [9], Kamoun et al. [10]. The resequencing system with finite server queue was considered by Gun and Marie [11],Yum and Ngai [12]. All these works have centered on obtaining the distribution of resequencing delay. Buffer probability distribution in resequencing of an infinite queue with block diagonal structure was first studied by Verma [13]. Recently Takine et al. [14] investigated queue length diribution of a resequencing buffer fed by a homogeneous M/M/2 queue.

In computer communication systems, there may be situations where communication between two nodes can be performed by two independent channels. After getting the required service, the job waits in a resequencing queue due to physical restrictions. The waiting space, i. e., the buffer for resequencing queue, is finite for such systems. Some other examples of this type of situations are common in distributed database systems, multi-link stores, and forword switching networks. Earlier work on resequencing queue has been based on the assumption of infinite buffer resequencing. In this paper, we consider two server queueing model and a finite resequencing queue. The queue size distribution is obtained in implicit form.

NOTATION AND STATES

- **a** Arrival rate
- $u_1(u_2)$ Service rate of fast (slow) server
- **n** Number of customer in the main queue buffer.
- e_1 1 (resp.0) if the faster server is busy (resp. idle)
- e_2 1 (resp. 0) if the slower server is busy (resp. idle)
- **m** Number of customers in the resequencing buffer.
- **Z= I** If the fast server is serving the customer which enters the system earlier. It will be referred to as *in-sequence* state.
- **Z=0** If the slow server is serving the customer who entered the system earlier. It will be referred to as *out-sequence* state.

These notations hold true even if there is a single customer in the system. It can be easily varified that the state variables (n, e_1, e_2, m, Z) belong to the space

$$E= \{0\} \cup N \times \{0, 1\} \times \{0, 1\} \times N \times \{I, 0\}$$

where {0} is the empty state. It is noted that E provides a complete markovian state space description of the system. The future states of the space E depend only on the present state and it contains all the states discribing the system. Hence E provides a complete markovian state space discription of the system (Cox and Miller) [15].

THE STEADY STATE EQUATIONS GOVERNING THE MODEL

For our model we have equilibrium equation for various states as follows:

At origin

$$aP(0)=u_1\sum_{j=0}^{M}P(0,1,0,J,I)=u_2\sum_{j=0}^{M}P(0,0,1,J,0) \qquad (1)$$

For states when Z=I

$$(a+u_1+u_2)P(1,1,1,j,I)=u_2P(1+1,1,1,j-1,I)+aP(i-1,1,1,j,I)$$
For $0<i<B,\ j>0,\ e_1=1,\ e_2=1$ (2.1)

$$(u_1+u_2)\ P(B,1,1,j,I)= aP(B-1,1,1,j,I)$$
For $i=B,\ j>0,\ e_1=1,\ e_2=1$ (2.2)

$$(a+u_1+u_2)P(0,1,1,j,I)=u_2P(1,1,1,j-1,I)+aP(0,1,0,j,I)$$
For $i=0,\ j>0,\ e_1=1,\ e_2=1$ (2.3)

$$(a+u_1)\ P(0,1,0,j,I)= u_2P(0,1,1,j-1,I)$$
For $i=0,\ j>0,\ e_1=1,\ e_2=0$ (2.4)

$$(a+u_1+u_2)P(i,1,1,0,I)=u_2\sum_{j=0}^{M}P(i+1,1,1,j,0)+aP(i-1,1,1,0,I)$$
For $0<i<B,\ j=0,\ e_1=1,\ e_2=1$ (2.5)

$$(a+u_1+mu_2)P(0,1,1,0,I)=u_2\sum_{j=0}^{M}P(1,1,1,j,0)+aP(0,1,0,0,I)$$

For $i=0, j=0, e_1=1, e_2=0$

(2.6)

$$(a+u_1)\, P(0,1,0,0,I)= u_2\sum_{j=0}^{M} P(0,1,1,j,0)+aP(0,0,0,0)$$

For $i=0, j=0, e_1=1, e_2=0$

(2.7)

For states when Z= 0

$$(a+u_1+u_2)\, P(i,1,1,j,0)= u_1P(i+1,1,1,j-1,0)+aP(i-1,1,1,j,0)$$

For $0<i<B, j>0, e_1=1, e_2=1$

(3.1)

$$(u_1+u_2)\, P(B,1,1,j,0)= aP(B-1,1,1,j,0)$$

For $i=B, j>0, e_1=1, e_2=1$

(3.2)

$$(a+u_1+u_2)\, P(0,1,1,j,0)= u_1P(1,1,1,j-1,0)+aP(0,0,1,j,0)$$

For $i=0, j>0, e_1=1, e_2=1$

(3.3)

$$(a+u_2)\, P(0,0,1,j,0)= u_1P(0,1,1,j-1,0)$$

For $i=0, j>0, e_1=0, e_2=1$

(3.4)

$$(a+u_1+u_2)P(i,1,1,0,0)=u_1\sum_{j=0}^{M} P(i+1,1,1,j,0)+aP(i-1,1,1,0,0)$$

For $0<i<B, j=0, e_1=1, e_2=1$

(3.5)

$$(a+u_1+u_2)P(0,1,1,0,0)=u_1\sum_{j=0}^{M} P(1,1,1,j,I)+aP(0,0,1,0,0)$$

For $i=0, j=0, e_1=1, e_2=1$

(3.6)

$$(a+u_2)\, P(0,0,1,0,0) = u_1\sum_{j=0}^{M} P(0,1,1,j,I)$$

For $i=0, j=0, e_1=0, e_2=1$

(3.7)

FOR B= 0

In this case, a customer who arrives when both servers are busy is not taken for service. Due to resequencing constraints, the customers leave the system in the same order in which they entered. It is assumed that the resequence box has a finite waiting space. Noting that n= 0 due to B= 0, we can omit n from the notation. Now Equations 1 - 3 reduce to:

$$aP(0)=u_1\sum_{j=0}^{M} P(1,0,j,I)=u_2\sum_{j=0}^{M} P(0,1,j,I);\ j=0,1,\ldots,M \quad (4)$$

$$(a+u_1)\, P(1,0,j,I)= u_2P(1,1,j-1,I);\ j=1,2,\ldots,M \quad (5.1)$$

$$(a+u_2)\, P(0,1,j,0)= u_1P(1,1,j-1,0);\ j=1,2,\ldots,M \quad (5.2)$$

$$(u_1+u_2)\, P(1,1,j,I)= aP(1,0,j,I);\quad j=0,1,\ldots,M \quad (6.1)$$

$$(u_1+u_2)\, P(1,1,j,0)= aP(0,1,j,0);\quad j=0,1,\ldots,M \quad (6.2)$$

$$(a+u_1)\, P(1,0,0,I) = aP(0) + u_2\sum_{j=0}^{M} O(1,1,j,0) \quad (7.1)$$

$$(a+u_2)\, P(0,1,0,0) = u_1\sum_{j=0}^{M} P(1,1,j,I) \quad (7.2)$$

Using Equations 5 and 6, we have

$$P(1,0,j,I)=\left(\frac{u_1}{a+u_1}\right)^j\left(\frac{u_1}{u_1+u_2}\right)^j P(1,0,0,I);\ j=0,1,\ldots,M \quad (8)$$

$$P(1,1,j,I)=\left(\frac{u_2}{a+u_1}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1} P(1,0,0,I);\ j=0,1,\ldots,M \quad (9)$$

$$P(0,1,j,0)=\left(\frac{u_1}{a+u_2}\right)^j\left(\frac{a}{u_1+u_2}\right)^j P(0,1,0,0);\ j=0,1,\ldots,M \quad (10)$$

$$P(1,1,j,0)=\left(\frac{u_1}{a+u_2}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1} P(0,1,0,0);\ j=0,1,\ldots,M \quad (11)$$

Putting Equation 11 into Equation 7.1, we obtain

$$(a+u_1)P(1,0,0,I)=aP(0)+u_2\sum_{j=0}^{M}\left(\frac{u_1}{a+u_2}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1} P(0,1,0,0) \quad (12)$$

$$\text{Where } r_1=\sum_{j=0}^{M}\left(\frac{u_1}{a+u_2}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1}$$

$$=\frac{a(a+u_2)}{u_2(a+u_1+u_2)}\left[1-\left(\frac{au_1}{(a+u_2)(u_1+u_2)}\right)^j\right]$$

Since $\frac{u_1}{a+u_2}<1$ and $\frac{u_1}{u_1+u_2}<1$, r_1 is always finite

so that Equation 12 can be written as

$$(a+u_1)P(1,0,0,I)= aP(0)+ r_1u_2P(0,1,0,0) \quad (13)$$

Putting Equation 9 into Equation 7.2,

$$(a+u_2)P(0,1,0,0)=u_1\sum_{j=0}^{M}\left(\frac{u_2}{a+u_1}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1}P(1,0,0,I) \qquad (14)$$

Denote $r_2=\sum_{j=0}^{M}\left(\frac{u_2}{a+u_1}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1}$

$$=\frac{a(a+u_1)}{u_1(a+u_1+u_2)}\left[1-\left(\frac{au_2}{(a-u_1)(u_1+u_2)}\right)^j\right]$$

Also we note r_2 always finite because $\frac{u_2}{a+u_1}<1$ and $\frac{u_1}{u_1+u_2}<1$

Now Equation 14 can be written as

$$(a+u_2)P(0,1,0,0)=r_2u_1P(1,0,0,0,1) \qquad (15)$$

The three unknown values P(0), P(1,0,0,I) and P(0,1,0,0) can be obtained by using Equations 13, 15 and normalizing condition which is given by

$$P(0)+\sum_{j=0}^{M}P(1,0,j,I)+P(1,1,j,I)+P(0,1,j,0)+P(1,1,j,0)=1 \qquad (16)$$

Putting values from Equations 8 and 11 into Equation 16, we have

$$P(0)+(r_2+r_3)P(1,0,0,I)+(r_1+r_4)P(0,1,0,0)=1 \qquad (17)$$

where $r_3=\sum_{j=0}^{M}\left(\frac{u_2}{a+u_1}\right)^j\left(\frac{a}{u_1+u_2}\right)^j$

$$=\frac{(a+u_1)(u_1+u_2)}{u_1(a+u_1+u_2)}\left[1-\left(\frac{au_2}{(a+u_1)(u_1+u_2)}\right)^j\right]$$

and $r_4=\sum_{j=0}^{M}\left(\frac{u_1}{a+u_2}\right)^j\left(\frac{a}{u_1+u_2}\right)^j$

$$=\frac{(a+u_1)(u_1+u_2)}{u_2(a+u_1+u_2)}\left[1-\left(\frac{au_1}{(a+u_2)(u_1+u_2)}\right)^j\right]$$

To obtain the values of P(0), P(1,0,0I) and P(0,1,0,0) we solve the system of linear equations

$$\begin{pmatrix} a & -(a+u_1) & r_1u_2 \\ 0 & r_2u_1 & -(a+u_2) \\ 1 & (r_2+r_3) & (r_1+r_4) \end{pmatrix}\begin{pmatrix} P(0) \\ P(1,0,0,I) \\ P(0,1,0,0) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (18)$$

which gives

$$P(0)=\frac{(a+u_1)(u_1+u_2)}{\phi}-\frac{r_1r_2u_1u_2}{\phi}=\left[(a+u_1)(a+u_2)-r_1r_2u_1u_2\right]/\phi \qquad (19)$$

$$P(1,0,0,I)=a(a+u_2)/\phi \qquad (20)$$

$$P(0,1,0,0)=a\,r_2u_1/\phi \qquad (21)$$

where $\phi=(a+u_1)(a+u_2)\ r_1r_2u_1u_2+a[(r_2u_1(r_1+r_4)+(a+u_2)(r_2+r_3)$ (22)

RESULTS

1. The steady state probability of j customers in the resequencing buffer and the customer who has arrived earlier is being served by the fast server is

$$P(j,I)=\left(\frac{a+u_1+u_2}{u_1+u_2}\right)\left(\frac{u_2}{(a+u_1)}\right)^j\left(\frac{a}{(u_1+u_2)}\right)^j\left(\frac{a(a+u_2)}{\phi}\right) \qquad (23)$$

2. The steady state probability of j customers in the resequencing buffer and the customer who has arrived earlier is being served by the slow server is

$$P(j,0)=\left(\frac{a+u_1+u_2}{u_1+u_2}\right)\left(\frac{u_1}{(a+u_2)}\right)^j\left(\frac{a}{u_1+u_2}\right)^j\left(\frac{ar_2u_1}{\phi}\right) \qquad (24)$$

3. The steady state probability of j customers in the resequencing buffer is given by

$$q_j=\begin{cases} P(0)\frac{a+u_1+u_2}{u_1+u_2}\left[(a+u_2)+r_2u_1\right]\frac{a}{\phi} \text{ if } j=0 \\ \left(\frac{a}{u_1+u_2}\right)^j\left(\frac{a+u_1+u_2}{u_1+u_2}\right)\left[\left(\frac{u_2}{a+u_1}\right)^j\left(\frac{a+u_2}{r_2u_1}\right)+\left(\frac{u_1}{a+u_2}\right)^j\frac{a}{\phi}\right. \\ \text{for } j=1,2,\ldots,M \end{cases}$$

In limiting case, when M tends to infinity, Equations 23, 24 and Equation 25 reduce to

$$P(j,1)=\left(\frac{a+u_1+u_2}{u_1+u_2}\right)\left(\frac{a+u_2}{\sigma_2u_1\psi}\right)\left(\frac{u_2}{a+u_1}\right)^j\left(\frac{a}{u_1+u_2}\right)^j \quad j=0,1,\ldots \tag{26}$$

$$P(j,0)=\frac{a+u_1+u_2}{u_1+u_2}\frac{1}{\psi}\left(\frac{u_1}{a+u_2}\right)^j\left(\frac{a}{u_1+u_2}\right)^j \quad j=0,1,\ldots \tag{27}$$

$$q_j=\begin{cases} P(0)+\frac{a+u_1+u_2}{u_1+u_2}\left[\frac{a+u_2}{\sigma_2u_1}+1\right]\frac{1}{\psi} \text{ if } j=0 \\ \left(\frac{a}{u_1+u_2}\right)^j\frac{a+u_1+u_2}{u_1+u_2}\left[\left(\frac{u_2}{a+u_1}\right)^j\frac{a+u_2}{\sigma_2u_1}+\left(\frac{u_1}{a+u_2}\right)^j\right]\frac{1}{\psi} \\ \text{for } j=1,2,\ldots \end{cases} \tag{28}$$

where

$$\sigma_1=\sum_{j=0}^{\infty}\left(\frac{u_1}{a+u_2}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1}=\frac{a(a+u_2)}{u_2(a+u_1+u_2)} \tag{29}$$

$$\sigma_2=\sum_{j=0}^{\infty}\left(\frac{u_1}{a+u_1}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1}=\frac{a(a+u_1)}{u_1(a+u_1+u_2)} \tag{30}$$

$$\sigma_3=\sum_{j=0}^{\infty}\left(\frac{u_1}{a+u_1}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j+1}=\frac{(a+u_1)(u_1+u_2)}{u_1(a+u_1+u_2)} \tag{31}$$

$$\sigma_4=\sum_{j=0}^{\infty}\left(\frac{u_1}{a+u_2}\right)^j\left(\frac{a}{u_1+u_2}\right)^{j}=\frac{(a+u_2)(u_1+u_2)}{u_2(a+u_1+u_2)} \tag{32}$$

$$\psi=\frac{(a+u_1)(a+u_2)}{\sigma_2u_1a}-\frac{\sigma_1u_2}{a}+\frac{(\sigma_2+\sigma_3)(a+u_2)}{\sigma_2u_1}+\sigma_1+\sigma_4 \tag{33}$$

It should be noted that in this particular case the model describes the same results as obtained by Verma [13] and Equations 26, 27, 28, telly with Equations 23, 24 and 25 of Verma's model [13].

DISCUSSION

In this investigation, an M/M/2/B queueing model with two types of servers are developed. In a particular case when B=0, various steady state probabilities have been established. Due to resequencing constraints, the presented model is important from practical point of view and may be helpful to system designer to decide optimal buffer size in many information systems such as distributed database and communication networks.

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