



## Non-linear Forced Vibration Analysis of Piezoelectric Functionally Graded Beams in Thermal Environment

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### PAPER INFO

#### Paper history:

Received 29 January 2021

Received in revised form 13 March 2021

Accepted 31 March 2021

#### Keywords:

*Geometrical Non-linearity*

*Functionally Graded Beams*

*Thermo-piezoelectric*

*Forced Vibrations*

*Multimode Approach*

### ABSTRACT

This work proposes a geometrically non-linear vibratory study of a functional gradation beam reinforced by surface-bonded piezoelectric fibers located on an arbitrary number of supports, subjected to excitation forces and thermoelectric changes. The non-linear formula is based on Hamilton's principle combined with spectral analysis and developed using Euler-Bernoulli's beam theory. In the case of a non-linear forced response, numerical results of a wide range of amplitudes are given based on the approximate multimodal method close to the predominant mode. In order to test the methods implemented in this study, examples are given and the results are very consistent with those of the literature. It should also be noted that the thermal charge, the electrical charge, the volume fraction of the structure, the thermal properties of the material, the harmonic force and the number of supports have a great influence on the forced non-linear dynamic response of the piezoelectrically functionally graded structure.

doi: 10.5829/ije.2021.34.11b.02

## 1. INTRODUCTION

Functionally graded piezoelectric structures are heterogeneous composite materials with excellent mechanical and electrical properties, making them potentially useful for many applications in structural mechanics, electronics and other engineering fields. These structures are mainly composed of functionally graded materials (FGMs) and piezoelectric materials and are very useful in practice as they are related to structural vibration control and thermoelectric stress control. FGM is an advanced composite material that can change continuously between surfaces according to a certain distribution law. In general, FGM consists of a mixture of metal and ceramics. Refractory ceramics have a heat-insulating effect due to their low thermal conductivity, while ductile metals have higher mechanical properties and reduce the risk of fracture.

This gives FGMs the following advantages: they can withstand harsh streets in high-temperature environments while maintaining their structural integrity, as demonstrated by Hosseini Hashemi et al. [1]. Piezoelectric structures are another class of advanced materials: they are smart structures that can be used as actuators for piezoelectric transformers and sensors to control structural vibrations. The main advantage of piezoelectric materials is that they can affect the mechanical state of the structure by changing the electric field applied to the material, as shown in the document of Tadi Beni et al. [2]. Therefore, the piezoelectric FGM structure has the advantage of combining the characteristics of the FGM material and the piezoelectric material.

In recent years, research activities related to this topic have been carried out. Demir et al. [3] have committed to solve the problem of bending nano/micro beams under concentrated and dispersed loads, and target various boundary conditions, i.e. cantilevered, tight, cantilevered and simply supported. Habibi et al.

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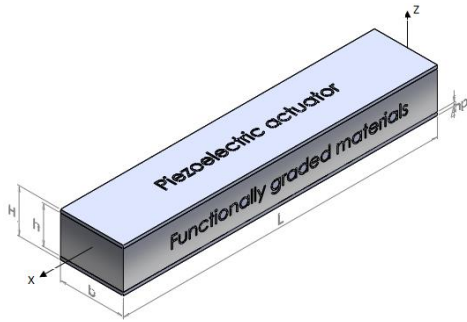
[4] developed the size dependent non-linear formulation for the Euler-Bernoulli nano-beam using the size dependent coherent piezoelectric theory. In this analysis, the properties of the FGM piezoelectric beam on bending, buckling, and free vibration responses were obtained and discussed. Samani and Beni [5] studied the static behavior and nonlinear free vibrations of Timoshenko's piezoelectric nanobeams under mechanical and electrical loads. In this analysis, they found that the size-dependent derivative formulation and the results of the formula were compared with the results of the linear torque stress theory and the classical linear and non-linear theories [5]. Tadi Beni [6] studied the mechanical and thermal buckling of the flexoelectric nanobeam. The results of this study indicate that as the thickness and length scale parameter increased the critical load and the critical temperature change increased. In addition, the results showed that a decrease in flexoelectric coefficient related to beam softening, critical load and critical temperature is generally reduced. Tadi Beni [7]. studied the high-order electromechanical coupling of free-vibrating nanoparticles based on Euler-Bernoulli beams in a thermal environment. In this study, the influence of parameters (such as size, length and temperature) on the natural frequencies of isotropic and anisotropic nanobeams were investigated. Tadi Beni et al. [8] used the coherent torque stress theory to study the non-linear analysis of the free and forced vibration of isotropic piezoelectric/viscoelastic nanobeams in a piezoelectric process. Nowadays, FGM structures that couple with piezoelectric materials are one of the most important engineering elements that are used in various types of systems. They also play an important role in the field of active control and intelligent detection. Li and Cheng [9] have proposed a vibration analysis method used to reinforce the static thermal post-bending of FGM stamped beams with a piezoelectric layer bonded to the surface. Use of numerical methods to solve ordinary differential equations. Li et al. [10] studied the static bending and free vibration of the cantilevered piezoelectric FGM beam using the modified stress gradient theory and Timoshenko's beam theory. Kiani and Eslami [11] studied the buckling of FGM beams. In this analysis, they assumed that the buckling surface of the beam has several piezoelectric layers and is affected by the temperature changes and constant tension. Rafiee and coworkers [12] studied the non-linear free vibration of carbon nanotube-reinforced FGM materials with a piezoelectric layer on the surface, which can withstand the combined effects of heat and electric charge. The results showed that the ratio between non-linear and linear frequencies increased with increasing the volume fraction and temperature. In another study, the same authors investigated the nonlinear thermal bifurcation buckling of carbon nanotube-reinforced composites, in

which a piezoelectric layer is bonded to the surface of a carbon nanotube structure [13]. Yuan [14] proposed an active vibration and sound control law based on an intelligent panel structure of dynamic vibration damper (DVA) type. Tang and Ding [15] analyzed the non-linear dynamic response of the coupling of transverse and longitudinal deformations of a functional gradient bi-directional beam. In this investigation, they assumed that material properties, moisture and heat distribution change progressively in the thickness and length directions. Their results showed that the non-linear frequency increased with increasing temperature and humidity concentration. They also showed that moisture concentration has a great influence on the thermal vibration of the FGM beam. More recently, Liu and coworkers [16] have studied the non-linear vibration of piezoelectric nanoplate materials subjected to thermal loading under various boundary conditions. The analysis is based on the theory of non-native Mindlin Patch Theory. However, it should be noted that the proposed literature review reveals the following conclusions: most research on the geometric non-linearity of FGM beams with surface-bound piezoelectric layers is limited to the use of numerical methods to solve the guiding equations. In addition, we have noticed that there are few studies on the forced vibration of piezoelectric FGM beams in thermal environments, and most research is based on linear theory.

In this paper, for the first time, attempts are made to exploit the approximate multimodal method that is close to the dominant mode to solve the guiding equations of the forced vibration of geometrically non-linear FGM piezoelectric beams. The paper also aims to carry out a numerical study of the free and forced non-linear vibrations of FGM beams reinforced with surface-fixed piezoelectric layers. The latter is placed on any number of supports and subjected to mechanical, thermal and electrical loads. In addition, the research covers a wide range of thermal loads ( $300 \leq T \leq 500$ ), electrical loads ( $-400 \leq V \leq +400$ ) and volume fractions ( $0 \leq n \leq 5$ ).

## 2. FUNDAMENTAL EQUATIONS

Consider the piezoelectric FGM beam shown in Figure 1. The length of the rectangular cross-section of the beam is  $L$  and the thickness is  $H$ . It consists of an FGM core of thickness  $h$  and a layer of piezoelectric material of thickness  $h_p$ . It is assumed that the piezoelectric actuator is symmetrical and perfectly adhered to the upper and lower surfaces of the FGM beam. The effective characteristics of the FGM beam are defined by the Voigt mixing rule [17], and the volume fraction is distributed using the power law [18], a technique commonly used by researchers because of its high



**Figure 1.** Coordinate system and schematic diagram of a piezoelectric functionally graded beams

accuracy. The characteristics of FGMs are illustrated below [19]:

$$P(z,T) = P_m + (P_c - P_m) \left( \frac{1+z}{2} \right)^n \tag{1}$$

Most FGMs are used at high-temperature environments, and the characteristics of the constituent materials depend on temperature, which can be written according to the definition in the literature [20]:

$$P = P_0(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3) \tag{2}$$

The piezoelectric material is assumed to have temperature-independent characteristics, where  $C_{11}$  and  $\alpha_p$  are the reduced elastic constant and thermal expansion coefficient, respectively, as summarized in Table 1.  $P$  is the temperature correlation coefficient of the FGM layer stated in Table 2. In this analysis, it is assumed that Young's modulus  $E_f$  and coefficient of thermal expansion  $\alpha_f$  are temperature dependent and can be evaluated at any temperature. However, the density  $\rho_f$ , thermal conductivity  $k_f$  and Poisson's ratio are independent of temperature [21].

**2. 1. Linear Formulation** The linear vibration equation of the piezoelectric FGM beam can be obtained:

$$\left( D_{11} - \frac{B_{11}^2}{A_{11}} \right) \frac{\partial^4 w}{\partial x^4} + (N_x^T + N_x^P) \frac{\partial^2 w}{\partial x^2} + I_0 \frac{\partial^2 w}{\partial t^2} = 0 \tag{3}$$

$N_x^T$  and  $N_x^P$  are the thermal resultant and the electrical force, respectively. They are calculated using the relations given in documents [22-23]:

$$N_x^T = \int_{-H/2}^{H/2} E\alpha\Delta T dz, N_x^P = \int_{-H/2}^{H/2} Ed_{31}E_z dz \tag{4}$$

$A_{11}$ ,  $B_{11}$  and  $D_{11}$  are extension-extension, flexion-extension-flexion and flexion-flexion coupling

coefficients, respectively; which can be evaluated using the classical FMS beam theory, as reported in the literature [24-25]. Their definitions are as follows:

$$A_{11} = \int_{-\frac{H}{2}}^{\frac{H}{2}} Edz, B_{11} = \int_{-\frac{H}{2}}^{\frac{H}{2}} E(z - z_0) dz, \tag{5}$$

$$D_{11} = \int_{-\frac{H}{2}}^{\frac{H}{2}} E(z - z_0)^2 dz$$

Equation (3) can be written in a slightly more complicated way, and the result is:

$$w'''' + \lambda w'' - \beta^4 w = 0 \tag{6}$$

In formula (6), the new symbol represents the following functional relationship:

$$\lambda = \frac{N^T + N^P}{D_{11} - \frac{B_{11}^2}{A_{11}}}, \beta^2 = \frac{\omega}{c}, c^2 = \frac{D_{11} - \frac{B_{11}^2}{A_{11}}}{I_0} \tag{7}$$

The lateral displacement of the beam can be defined as the correlation between several functions [26]. We can write the general solution of equation (6) at the  $j$ th support as follows:

$$w_{ji}(x^*) = A_j \sin \left( \sqrt{\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 + 4\beta_i^4}} (x^* - \xi_{j-1})L \right) +$$

$$B_j \cos \left( \sqrt{\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 + 4\beta_i^4}} (x^* - \xi_{j-1})L \right) +$$

$$C_j \sinh \left( \sqrt{-\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 + 4\beta_i^4}} (x^* - \xi_{j-1})L \right) +$$

$$D_j \cosh \left( \sqrt{-\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 + 4\beta_i^4}} (x^* - \xi_{j-1})L \right) \tag{8}$$

$x^*$  is the dimensionless coordinate, which can be written  $x^* = \frac{x}{L}$  and  $\xi_j = \frac{x_j}{L}$  is the dimensionless position of the support. The index  $i$  changes from 1 to  $n$ , where  $n$  is the number of functions. The constants  $A_j$ ,  $B_j$ ,  $C_j$  and  $D_j$  are determined by the boundary and continuity conditions of the beam. We point out that due to the applied thermoelectric axial loading, this system of equations allows us to obtain the natural frequency which is solved iteratively using the Newton-Raphson algorithm and the shape of the vibration mode.

**2. 2. Non-linear Formulation** Taking into account von-Karman's geometrical non-linearity (explaining the tension of the beam in the median plane), the relationship between deformation and displacement can be written as follows:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} - (z - z_0) \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \tag{9}$$

The Von-Karman geometric non-linearity considered in Equation (9) is applicable to displacement amplitudes of the order of the thickness of the rolled beam. This hypothesis is generally considered in the literature and mentioned in literature [27]. Therefore, the kinetic energy  $T_e$  of the piezoelectric FGM beam is equal to [28]:

$$T_e = \frac{I_0}{2} \int_0^L \left( \frac{\partial w}{\partial t} \right)^2 dx \tag{10}$$

For our case, the total elastic deformation energy of the Euler-Bernoulli beam is defined as follows:

$$V = \frac{1}{2} \int_0^L N_x \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + M_x \left( -\frac{\partial^2 w}{\partial x^2} \right) dx \tag{11}$$

The forces generated by the stresses  $N_x$  and  $M_x$  are the internal axial force and the bending moment acting on the median plane of the beam, respectively [29]. The lateral displacement function develops into a series of basic spatial functions, while the time function is considered as harmonics, as shown in literature [30]:

$$w(x,t) = a_i w_i(x) \sin \omega t \tag{12}$$

The expressions of kinetic energy and potential energy that vary with the lateral displacement defined above can be expressed as follows:

$$T_e = \frac{1}{2} \omega^2 a_i a_j m_{ij} \cos^2 \omega t \tag{13}$$

$$V = \frac{1}{2} a_i a_j a_k a_l b_{ijkl} \sin^4 \omega t + \frac{1}{2} a_i a_j k_{ij} \sin^2 \omega t + \frac{1}{2} a_i a_j a_k V_{ijk} \sin^3 \omega t \tag{14}$$

$m_{ij}$ ,  $k_{ij}$ ,  $b_{ijkl}$  and  $V_{ijk}$  are mass tensor, linear and non-linear stiffness tensors. For the piezoelectric FGM beam excited by the force  $F(x,t)$ , non-linear vibration equations are studied. The physical force  $F(x,t)$  excites the transverse mode of the structure by a set of generalized forces  $F_i(t)$ . These forces depend on the expression of  $F(x,t)$ , the point of excitation of the concentrated force, the repair in the range  $S$  representing the length of the beam or part of the beam, and the mode considered. The generalized force  $F_i(t)$  is given by:

$$F_i(t) = \int_S F(x,t) w_i(x,t) dx \tag{15}$$

In our case, the force  $F(x,t)$  can be considered as the distributed harmonic force  $F^d(x,t)$  or the concentrated harmonic force  $F^c(x,t)$  acting on the point  $x_f$ . We can write [31]:

$$F_i^d(t) = F^d \sin \omega t \int_S w_i(x) dx = f_i^d \sin \omega t \tag{16}$$

$$F_i^c(t) = F^c \sin(\omega t) w_i(x_f) = f_i^c \sin \omega t$$

According to Hamilton's principle, the dynamic behavior of the structure is expressed as follows:

$$\delta \int_0^{2\pi} \omega (V - T_e + W_T) dt = 0 \tag{17}$$

Given the symmetry of the matrices, the non-linear algebraic equations are calculated using tensor notation:

$$a_i k_{ir} + \frac{3}{2} a_i a_j a_k b_{ijk r} - \omega^2 a_i m_{ir} = f_r^d, r = 1, \dots, n \tag{18}$$

$$a_i k_{ir} + \frac{3}{2} a_i a_j a_k b_{ijk r} - \omega^2 a_i m_{ir} = f_r^c, r = 1, \dots, n \tag{19}$$

In order to carry out a general parametric study, we used a non-dimensional formulation by setting up:

$$x^* = \frac{x}{L}, w_i(x) = r w_i^*(x^*), r^2 = \frac{\int_{-H/2}^{H/2} z^2 dz}{\int_{-H/2}^{H/2} dz} \tag{20}$$

where  $m_{ij}^*$ ,  $k_{ij}^*$  and  $b_{ijkl}^*$  are the general non-dimensional matrices which are defined by:

$$m_{ij}^* = \int_0^1 w_i^* w_j^* dx^*, \tag{21}$$

$$k_{ij}^* = \int_0^1 \left( \frac{\partial^2 w_i^*}{\partial x^{*2}} \right) \left( \frac{\partial^2 w_j^*}{\partial x^{*2}} \right) dx^* +$$

$$\alpha_2 \int_0^1 \left( \frac{\partial^2 w_i^*}{\partial x^{*2}} \right) \left( \frac{\partial^2 w_j^*}{\partial x^{*2}} \right) dx^* + \alpha_3 \int_0^1 \left( \frac{\partial w_i^*}{\partial x^*} \right) \left( \frac{\partial w_j^*}{\partial x^*} \right) dx^*$$

$$b_{ijkl}^* = \alpha_1 \int_0^1 \left( \frac{\partial w_i^*}{\partial x^*} \right) \left( \frac{\partial w_j^*}{\partial x^*} \right) dx^* \int_0^1 \left( \frac{\partial w_k^*}{\partial x^*} \right) \left( \frac{\partial w_l^*}{\partial x^*} \right) dx^*$$

$$\alpha_1 = \frac{A_{11}^2}{48(D_{11}A_{11} - B_{11}^2)} H^2, \alpha_2 = \frac{B_{11}^2}{(D_{11}A_{11} - B_{11}^2)}, \tag{22}$$

$$\alpha_3 = \frac{(N^T + N^P) A_{11}}{2(D_{11}A_{11} - B_{11}^2)} L^2$$

The generalized dimensionless force  $f_i^{*d}$  corresponds to the uniformly distributed force in the range of  $S^*$  ( $0 \leq S^* \leq 1$ ) on one side, and the generalized dimensionless force  $f_i^{*c}$  on the other side corresponds to the force concentrated at any point of the beam, defined in literature [32] as follows:

$$f_i^{*d} = \frac{L^4 F^d}{r \left( D_{11} - \frac{B_{11}^2}{A_{11}} \right)} \int_{S^*} w_i^*(x^*) dx^*,$$

$$f_i^{*c} = \frac{L^3 F^c}{r \left( D_{11} - \frac{B_{11}^2}{A_{11}} \right)} w_i^*(x^*)$$
(23)

The numerical solutions of Equations (18) and (19) are obtained using the approximate method described by El Kadiri et al. [33]. This approximation consists of ignoring the second-order terms provided by the relevant mode. As mentioned in the literature, in the non-linear  $a_{i,j} a_k b_{ijk}$  expression of Equations (18) and (19), the second order term of  $\epsilon_i$  will be ignored, resulted in:

$$a_i a_j a_k b_{ijk} = a_1^3 b_{111r} + a_1^2 \epsilon_i b_{11ir}$$
(24)

Formulas (18) and (19) can be expressed as matrices:

$$\left( \left[ K_R^* \right] - \omega^{*2} \left[ M_R^* \right] \right) \{ A_R \} + \frac{3}{2} \left[ \alpha^* \right] \{ A_R \} = \left\{ f_i^{*d} - \frac{3}{2} a_1^3 b_{111}^* \right\}$$
(25)

$$\left( \left[ K_R^* \right] - \omega^{*2} \left[ M_R^* \right] \right) \{ A_R \} + \frac{3}{2} \left[ \alpha^* \right] \{ A_R \} = \left\{ f_i^{*c} - \frac{3}{2} a_1^3 b_{111}^* \right\}$$
(26)

The index  $i$  changes from 2 to  $n$ , where  $\left[ \alpha^* \right]$  is the matrix defined by  $a_1^2 b_{11ir}^*$ , and the vector  $\{ A_R \} = \left[ \epsilon_2, \epsilon_3, \dots, \epsilon_{10} \right]$  is a vector modeling the contribution coefficient, which can be determined by solving approximate linear Equations (25) and (26).

### 3. PRESENTATION AND DISCUSSION OF NUMERICAL RESULTS

In this numerical analysis, we consider that the length of the beam is  $L = 200$  mm, the thickness  $H = 10$  mm and the thickness of the FGM layer is  $h = 8$  mm. The piezoelectric fibers are manufactured on the basis of

PZT 5A, assuming that they are not affected by temperature, their characteristics are defined in Table 1 according to literature [34]. The FGM layer is based on silicon nitride ( $Si_3N_4$ ) and stainless steel (SUS304). Their Young's modulus and coefficient of thermal expansion are temperature dependent and are therefore listed in Table 2 according to literature [35-36]. Unless otherwise stated, we assume that the reference temperature is the same as the temperature of the lower surface of the piezoelectric FGM beam, while the temperature of the upper surface is variable and the Poisson's ratio of the FGM layer is a constant equal to 0.28. In addition, in order to ensure the accuracy and validity of the results obtained from this analysis and approximation, verification and validation studies will be conducted in the following section. Subsequently, a comprehensive parameter study was conducted to evaluate the influence of different parameters on the non-linear vibratory behaviour of the piezoelectric FGM beam.

#### 3. 1. Comparison with Previous Results

The first application presents a non-linear vibratory analysis of the results of a homogeneous isotropic beam which is compared to the predictions reported in literature [37-40]. Table 3 shows the ratio of the non-linear frequency to the linear frequency of the isotropic beam under different vibration amplitudes  $W_{max}/r$ . The results presented in Table 3 show that there is a good agreement between the predicted value of the current method and the other published data in the literature. In another verification study, under the conditions corresponding to a thermal load  $T_c = 300K$  and the absence of electric charge ( $V = 0V$ ), the Backbone Curves of the piezoelectric FGM beam with different volume fractions of  $n$  were considered in Figure 2. The figure shows that the results of this study are consistent with those obtained by Fu et al. [37]. Moreover, Figure 3 clearly shows that the influence of the volume fraction index  $n$  significantly affects the frequency ratio, and the non-linear frequency increases with increasing vibration amplitude. According to Figure 3, when the volume fraction remains constant  $n = 1$ , the voltage

**TABLE 3.** Comparisons of non-linear and linear frequency ratios of a homogeneous isotropic beam

$W_{max}/r$	Present	Ref [37]	Ref [38]	Ref [39]	Ref [40]
1	1.0222	1.0231	1.0222	1.0252	1.0222
2	1.0868	1.0892	1.0857	1.0899	1.0857
3	1.1880	1.1902	1.1831	1.1885	1.1833
4	1.3187	1.3178	1.3064	1.3140	1.3065
5	1.4723	1.4647	1.4488	1.4597	1.4477

variations  $V$  are respectively equal to  $V=400V$ ,  $0V$  and  $-400V$  have little influence on the backbone curves. Figure 4 shows the effect of temperature changes ( $T_c = 300,400,500K$ ) when the volume fraction is kept constant at  $n=1$ . The influence of temperature and volume fraction on the amplitude-frequency response curve is more severe than that of electrical charge. This can be predicted by formula (4), the value of the piezoelectric deformation constant is much less

than the thermal expansion coefficient. At the same time, the difference in thickness between the piezoelectric layer and the FGM layer is another factor. It can also be seen that the results of this study are consistent with those of the literature. It should also be noted that an increase in temperature causes an increase in the ratio of the non-linear frequency to the linear frequency.

TABLE 1. Properties of PZT 5A

Properties	$E_p$ (Gpa)	$\rho_p$ (Kg/m3)	$\kappa_p$ (W/mK)	$\nu_p$	$\alpha_p$ (1/K)	$d_{31}$ (m/V)
Values	63	7600	2.1	0.3	0.9e-6	2.54e-10

TABLE 2. Coefficients material properties as a function of temperature for  $Si_3N_4$  and SUS304

Materials	Properties	$P_0$	$P_{-1}$	$P_1$	$P_2$	$P_3$
$Si_3N_4$	$E_c$ (Pa)	348.43e+9	0	-3.07e-4	2.160e-7	-8.964e-11
	$\alpha_c$ (1/K)	5.8723e-6	0	9.095e-4	0	0
SUS304	$E_m$ (Pa)	201.04e+9	0	3.079e-4	-6.534e-7	0
	$\alpha_m$ (1/K)	12.33e-6	0	8.086e-4	0	0

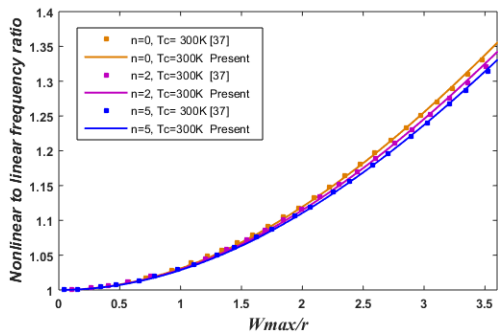


Figure 2. Comparison of the frequency ratio of the piezoelectric FGM beam with variation of the volume fraction  $n$

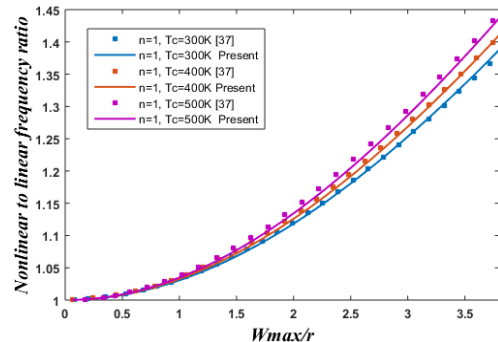


Figure 4. Comparison of the frequency ratio of the piezoelectric FGM beam under thermal load

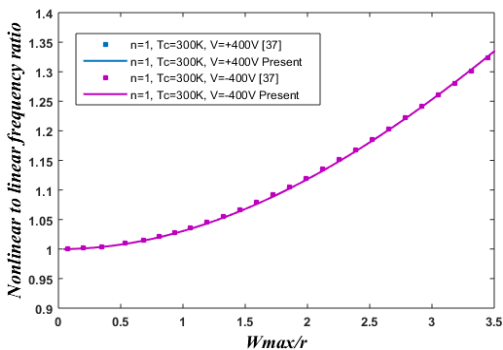


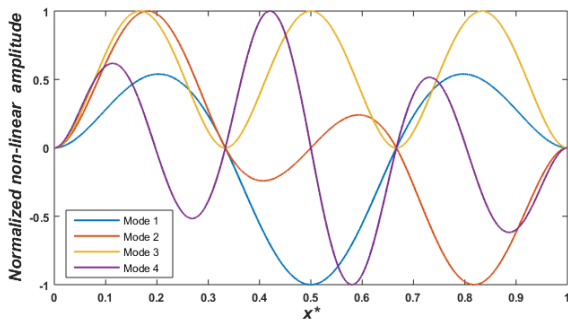
Figure 3. Comparison of the frequency ratio of the piezoelectric FGM beam under electrical load

### 3. 2. Numerical Results and Discussion

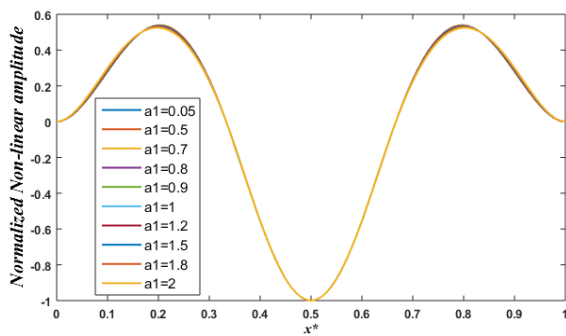
The numerical results presented in this section are obtained for embedded beams resting on two supports. The positions of the supports are chosen as follows:  $\zeta_1 = \frac{1}{3}$ ,

$\zeta_2 = \frac{2}{3}$ . Figure 5 shows the typical shape of the first four

modes of an isotropic beam. However, Figure 6 uses the current formula to present the shape of the first non-linear mode of the piezoelectric FGM, where the volume fraction of  $n=1$ , the thermal charge of  $T_c = 300K$  and the electrical charge of  $V=0V$ . It can be clearly seen in Figure 6 that for different vibration



**Figure 5.** The first normalized linear modes of a clamped beam resting on two simple supports

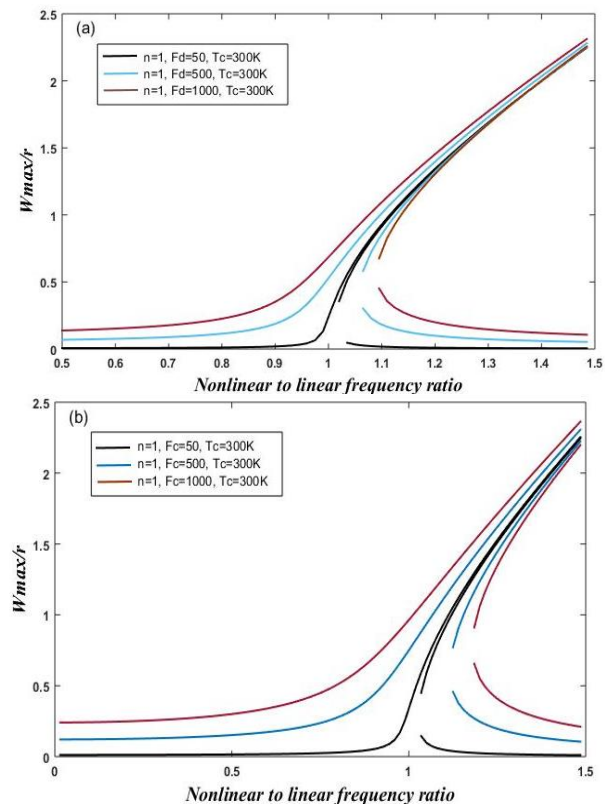


**Figure 6.** The first normalized non-linear mode of piezoelectric FGM beam, which rests on two simple supports and has several vibration amplitudes values

amplitudes, the effect of geometrical non-linearity can be observed.

As shown in Figures 7-10, in the case of a non-linear forced-vibration system, the resonance curve shows the jump phenomenon [40]. This behavior indicates that as the excitation frequency increases or decreases, the amplitude of the vibration may increase or decrease. This leads to the creation of a frequency range in which there are three amplitudes for a given frequency, resulting in frequency jumps. In this part of the numerical analysis, two typical excitations are verified, namely that the harmonic force uniformly distributed along the length of the beam is given by (a), while (b) presents the case of a force concentrated in the center of the beam. All frequency response curves show that the resonance area of the concentrated harmonic force is wider than that of the uniformly distributed harmonic force. In fact, this behavior indicates that the way to widen the resonance band is to add stiffness characteristics. A hardening or softening stiffness can produce the wider resonant frequency band [41]. As shown in Figures 7-10, the action of the concentrated harmonic force causes the widening of the resonant frequency band. In other words, beams subjected to concentrated harmonic forces exhibit a softening behavior compared to beams subjected to uniformly distributed harmonic forces. For the three excitation

levels corresponding to  $F = 50, 500$  and  $1000$ , Figures 7a and 7b show the influence of the uniformly distributed harmonic excitation and the concentrated harmonic force on the amplitude-frequency response curve of the beam, respectively. In these figures, we can see that the peak amplitude increases with excitation. In addition, for higher excitation values, the frequency range of the solution is wider. Figure 8 shows the effect of thermal loading on the amplitude-frequency response curve when the force is set to  $F = 500$ . The results show that as the thermal load increases, the amplitude of the frequency response decreases, while the amplitude-frequency curve tends to the right. In fact, this behavior indicates that the presence of non-linear terms can bend the amplitude-frequency response curve. In addition, as the temperature decreases, the hardening effect is greatly enhanced. Therefore, it can be deduced that thermal loading has a significant influence on the frequency response and the hardening of the beam. Figure 9 shows the effect of the volume fraction on the resonance response when the force is set at  $F=500$ . As shown in Figures 9a and 9b, for both types of excitation, an increase in the volume fraction index leads to an increase in the dimensionless frequency and a decrease in the maximum amplitude. Figure 10 shows the effect of the electrical voltage on the forced dynamic response of the beam when the force is set at  $F = 500$  and the volume fraction is set at  $n = 1$ .



**Figure 7.** Resonance curves of three levels of excitation

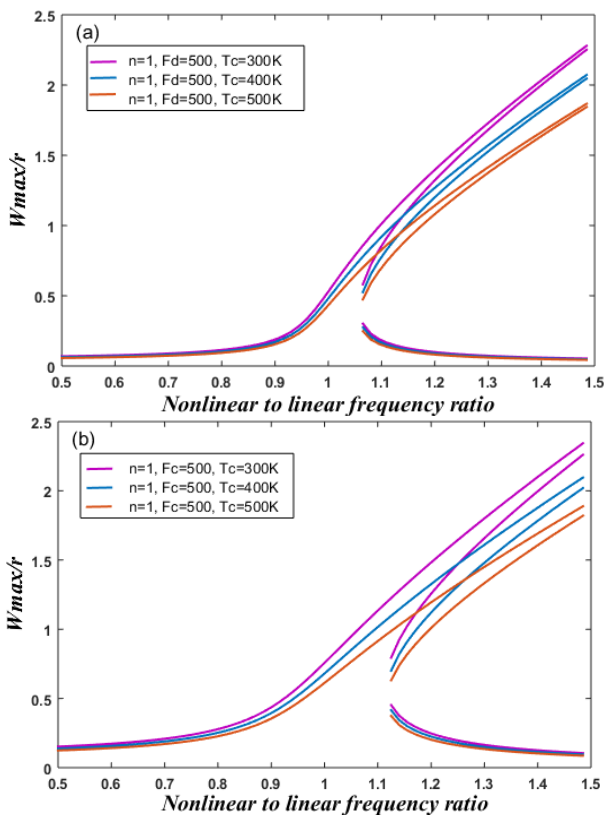


Figure 8. Resonance curves of thermal loads

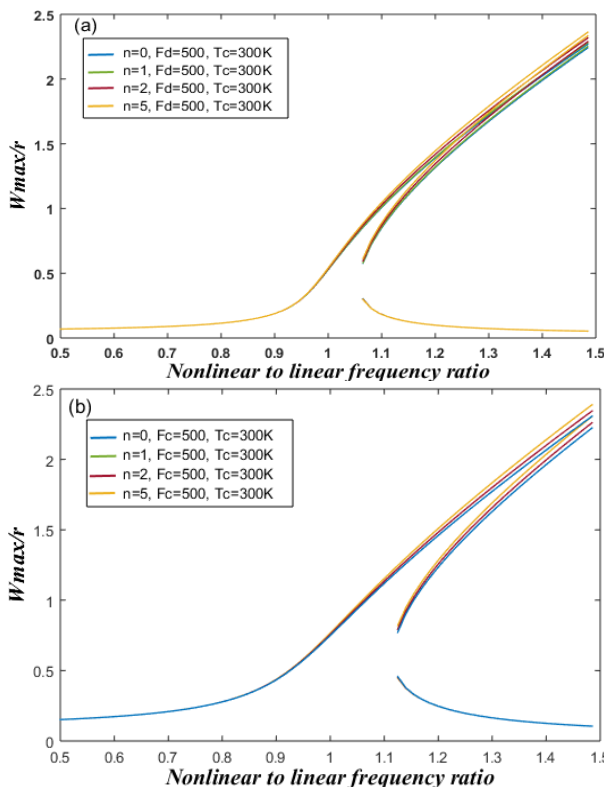


Figure 9. Resonance curves with different volume fraction values

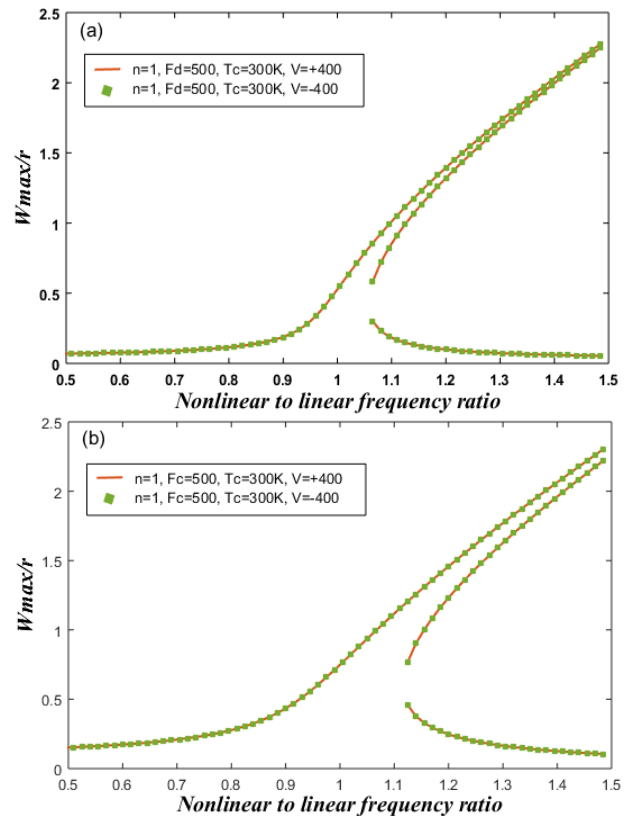


Figure 10. Resonance for different values of the electric charge

Obviously, since the value of the piezoelectric strain constant is much smaller than the coefficient of thermal expansion, the variation of the electric voltage has little effect on the resonance response curve, and the same phenomenon is observed in Figure 3. In addition, Figure 10 shows the unstable region of the non-linear forced vibration, in which the discontinuous part is the unstable boundary, the solution between the boundaries is unstable and the other regions are stable. In fact, this behavior indicates that the amplitude number varies with the type and value of the external excitation frequency. According to the forced non-linear response, the presence of regions with multiple values will cause a jump phenomenon. In the case of uniformly distributed external excitation, the unstable region is offset from the concentrated excitation.

#### 4. CONCLUSIONS

On the basis of Euler-Bernoulli's beam theory and von Kármán's displacement-deformation relationship, we have studied the free vibration and the geometrically non-linear forced vibration of the piezoelectric FGM beam under the action of a thermoelectric field. The Hamiltonian principle and spectral analysis are used to



obtain the guiding equations that control the free and forced non-linear behavior. The latter was adopted for the case of a uniform distribution over the length of the beam and also for the case of a force concentrated in the center of the beam. In addition, the analytical response of the non-linear forced vibration is obtained by introducing an approximation function based on the multimode method close to the main mode. This approximation makes it possible to obtain the dynamic behavior of beams resting on several supports. Then, the methods used in this study are verified by referring to the results of the literature. Finally, the numerical results revealed an impact on the resonance curve through several variances: the volume fraction index, the thermal load, the effect of the beam supports and the thermal characteristics of the constituent materials. The above analysis clearly highlights the following points:

- In the resonance response curve, an increase in the distributed or concentrated harmonic excitation force applied to the piezoelectric FGM beam causes the curve to gradually increase as the frequency increases, and this increase in force also widens the resonance curve.
- As the temperature decreases, the ratio of non-linear to linear frequency increases, and the amplitude-frequency response curve shows an improvement in peak amplitude.
- As the volume fraction increases, the amplitude of the forced vibration system gradually increases.
- Compared to the thermal load, the electrical load has little effect on the behavior of free and forced non-linear vibrations. The results also confirmed that the addition of reinforcing mounts has an important influence on the non-linear vibration behavior of the piezoelectric FGM beam.

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**Persian Abstract**

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**چکیده**

این کار یک مطالعه ارتعاشی غیر خطی هندسی از یک پرتو درجه بندی عملکردی را تقویت می کند که توسط الیاف پیزوالکتریک پیوندی روی سطح واقع در تعداد دلخواه پشتیبانی ، تحت نیروهای تحریک و تغییرات ترموالکتریک تقویت شده است. فرمول غیر خطی بر اساس اصل همپلتون همراه با تجزیه و تحلیل طیفی و با استفاده از نظریه پرتو اویلر-برنولی ساخته شده است. در مورد پاسخ اجباری غیر خطی ، نتایج عددی طیف گسترده ای از دامنه ها بر اساس روش چند مدی تقریبی نزدیک به حالت غالب داده می شود. به منظور آزمایش روشهای پیاده سازی شده در این مطالعه ، مثالهایی آورده شده است و نتایج با داده های گزارش شده بسیار مطابقت دارد. همچنین باید توجه داشت که بار حرارتی ، بار الکتریکی ، کسر حجمی سازه ، خصوصیات حرارتی ماده ، نیروی هارمونیک و تعداد تکیه گاه ها تأثیر زیادی در پاسخ دینامیکی غیر خطی دینامیکی ساختار پیزوالکتریک دارای درجه بندی عملکردی است.

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