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Analytical Investigation on the Stress Distribution in Structural Elements Reinforced with Laminates Subjected to Axial Loads

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ABSTRACT

The present paper investigates the stress distribution in structural elements reinforced with laminates and subjected to axial loads. The proposed analysis provides an analytical solution for the shear stress distribution between the substrate and the reinforcement, and for the normal stresses in the cross-section of the reinforcement. The stress analysis assumes a linear elastic mechanical behaviour of the adherents. The solutions obtained are applied to the problem of the elastic stability of a beam/column supported at its ends. The manifestation of the buckling phenomena leads to the failure of the structural el-ement for loads lower than the characteristic strengths of the materials. In particular, the critical load represents the watershed value between the purely compressive and flexural behaviour of the element considered. In the case of stiff substrate, the buckling phenomenon leads to delami-nation of the reinforcement in the compressed area. Since buckling is an elastic problem, the stress distribution in the elastic range is used here to determine the value of the delamination length of the reinforcement at the critical load. In this way an analytical analysis which may be useful in the design phase of the reinforcement is proposed.

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1. INTRODUCTION

Structural composite materials have recently seen increasing use in a number of industrial fields, such as aeronautical, civil and automotive applications [1-3]. This is due to many advantages of composite technology, which is characterised by high mechanical properties and the possibility of increasing the strength of structural elements without adding weight to the structure [4,5].

Composite laminates are widely used for reinforcement or repair works of cracked structural elements or characterised by mechanical performance lower to those prescribed by the standards [6,7]. The reinforcement applications through the use of laminates have the objective of reducing the concentration of stresses in the structural element of the substrate, transferring the stresses to the reinforcement layer, to avoid the propagation of the fracture [8].

Unlike traditional mechanical reinforcement technologies (e.g. reinforced injections, tying, etc.), reinforcement by adhesive application of laminates involves less preventive work and no removal of the substrate material, resulting in a more uniform distribution of stresses.

Xiong and Shenoi [9] developed an experimental procedure based on static and fatigue strength concepts for an alloyed composite repair scheme for cracked aluminium alloy panels. The results showed that due to the reduction of the stress concentration caused by the repair, there is an improvement of the residual strength of the repaired specimens; thus, the residual tensile strengths of the repaired specimens are higher than those of the unrepaired specimens.

Jones and Chiu [10] carried out a series of experimental and numerical studies on crack repair in thick structural components. For this purpose, various problems were considered, such as semi-elliptical surface defects, integer surface defects, cracked fixing holes and cracked fins.

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The use of patches for reinforcing substrate materials also has applications in microelectronics and the medical field, where thin films are made mechanically cooperating in the substrate. The most frequent failure mechanism in this type of application is buckling of the patch.

Two buckling failure modes of the reinforcement could be observed: delamination and wrinkling. The first type consists in the detachment of the reinforcement from the substrate; the second occurs when the deformation of the patch exceeds that of the substrate. The occurrence of either type depends on the stiffness ratio between the substrate and the reinforcement; in particular, delamination occurs for stiff substrates and wrinkling for elastic substrates, as stated by Mei et al. [11].

The study of the reinforcement system of existing structures is an important issue [12]. Ozturk et al. [13] studied the effect of applying FRP reinforcements on historic buildings in Nigde, Turkey. Further work by Ozturk and Yilmaz [14] also provided an analytical solution for the modelling of FRP structural reinforcement by analysing a case study of historic buildings in Cappadocia.

In the present paper, the special case of a longitudinally developed structural element reinforced on both major faces is studied. In particular, the load transfer mechanism between the substrate and the reinforcement is analysed by means of a shear-lag model, through a parametric study on the influence of the elastic moduli of the materials and their geometries. Then, through the energy formulation for the critical delamination stress, a method of determining the delamination length at the critical axial load is proposed.

2. THEORY AND MODELING

In this section the theoretical analysis of the mechanical model is presented. By means of a shear-lag model the governing equations related to the interfacial shear stresses between patch and substrate and those related to the normal stresses in the reinforcement layer are exposed and analyzed through a parametrical study.

2.1 Analytical model

Fig. 1 illustrates the shear-lag model analyzed. In particular, this model considers the case of a composite beam reinforced by two laminates on the external faces and subjected to uniform tensile stresses σ_0. These stresses are applied directly to the structural substrate element and are transferred to the reinforcing laminates through shear stresses.

The assumptions underlying the calculation are:

i. all structural elements are homogeneous, elastic, linear and isotropic;

ii. normal (peel) stresses between patch and substrate are neglected;

iii. laminates are made of the same material;

iv. the tensile load is applied uniformly at the ends of the substrate.

Fig. 2 shows the free body diagram of the overlap region between the patch and the substrate. The equilibrium of the longitudinal loads yields:

$$d\sigma_f \cdot t_f \cdot b + \tau \cdot b \cdot dx = 0 \qquad (1.1)$$

$$\frac{d\sigma_f}{dx} = -\frac{\tau}{t_s} \qquad (1.2)$$

The shear strain is equal to:

$$\gamma = \frac{u_s - u_f}{t_s} \qquad (2)$$

Given the relationship:

$$\gamma = \frac{\tau}{G_s} \qquad (3)$$

Replacing Eqs. (2) and (3) in (1.2) the following could be obtained:

$$\frac{d^2\sigma_f}{dx^2} = -\frac{G_s}{t_f \cdot t_s}\left(\frac{du_s}{dx} - \frac{du_f}{dx}\right) = -\frac{G_s}{t_f \cdot t_s}\left(\frac{\sigma_s}{E_s} - \frac{\sigma_f}{E_f}\right) \qquad (4)$$

Considering the symmetry of the applied reinforcements, the following could be obtained:

$$\sigma_0 \cdot t_s = \sigma_s \cdot t_s + 2 \cdot \sigma_f \cdot t_f \qquad (5)$$

$$\sigma_s = \sigma_0 - \frac{2 \cdot \sigma_f \cdot t_f}{t_s} \qquad (6)$$

Combining Eqs. (4) and (6) the following is obtained:

$$\frac{d^2\sigma_f}{dx^2} = -\frac{G_s}{t_f \cdot t_s}\left(\frac{\sigma_0}{E_s} - 2\frac{\sigma_f}{E_f}\frac{t_f}{t_s} - \frac{\sigma_f}{E_f}\right) \qquad (7)$$

The result of the differential Eq. (7) is given as follows:

$$\sigma_f(x) = K_1 \cdot \sinh(\lambda x) + K_2 \cdot \cosh(\lambda x) + \frac{E_f\, t_s}{E_s\, t_s + E_f\, t_f}\sigma_0 \qquad (8)$$

where:

$$\lambda = \sqrt{\frac{G_s}{t_f \cdot t_s}\left(\frac{1}{E_f} + 2\frac{t_f}{E_s \cdot t_s}\right)} \qquad (9)$$

The boundary conditions relative to Eq. (7), are:

$$x = \pm L;\ \sigma_f = 0 \qquad (10)$$

From Eq. (8) the following could be obtained:

$$\sigma_f(x) = \frac{E_f\, t_s}{E_s\, t_s + E_f\, t_f}\sigma_0\left[1 - \frac{\cosh(\lambda x)}{\cosh(\lambda L)}\right] \qquad (11)$$

Shear stresses are given by the following:

$$\tau(x) = \frac{E_f\, t_s\, t_f}{E_s\, t_s + E_f\, t_f} \sigma_0 \frac{\lambda \sinh(\lambda x)}{\cosh(\lambda L)} \tag{12}$$

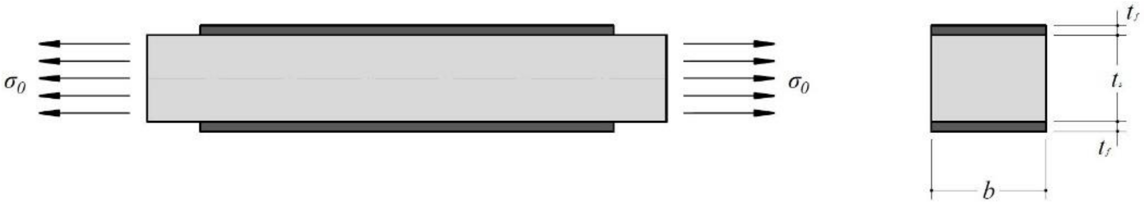


Figure 1. Reinforced structural element, longitudinal and cross-section

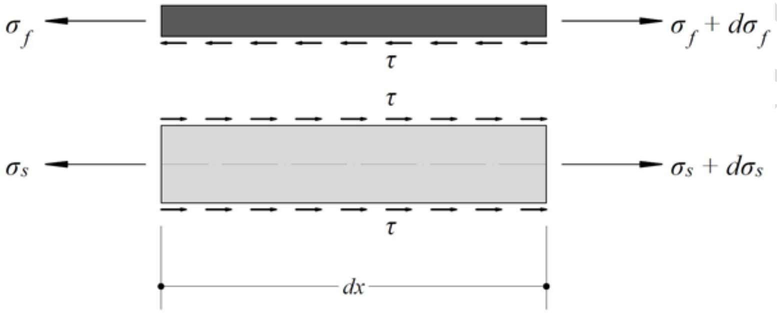


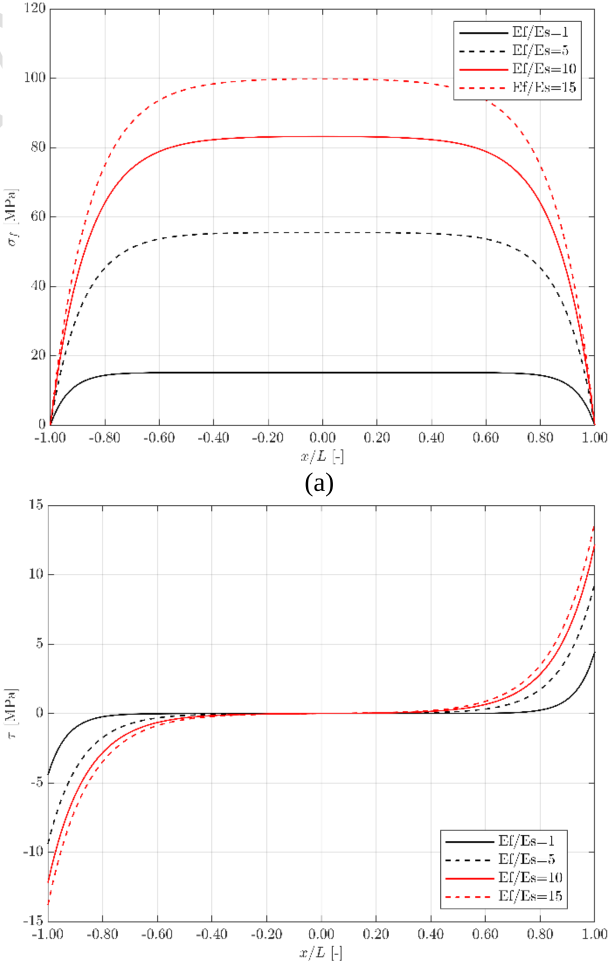
Figure 2. Free body diagram

2.2 A parametric study on the stress distribution

The following section deals with a parametric study on the distribution of normal stresses in the reinforcement layer and of the interfacial shear stresses between the reinforcement and the substrate. In particular, the effects of the elastic modulus of the adherents, the length and thickness of the reinforcement are investigated.

2.2.1 Elastic modulus

The elastic modulus of the adherents is a crucial element in the design phase for the correct choice of reinforcement elements. It is necessary to choose reinforcement elements with stiffness values compatible with those of the substrate to avoid the occurrence of unwanted stress peaks. The result of the calculation for different values of elastic modulus is graphically illustrated in Fig. 3(a-b).



(a)

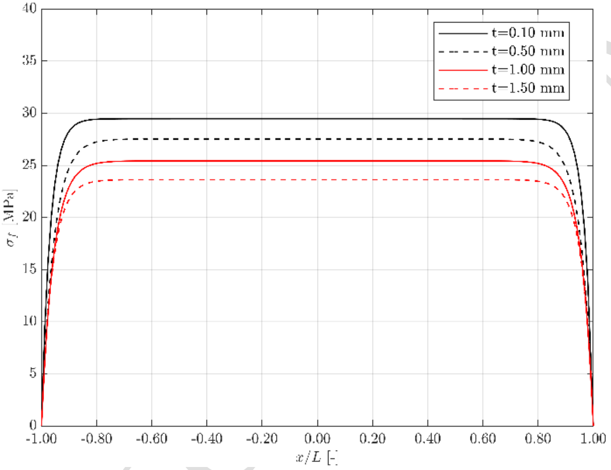
(b)

Figure 3. Effect of the Young moduli ratio of the adherents on the normal stress distribution in the reinforcement (a) and on the interfacial shear stress distribution (b)

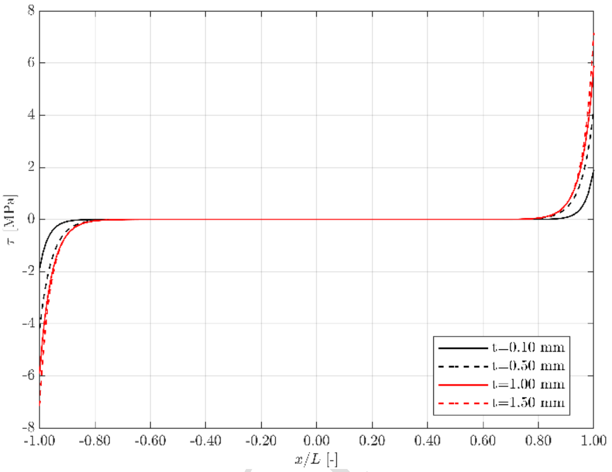
It could be observed that for each configuration the normal and shear stresses show the same qualitative trend. For the normal stresses, it can be observed that an increase in the ratio between the elastic moduli of the reinforcement and the substrate leads to an evident increase of the normal stresses near the central zone of the reinforcement. On the other hand, for the shear stresses, it is noted that the concentration of the stresses tends to flatten towards zero values for lower elastic moduli ratios. As the ratio of elastic moduli increases, it tends towards the extreme of the reinforcement.

2.2.3 Reinforcement thickness

In the present work, a further geometrical factor influencing the stress distribution, the thickness of the adherents was investigated. In fact, the stress distribution can be considered as constant through the width of the adherents. The effect of the anchorage thickness of the reinforcement to the substrate is shown in Fig. 4(a-b).



(a)



(b)

Figure 4. Effect of the reinforcement's thickness on the normal stress distribution in the reinforcement (a) and on the interfacial shear stress distribution (b)

It could be observed that in the case of normal stresses, an increase in the thickness of the reinforcement layer leads to a reduction in the stress peak; on the contrary, an increase in thickness leads to a reduction in normal stresses. On the contrary, in the case of shear stresses, there is an increase in the stress peaks as the thickness of the reinforcement increases.

2.3 Buckling Delamination

The mechanical contribution to the stability of a beam given by the reinforcement in the stretched zone could be investigated by means of the Winkler elastic model of the beam on elastic soil. The value of the critical load was determined by Timoshenko and Gere [15]:

$$P_{cr,RB} = \frac{\pi^2 EI}{l^2} + \frac{kl^2}{\pi^2} = P_{cr,E} + \frac{kl^2}{\pi^2} \qquad (13)$$

Where k is the flexural stiffness constant of the reinforcement and can be determined experimentally. The value of the critical load of the reinforced beam is greater than the classical Eulerian critical load by an amount proportional to the length of the beam and the stiffness of the reinforcement [15].

Considering the energy criterion, as reported by Mei et al. [11], the delamination stress for reinforcement could be expressed by the following experession:

$$\sigma_{cr} = \frac{\pi^2 E_f}{3(1-\nu_s{}^2)}\left(\frac{t_f}{2x}\right)^2 \qquad (14)$$

Where E is the Young modulus and **υ** is the Poisson's modulus, h is the reinforcement thickness and x is the half delamination length.

Since the occurrence of elastic buckling is an elastic phenomenon, it is possible to derive the delamination length by equating the critical elastic stress to the elastic stress distribution shown in the previous section.

Equalizing Eqs. (14) and (11), the following could be obtained:

$$x^2\left(1-\frac{\cosh(\lambda x)}{\cosh(\lambda L)}\right)=\frac{\pi^2\left(E_s\,t_s+E_f\,t_f\right)}{12\,(1-\nu_s{}^2)\,\sigma_0}\frac{t_f{}^2}{t_s} \qquad (15)$$

By solving Eq. (15) by iteration, the delamination length of the reinforcement at the critical load is obtained.

3. NUMERICAL APPLICATION

The following is a numerical application carried out by solving Eq. (15), considering different values of flexural stiffness k. The application herein reported provides an analytical analysis in alternative to the one presented in literature [16]. Fig. 5 shows, when varying the ratio of the elastic moduli of the support and of the reinforcement (E_f/E_s), the trend of the percentage of the delamination semi-length in correspondence of the critical load with respect to the total length of the reinforcement (*x/L*).

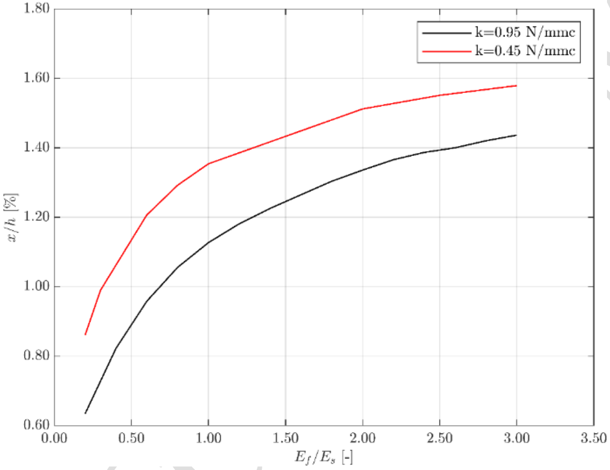


Figure 5. Delamination length percentage vs stiffness ratio (adimensional graph).

Fixed values – obtained from experimental campaign conducted by Capozucca et al. [16] – are considered for the elastic moduli of the reinforcement (also characterised by a constant k) and as the stiffness of the substrate varies, the percentages of the delamination length are obtained.

It is observed that as the E_f/E_s ratio increases, the behaviour of the curve is creepy. In particular, increasing the stiffness of the reinforcement with respect to that of the substrate, the laminate absorbs an increasing amount of strain and this determines an increase in the delamination length in correspondence with the critical load.

4. CONCLUSIONS

In the present paper the elastic stress distribution in a structural element subjected to axial loads is treated. By means of a shear-lag model, the distribution of both the interfacial shear stresses between the reinforcement and the substrate and the normal stresses inside the reinforcement layer are analysed. The study of the stresses is deepened through a parametric study, highlighting the influence of material and geometric factors on the distribution of the stresses. The results show that the greatest influence is exerted by varying the ratio between the elastic moduli of the reinforcement and the substrate.

The proposed analysis is then extended to the application case of a reinforced structural element subjected to axial buckling loads. Since buckling is an elastic phenomenon, the stress analysis previously exposed is extended - through the energetic formulation - to the determination of the initial delamination length of the reinforcement layer. These considerations are deepened through a parametric study that correlates the stiffness ratio to the initial delamination length.

The main objective of this paper is to provide an accurate analysis of the stress distribution, which allows the designer to assess the possible damage caused by delamination under compressive axial loads. An experimental campaign is under process, in order to validate the theoretical analysis here presented.

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Persian Abstract

مقاله حاضر به بررسی توزیع تنش در عناصر سازه‌ای تقویت‌شده با لایه‌های لایه‌ای و تحت بارهای محوری می‌پردازد. تحلیل پیشنهادی یک راه حل تحلیلی برای توزیع تنش برشی بین زیرلایه و تقویت‌کننده و برای تنش‌های نرمال در مقطع آرماتور ارائه می‌کند. تحلیل تنش رفتار مکانیکی الاستیک خطی چسبنده ها را فرض می کند. راه‌حل‌های به‌دست‌آمده برای مسئله پایداری الاستیک تیر/ستون که در انتهای آن پشتیبانی می‌شود، اعمال می‌شود. تجلی پدیده کمانش منجر به شکست عنصر ساختاری برای بارهای کمتر از مقاومت مشخصه مواد می شود. به طور خاص، بار بحرانی نشان دهنده مقدار حوضه بین رفتار صرفاً فشاری و خمشی عنصر در نظر گرفته شده است. در مورد بستر سفت، پدیده کمانش منجر به جدا شدن آرماتور در ناحیه فشرده می شود. از آنجایی که کمانش یک مشکل الاستیک است، در اینجا از توزیع تنش در محدوده الاستیک برای تعیین مقدار طول لایه لایه شدن آرماتور در بار بحرانی استفاده می شود. به این ترتیب یک تحلیل تحلیلی - که ممکن است در مرحله طراحی تقویت‌کننده مفید باشد - پیشنهاد می‌شود.