A NUMERICAL DESIGN TECHNIQUE FOR A RELAY–TYPE FEEDBACK CONTROL SYSTEM

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Received July 1988

Abstract An efficient numerical method for the design and synthesis of compensator for a relay type control system is developed and discussed. Previous works based on the interactive graphic method are reviewed and it is shown that the combination of the trequency and time domain numerical techniques provide a powerful tool in design of a wide class of relay control systems. An example is presented to demonstrate the efficiency of the procedure.

چکیده : در این مقاله یک روش عددی کارآمد برای طراحی جبران‌کننده در سیستمهای کنترل بازخوز از نوع رله مورد بحث و توسعه قرار گرفته است . مطالب براساس روش محاوره‌ائی گرافیکی مرورشده‌است و نشان داده شده است که ترکیبی ازتکنیکهای عددی پاسخ فرکانسی وزمانی ابزاری پرقدرت جهت طراحی دسته وسیعی از سیستم‌های کنترل رله‌ای را تشکیل می‌دهند . یک مثال برای اثبات کارآئی تکنیک طراحی ارائه شده است .

INTRODUCTION

The computer based graphical techniques described previously [1] are extended and refined to generate a more efficient and robust numerical method for the synthesis of compensator for a class of relay-type feedback control systems. The separable relay element may be single or multi-valued, exhibiting dead-zone and hysteresis. It has been shown [2] that for such a nonlinear system different graphical computational procedures based on the sinusoidal input describing function may be adopted for analysis and design.

On the other hand the techniques of nonlinear programming may be used to obtain solution of optimal control problems [3]. These techniques often work satisfactorily in cases where the desired system performance can be expressed by a single objective function. However, the formulation of this cost function and the resulting mathematical manipulations and algorithms in a complex nonlinear system with highly conflicting parameters can present the designer with a difficult task. In this work attention is focused on the classical problem of feedback relay control systems where the desired specification is not easily expressed by a single performance index. To meet the need for computer aided design technique, a new approach based on the method of inequalities is adopted. This approach differs from nonlinear programming in that no attempt is made to formulate the performance index and instead each aspect of desired performance is expressed as a separable inequality. A moving boundary process based on the Hill-climbing technique, [7, 9] is then applied to solve the systems of inequalities.

The method provides the designer with a facility for expressing his requirements and impose constraints easily and directly. When programmed in conversational mode, the designer uses his judgment to effect trade-offs and arrive at a realistic formulation of the performance inequalities. In this work the inequalities are formulated as a set of time and frequency domain specifications and the parameters of the controller are found to meet, if possible, the desired performance specifications subject to the imposed constraints.

DESIGN BY THE METHOD OF INEQUALITIES

A design problem can be formulated as a set of inequalities

$$\phi_i (p) < c_i, i=1, 2 \dots m \tag{1}$$

where c_i is a set of real numbers and p denotes a real vector $[p_1 \dots p_n]$ and the ϕ_i are real functions of p. The components of p represent system parameters and the inequalities (1) can represent performance specifications and system constraints, the latter usually being fixed while the former are subject to trade offs during the design procedure. If a direct time domain approach is to be considered, then the dynamical behaviour of the system may be specified by a set of time functions f(t) for $0 < t < \infty$ where f(t) can represent such functionals as system rise time, settling time and percentage overshoot in response to a specified input time function. In the non-linear case it is valuable to consider additional functionals in the frequency domain such as

$$\phi_i (p)=f(\omega, p), \qquad \omega_1 < \omega < \omega_2 \tag{2}$$

$$\phi_i (p)=f(a, p) \qquad a_1 < a < a_2$$

to ensure avoidance of critical regions of possible limit cycle operation or to set bounds to unavoidable limit cycle behaviour.

Each inequality in expression (1) defines a set S_i of points in the n dimensional space R^n, the co-ordinates of this space being p_1, $p_2 \dots p_n$ such that

$$\phi_i(p)$$

$$S_i=[p : \phi_{i(p)} \leqslant c_i \tag{3}$$

where $\phi_i(p)=c_i$ represents the boundary of the set S_i. If there is a point p in R^n that satisfies all the inequalities $\phi_i(p) < c_i$, i=1, 2 ... m, then p is inside every set S_i. Let S denote the intersection of all sets S_i then

$$S = \bigcap_{i=1}^{m} S_i \tag{4}$$

Thus p satisfies all the inequalities, if and only if, p is in S. We may say that S is an admissable set and that any p in S is an admissable point. The design problem can be stated as follows: it is required to find a value of $p \epsilon S$ where S is a non-empty subset of R^n defined by expressions (3) and (4) such that a point is said to be a solution to the set of inequalities and hence the design problem. If, however, there is no solution to the problem, i.e. when S is empty, then an approximate solution may be obtained by using the point p which is inside some of the sets S_i and as near as possible to the boundaries of the remaining sets.

Various methods may be used to solve the problem posed by the set of inequalities given in expression (1). The method adopted here is that based on the use of a moving boundary process [4] which uses a hill climbing routine.

Let p^k denote the value of p at the kth iteration.

define

$$S_i^k = \bigcap_{i=1}^{m} S_i^k \tag{5}$$

where

$$S_i^k=\left\{ p: \phi_{(p)} \leqslant c_i^k \right\} \tag{6}$$

$$k \qquad c_i \text{ if } \phi_i (p^k) \leqslant c \tag{7}$$

$$= \phi_i(p^k) \text{ if } \phi_i(p^k) > c_i \quad (8)$$

A change from p^k to some new point $\tilde{p}^k$ is called a trial and is judged a success if and only if

$$\tilde{c}^k \leqslant c^k_i, i=1, 2 \ldots m \quad (9)$$

If $\tilde{p}^k$ is a success then $p^{k+1}=\tilde{p}^k$ is taken as the new point in the iterative procedure. If p^k is not a success a new trial point is repeatedly chosen until success occurs. The computational procedure for generating a new trial point is as follows.

Let v^j_i, (i=1, 2 ... n) denote the orthogonal unit vector and e_i(i=1, 2, ... n) corresponding real numbers. A new trial point can be taken as

$$\tilde{p}^k = p^k + e_i v^j_i \quad (10)$$

If p^k is a failure, e_i is replaced by $-be_i$, $0 < b < 1$. If p^k is a success, e_i is replaced by de_i, $d > 1$. In either case i is set to (i+1) and when i=n, (i+1) is set to 1. This process is repeated until one success is followed by at least one failure for every e=1, 2, ... n. The vector v^{j+1} is chosen as follows. Let D_i be the algebraic sum of all successful values of e_i in the direction v^j during the jth stage and write

$$A_i = \sum_{q=i}^{n} D_q v^j_q , i = 1, 2 \ldots n \quad (11)$$

where A_i is the vector joining the initial and final points obtained by using the vectors $v^j_1, v^j_2, \ldots v^j_n$. A_2 is the sum of all the advances made in directions other than the first etc. If we now orthogonalise the vectors A_i by using the Gram-Schmidt procedure we have

$$B_1 = A_1$$

$$v^{j+1}_1 = \frac{B_1}{\| B_1 \|}$$

$$B_2 = A_2 - \langle A_2, v^{j+1}_1 \rangle v^{j+1}_1$$

$$v^{j+1}_2 = B_2 / \| B_2 \| \quad (12)$$

$$B_n = A_n - \sum_{k=1}^{n-1} \langle A_n , v^{j+1}_k \rangle v^{j+1}_k$$

$$v^{j+1}_n = B_n / \| B_n \|$$

Initially at j=0, e_i and v^j_i are chosen arbitrarily. As j increases, the rate of convergence of S^k to S improves because v^j_i becomes progressively orientated along the direction of most rapid advance. In this work the values d=3 and b=0.5 were used.

Evaluation of the Frequency Domain Functional

Applying the describing function technique to the autonomous system of Figure 1–b, harmonic balance equation governing limit cycle operation can in general be expressed as:

$$1 + G(j\omega)\, N(a, \omega) = 0 \quad (13)$$

where $G(j\omega)$ is the frequency transfer function of the linear part and $N(a, \omega)$ is the describing function gain representing the relay element and may in general be a function of both input amplitude and frequency.

Several graphical procedures based on the classical methods such as Nyquist and Bode plot have been suggested for the solution of the steady state harmonic balance equation (13). These methods are usually aimed at predicting and quantifying parameters of the possible limit cycle operation. However within the context of an iterative design program it is essential to have an efficient numerical method for the solution of equation (13).

The method adopted here is to search for solution in the space of a and ω over a specified grid of discrete values of a and ω.

Defining

$G_T = | N(a, \omega)\, G(j\omega) | = 1$

$Q_T = /N(a, \omega)\, G(j\omega) = -\pi \pm 2\pi k$, (k=1, 2,...

A two dimensional numerical search technique is used to search along the contours of G_T and Q_T over a specified discrete range of a and ω in the space a– ω. If a particular value of a and ω is found that satisfies equation set (14) simultaneously, then the solution exists and these values are the parameters of the predicted limit cycle; if no such values can be found then the relay system is free from limit cycling and by an extension of the Nyquist stability criterion is also stable.

Evaluation of the Time Domain Functional

A direct method of evaluation of time response utilizing state transition method is applied [5, 8].

Using the usual notation, the state-space equations of the relay feedback system of Figure 1–a can be expressed as:

$$\dot{X} = AX + BZ \qquad (15)$$
$$Y = CX + DZ \qquad (16)$$
$$Z = f(a, \dot{a}) \qquad (17)$$
$$\dot{X}_m = A_m X_m + B_m e \qquad (18)$$
$$a = C_m X_m + D_m e \qquad (19)$$
$$e = r - y \qquad (20)$$

Equations (15) and (16) relates to G(S) and (18), (19) relates to M(S, P) and the function $f(a, \dot{a})$ represents the relay characteristics which is assumed to be piece-wise continuous real function that is defined for all values of a and a. If r(t) and a(t) can be adequately represented by their piece-wise constant equivalent over interval T, then Z(t) is point-wise determinable.

Thus for any specified input signal r(t) and a set of initial conditions X(0), the closed loop response can be completely defined [8].

Computer Programs

The set of design programs are written in an interactive conversational mode, with communication between the routines being achieved mostly by the use of data files stored on high speed disks. A supervisor controls the organization of programs, calling the appropriate routine for execution.

Linear data being input in a natural manner as numerator and denomenator polynomial in "S" for G(S) and M(S, P). A frequency range

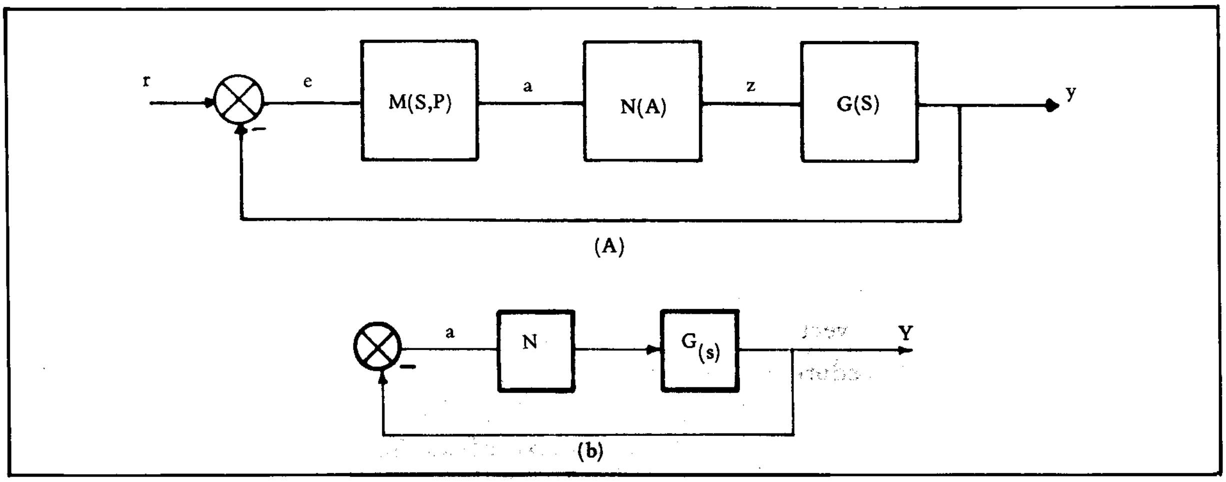

Figure 1. Block diagram of the relay control system.

of interest is also entered for evaluation of frequency response. A subprogram is then automatically called to obtain the state-space representation of G(S) and M(S, P), using a minimal realization routine [4].

The data entry program then requests a range of input amplitude and the relay characterities which can be of the form of an ideal relay, relay with dead-zone or a multivalued relay element. After the appropriate parameters of the relay are entered, another program is automatically called to calculate the describing function gain over the specified range of input amplitude.

Within the main body of the design program, several other subroutines exist which are executed in each iteration. A program examines equation set (14) for each discrete value of input frequency and amplitude for system with or without compensator, as required. Another program calculates the transient behaviour of the relay system for a specified amplitude of the step input. The formulation and testing of the systems of inequalities are carried in another routine and are solved by the method described in section [2].

The Design Procedure

In the design procedure the general structure of the compensator must be chosen. Although the program is versatile enough to allow any general structure, it is sensible to choose initially the simplest possible practical structure which can, if necessary, be augmented in any subsequent design. Thus a simple controller of the following form can initially be chosen:

$$M(S, P) = \frac{P_1 + P_2 S}{P_3 + P_4 S} \qquad (21)$$

If the uncompensated system is to be examined then, $P_1=P_3=1$ and $P_2=P_4=0$. Otherwise, the starting values of P_1, ... P_4 for iterative design is chosen by the designer, either on the basis of some knowledge and understanding of the system's dynamics or by the classical frequency domain techniques. It is also to be mentioned that intuition and "trial and error" is another factor in any such design procedure. Having decided the parameters of the compensator, the relay element is then represented by its describing function gain and a computational procedure [6] is employed to ensure stability. The magnitude of the step input signal r(t) is chosen and the desired form of the output y(t) is specified as rise time, settling time, peak overshoot and steady-state error.

Constraints can also be placed on the magnitude of the compensator element. A set of constraints is also specified for the frequency domain functionals such as magnitude and frequency of the possible limit cycle operation and bounds on the span of eigenvalues of G(jω). Starting from the initial values of the parameter of M(s, p) which guarantee asymptotic stability, the computation proceeds in an iterative manner to seek a solution set of parameters which will satisfy all the specified inequalities. If such a set is found the problem is solved. However if such a set can not be found then either the performance inequalities must be relaxed in a predetermined order of priority or a more versatile structure for M(s, p) must be used. To achieve efficient use of the computational facility in an iterative design procedure such as this, it is important for the user to be aware of the progress of the design and to be able to isolate quickly those particular inequalities which are proving most difficult to meet within the limitations imposed by the physical constraints. In the design program the result

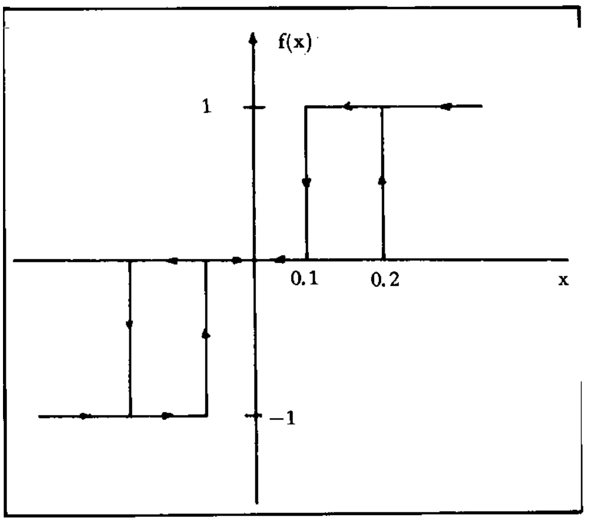

Figure 2. The relay element.

of each iteration is available if required.

Provided that a good initial structure is chosen for the compensator and that a realistic set of specifications and constraints is formulated, then convergence of the numerical design is usually quite rapid. However, if the inequality bounds are too severe the region of admissable points may be empty and the design will fail. Alternatively if the bounds are too wide, solution time will be short but an improved design will be possible and should be investigated. It is thus clear that the successful application of such a synthesis method requires a detailed knowledge of the system dynamics which allows realistic inequality bounds to be imposed, a good initial choice of compensator structure and a fast and accurate method of computing system transient performance.

Example

The feedback configuration is given by Figure (1–a) where the linear plant is

$$G(s) = \frac{1}{(s+1)^2}$$

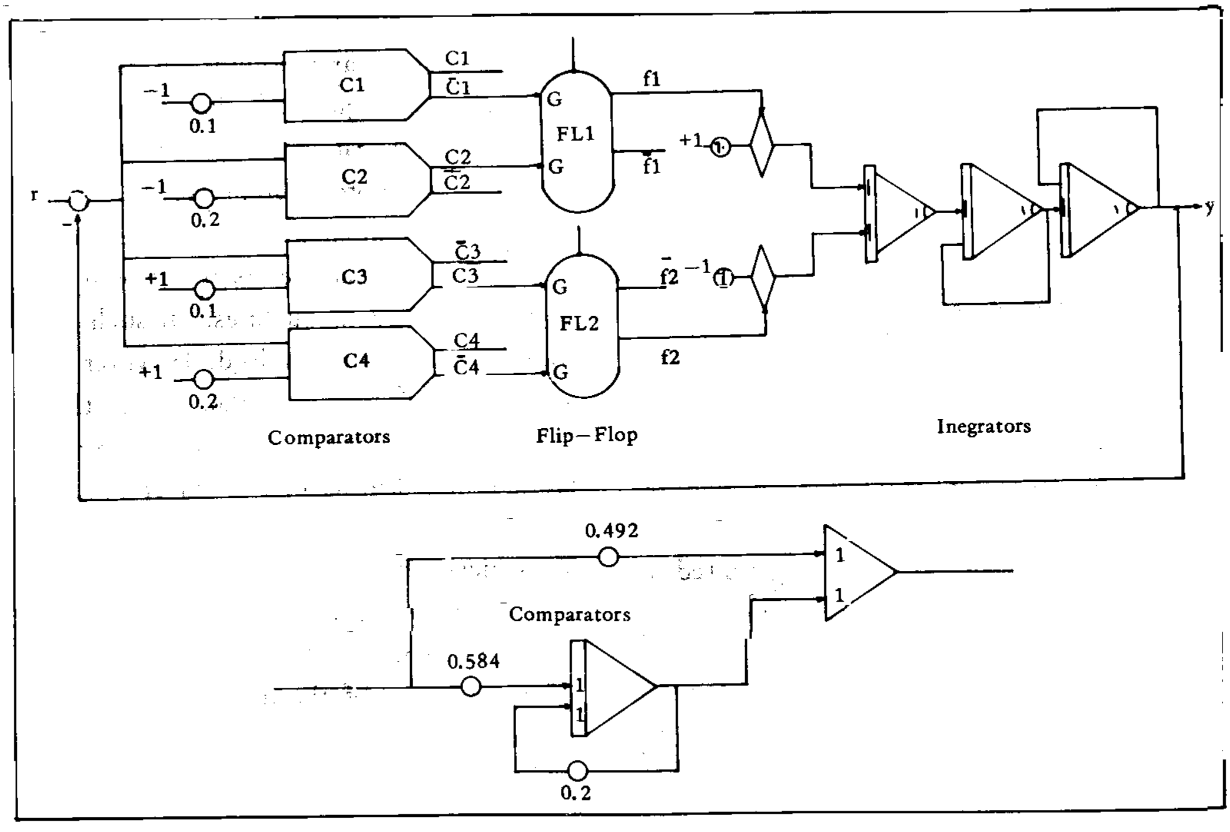

Figure 3. Analogue diagram of the uncompensated system.

and the relay element is one with hysteresis and dead-band as shown in Figure 2. The parameter of the limit cycle operation of the uncompensated system is predicted numerically and compared with the analogue simulation result in Table 1. The analogue computer diagram is shown in Figure 3 and the time response obtained digitally and by analogue simulation are shown in Figures (4, 5), respectively.

An initial compensator of the form

$$M(S, P) = \frac{1+s}{[illegible]}$$

was chosen and after 6 iterations of the design program the following compensator was obtained which eliminates limit cycle operation and satisfies the specified constraints in Table 2 for a unity step input.

$$M(S, P) = \frac{1 + 0.71s}{1 + 0.92s}$$

The time response of the compensated system obtained numerically is shown in Figure 6 and the result of analogue simulation in Figure 7.

The relay characteristic considered in this example has a general property in that it is discontinuous and without any linear segments. Thus it is a hard case for the time response method which uses the point-wise determination of the relay output. However by considering a small time interval (T=0.02 Sec.) it is shown that the time response is suf-

Table 1. Limit cycle predication.

Method	Frequency rad/sec.	Amplitude Units
Describing Function	$.9 < \omega < 1.1$	$0.4 < a < .8$
Digital time Response	0.98	0.77
Analogue Simulation	1.04	0.52

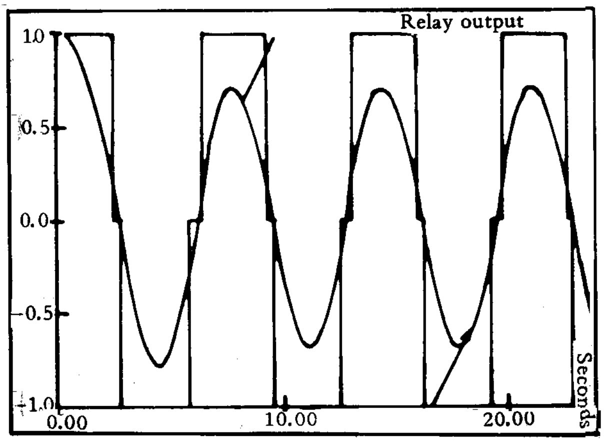

Figure 4. Digital time response of the uncompensated system.

Table 2. Time Response Specifications.

Unit step input to the system	Specified Constraints	Computed Response	Analogue Simulation
% steady-state error	5.0	4.2	3.5
% max. overshoot	10.0	9.98	9.0
Rise time (sec.)	5.0	4.6	4.1
Settling time (sec.)	10.0	6.81	7.6
Limit cycle ampl.	0.0	0.0	0.0
Limit cycle freq.	0.0	0.0	0.0

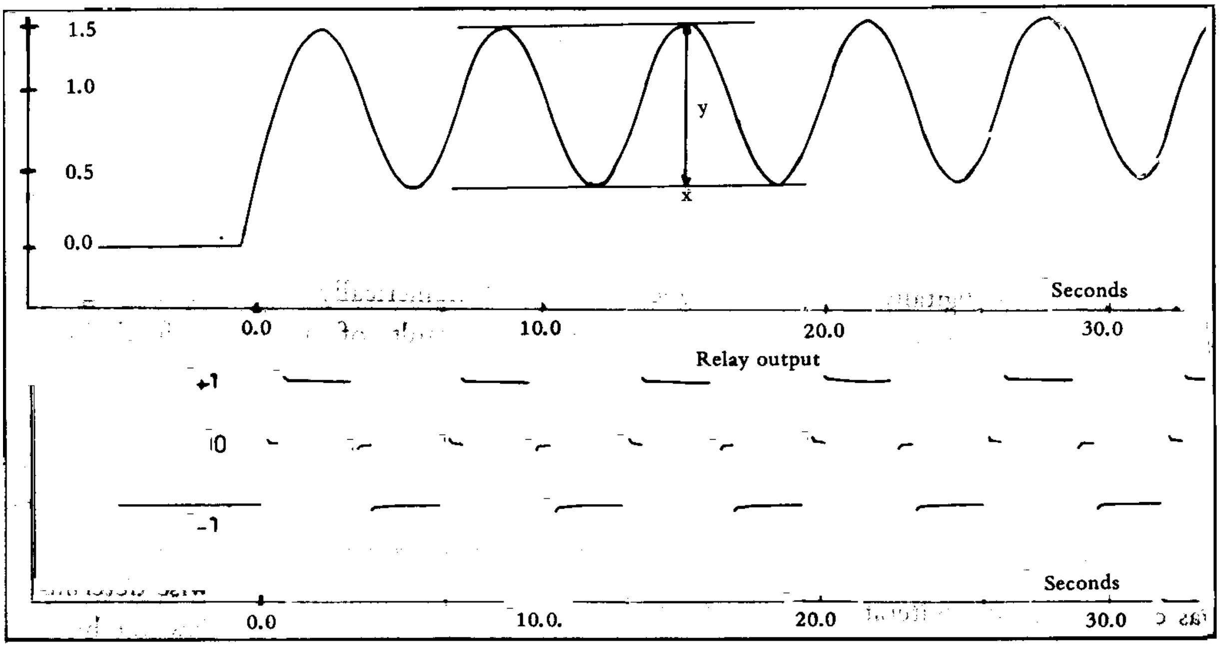

Figure 5. Analogue time response of the uncompensated system.

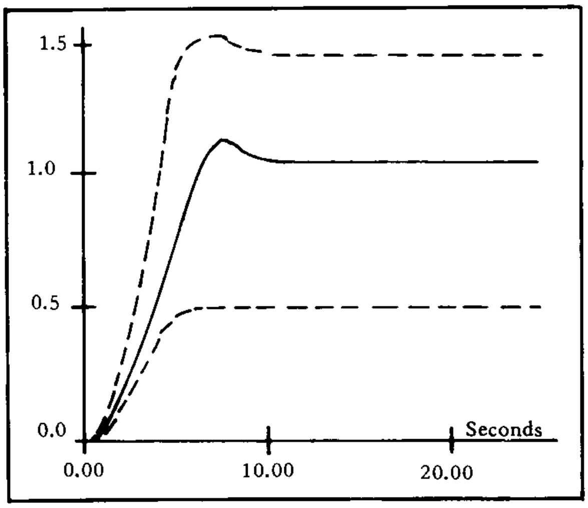

Figure 6. Digital time response of the compensated system.

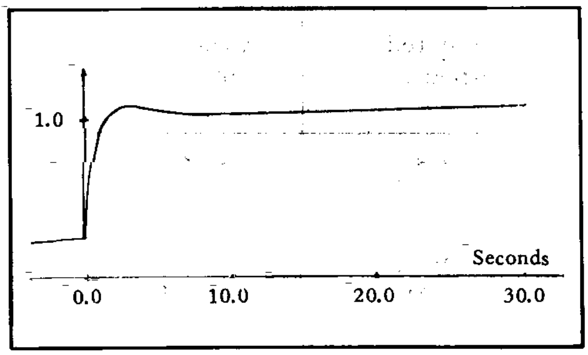

Figure 7. Analogue time response of the compensated system.

ficiently accurate, this is observed by the comparison of the results obtained numerically and by analogue computer.

In this example the dead-space in the relay eliminates the system dither and hence elimination of the limit cycle was possible by linear compensator. This dead-space, of course, causes an steady-state offset error which can not be completely eliminated. Thus a trade off can be effected between tolerable magnitude and frequency of limit cycle operation and steady-state error.

CONCLUSIONS

In this paper a powerful and efficient computer-aided numerical design and synthesis procedure for a relay-type control system is proposed. The method of inequalities employed utilizes the extension of both frequency and time domain to the nonlinear feedback systems. Performance criteria are formulated as a set of inequalities subject to a set of specified constraints and a nonlinear programming technique is used to solve them.

Provided that a set of realistic specifications and constraints has been chosen, the subsequent convergence of the numerical procedure is generally quite rapid. A worked example was given to illustrate the effectiveness of the computational numerical design method.

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