A General Solution for Implicit Time Stepping Scheme in Rate-dependant Plasticity

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ABSTRACT

In this paper derivation of the second differentiation of a general yield surface by implicit time stepping method along with its consistent elastic-plastic modulus is studied. Moreover, the explicit, trapezoidal implicit and fully implicit time stepping schemes are compared in rate-dependant plasticity. It is shown that implementing fully implicit time stepping scheme in rate-dependant plasticity predicts more accurate experimental results than other schemes.


1. INTRODUCTION

Viscoplasticity describes the rate-dependent inelastic behavior of solids. Rate-dependence in this context means that the deformation of the material depends on the rate at which loads are applied. The inelastic behavior of viscoplasticity is a plastic deformation and means that the material undergoes unrecoverable deformations when a certain load level is reached. Rate-dependent plasticity is usually an important task because of its transient plasticity calculations. The main difference between the rate-independent plastic and viscoplastic material models is that the latter exhibits not only permanent deformations, but continues to undergo a creep flow as a function of time under the applied load. Some studies of the previous researchers on the mentioned subject are reviewed here.

non-unified model to capture the irregular rate dependency including strain controlled inelastic responses of polymers at the glassy state. Graham et al. [15] presented a specimen that can be tested in combined tension and torsion to achieve low triaxiality over a range of Lode angle. Two important factors in time stepping schemes are accuracy and stability. In explicit time stepping schemes the time interval plays a crucial role in stability and accuracy, i.e. a very large time interval cannot guarantee the stability and accuracy of the algorithm. On the other hand, in implicit time stepping schemes, even by choosing a large time interval, the stability is guaranteed. In fact, this is the main advantage of implicit to explicit time stepping algorithms. However, choosing a large time interval may not give proper accuracy.

The main aim of the present study is to obtain a simplified matrix formulation of a general yield criterion for an implicit time stepping scheme in a global finite element method and comparing three time stepping scheming methods in rate-dependent plasticity. The current proposed method is obtained based on the development of techniques given in references [16-20] and [21-25] in FE and plasticity theories, respectively. To show the capability of the method, it is implemented for an internally elastic-viscoplastic pressurized thick walled cylinder with perfectly plastic and linear-isotropic hardening behavior. The provided program is coded in Compaq Visual Fortran Professional Edition 6.5.0.

There are two novel points in the current study for viscoplastic materials. First, the analytical derivation of the first and the second differentiation of a general criterion, and second, the comparison between the time stepping schemes for different criteria and also using perfect and linear hardening behaviors of materials. Figure 1 shows the flowchart of the program for two-dimensional elastic-viscoplastic applications for a global finite element program [16].

2. GENERAL FORM OF A YIELDING CRITERION

The general form of a yield surface for isotropic materials is \( F(J_1, J_2, J_3) = 0 \), where \( J_1 \) is the first invariant of stress and \( J_2 \) and \( J_3 \) are the second and the third invariants of deviatoric stresses, respectively. Whereas \( J_1 \) indicates the dependence of the yield surface to the hydrostatic pressure and \( J_2 \) and \( J_3 \) show the dependence of the yield surface to deviatoric stresses. Introducing \( \theta \) as the angle of loading in deviatoric plane we have [16]:

\[
\sin 3\theta = -\frac{2\sqrt{3}}{3} \frac{h_2}{(J_3)^{\frac{3}{2}}}
\]

(1)

The angle of \( \theta \) represents the direction of the loading vector in the deviatoric plane, see Figure 2. For an isotropic material it would be sufficient if the yield surface is studied only in the region of \(-\frac{\pi}{6} \leq \theta \leq +\frac{\pi}{6}\). Hence, because the Lode parameter is defined as \( \Gamma = -\sqrt{3} \tan \theta \), the yield surfaces can be studied in \(-1 \leq \Gamma \leq +1\). Then, for the pure shear we have \( \Gamma = \theta = 0 \), for pure tension \( \theta = -\frac{\pi}{6} \) and \( \Gamma = +1 \) and for pure compression \( \theta = +\frac{\pi}{6} \) and \( \Gamma = -1 \). The four well-known yield surfaces can be shown in the following form for the computational convenience, where \( \sigma_y \) is the uniaxial yield stress, \( \kappa \) is the hardening parameter, \( \Phi \) is the angle of internal friction and \( c \) is the cohesion [16], Table 1. Drucker-Prager yield surface has a form of circular cone. In order to make the Drucker-Prager circle coincide with the outer apices of the Mohr-Coulomb hexagon at any section, it can be shown that:

\[
\alpha = \frac{2 \sin \Phi}{\sqrt{3(3 - \sin \Phi)}} \quad \kappa' = \frac{6c \cos \Phi}{\sqrt{3(3 - \sin \Phi)}}
\]

(2)

where the parameters \( \alpha \) and \( \kappa' \) were employed in Drucker-Prager criterion as it shown in Table 1. In Figure 2 the presentation of the four criteria can be observed in deviatoric plane.

3. CALCULATION OF THE FIRST AND SECOND DIFFERENTIATION OF A YIELDING CRITERION

To employ the implicit time stepping scheme along with the consistent elastic-plastic modulus in global finite element method, we need to compute the first and the second differentiation of the yield surface \( \frac{\partial F}{\partial \sigma} \) and \( \frac{\partial^2 F}{\partial \sigma^2} \) as below. The symbols \( \ddot{\sigma} \) and \( \dddot{\sigma} \) are used for \( 6 \times 1 \) vector and \( 6 \times 6 \) matrix in three dimensional stress space, respectively.

3.1. Calculation of the First Differentiation

From Equation (1) it can be shown that:

\[
\sin 3\theta = \Phi(J_2, J_3),
\]

(3)

therefore, any yield criterion can take the following form:

\[
F = F(J_1, J_2, \theta).
\]

(4)

<table>
<thead>
<tr>
<th>TABLE 1. Four famous yield criteria [16].</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tresca</td>
</tr>
<tr>
<td>von Mises</td>
</tr>
<tr>
<td>Mohr-Coulomb</td>
</tr>
<tr>
<td>Drucker-Prager</td>
</tr>
</tbody>
</table>
Figure 1. Flow sequence for the two-dimensional elasto-viscoplastic stress analysis program [16]

By differentiating Equation (4):

\[
\frac{\partial \bar{\sigma}}{\partial \sigma} = \frac{\partial \sigma}{\partial L_1} \frac{\partial L_1}{\partial \sigma} + \frac{\partial \sigma}{\partial L_2} \frac{\partial L_2}{\partial \sigma} + \frac{\partial \sigma}{\partial L_3} \frac{\partial L_3}{\partial \sigma} + \frac{\partial \sigma}{\partial \sigma}\left( \frac{1}{\frac{\partial \sigma}{\partial \mu}} - \frac{3L_2}{(L_2^2 + \gamma)^2} \right).
\]  

(5)

Differentiating Equation (1) gives:

\[
\frac{\partial \bar{\sigma}}{\partial \sigma} = \frac{\partial \bar{\sigma}}{\partial \sigma} - \frac{3L_2}{(L_2^2 + \gamma)^2} \frac{\partial \bar{\sigma}}{\partial \mu}.
\]  

(6)

Inserting Equation (6) into Equation (5) and with the aid of Equation (1) and after some simplification, it is found that:

\[
\frac{\partial \bar{\sigma}}{\partial \sigma} = C_1 \frac{\partial L_1}{\partial \sigma} + C_2 \frac{\partial L_2}{\partial \sigma} + C_3 \frac{\partial L_3}{\partial \sigma}.
\]  

(7)

where values of \(C_1, C_2\) and \(C_3\) and vectors of \(\frac{\partial L_1}{\partial \sigma}, \frac{\partial L_2}{\partial \sigma}\) and \(\frac{\partial L_3}{\partial \sigma}\) are tabulated in Appendix I.

3.2. Calculation of the Second Differentiation

Considering \(\frac{\partial \bar{\sigma}}{\partial \sigma} = \bar{a}(J_1,J_2,J_3,\bar{\sigma})\), it is found that:

\[
\frac{\partial^2 \bar{\sigma}}{\partial \sigma^2} = \frac{\partial \sigma}{\partial L_1} \frac{\partial \sigma}{\partial \mu} + \frac{\partial \sigma}{\partial L_2} \frac{\partial \sigma}{\partial \mu} + \frac{\partial \sigma}{\partial L_3} \frac{\partial \sigma}{\partial \mu} + \frac{\partial \sigma}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma}.
\]  

(8)

Inserting Equation (6) in Equation (8) and with the aid of Equation (1) we have:

\[
\frac{\partial^2 \bar{\sigma}}{\partial \sigma^2} = C_1 \frac{\partial L_1}{\partial \sigma} + C_2 \frac{\partial L_2}{\partial \sigma} + C_3 \frac{\partial L_3}{\partial \sigma} + \frac{\partial \sigma}{\partial \sigma}.\]

(9)

Now, the last term in the right-hand side of Equation (9) can be found by the aid of Equation (7) as following:
\[ \frac{\partial \mathbf{a}}{\partial \sigma} = C_1 \frac{\partial \mathbf{j}_1}{\partial \sigma^2} + C_2 \frac{\partial \mathbf{j}_2}{\partial \sigma^2} + C_3 \frac{\partial \mathbf{j}_3}{\partial \sigma^2}, \]  
\[ (10) \]

where vectors \( \mathbf{C}_1, \mathbf{C}_2 \) and \( \mathbf{C}_3 \) and also matrices \( \frac{\partial \mathbf{j}_1}{\partial \sigma^2}, \frac{\partial \mathbf{j}_2}{\partial \sigma^2} \) and \( \frac{\partial \mathbf{j}_3}{\partial \sigma^2} \) are presented in Appendix II. After some simplifications, the general form of the second differentiation of any yield surface as \( f(J_1, J_2, J_3) \) takes the following form:

\[ \frac{\partial^2 \mathbf{F}}{\partial \sigma^2} = \left( \frac{\partial C_3}{\partial \sigma} - \frac{\tan \theta \beta}{2 \sigma} \right) \mathbf{F} \otimes \cdot \mathbf{e} + C_3 \mathbf{g} \otimes \mathbf{e} - \]
\[ \frac{\sqrt{\mathbf{F}}}{2 \cos \theta} \mathbf{f} \otimes \frac{1}{\sqrt{\mathbf{F}}} \frac{\partial C_3}{\partial \sigma} \mathbf{g} \otimes \mathbf{e} - \]
\[ \mathbf{g} \cdot \frac{1}{3} C_2 \mathbf{M} + \frac{1}{2} C_3 \mathbf{N}, \]
\[ (11) \]

where vectors \( \mathbf{e}, \mathbf{f} \) and \( \mathbf{g} \) along with matrices \( \mathbf{M} \) and \( \mathbf{N} \) are presented in Appendix III.

### 4. VISCOPLASTIC FLOW RULE

A common explicit form of an associated viscoplastic strain rate is offered by the following viscoplastic flow rule:

\[ \dot{\varepsilon}^{vp} = \gamma < \Phi(\sigma) > \frac{\partial \mathbf{F}}{\partial \sigma}, \]
\[ (12) \]

where \( \Phi(\sigma) \) is a yield surface and \( \gamma \) is a fluidity parameter controlling the plastic flow rate. The term \( \Phi(\sigma) \) is a positive monotonic increasing function for \( \sigma > 0 \) and the notation \( < \cdot > \) implies that:

\[ < \Phi(x) > = \Phi(x) \quad x > 0 \]
\[ < \Phi(x) > = 0 \quad x \leq 0 \]
\[ (13) \]

Different functions for \( \Phi \) have also been recommended as follows [16]:

\[ \Phi(\sigma) = e^{M(\sigma - \sigma_0) / \sigma_0} - 1, \]
\[ \Phi(\sigma) = \left( \frac{\sigma - \sigma_0}{\sigma_0} \right)^N, \]
\[ (14) \]

where \( M \) and \( N \) are arbitrary prescribed constants.

#### 4. 1. The Viscoplastic Strain Increment

With the strain rate law expressed by Equation (12) we can define a strain increment \( \Delta \varepsilon^{vp}_n \) occurring in a time interval \( \Delta t_n = t_{n+1} - t_n \) using implicit time stepping scheme as [16]:

\[ \Delta \varepsilon^{vp}_n = \Delta t_n \left( 1 - \theta \right) \varepsilon^{vp}_n + \theta \varepsilon^{vp}_{n+1}, \]
\[ (15) \]

For \( \theta = 0 \) the Euler time integration scheme is obtained which is also referred to as 'fully explicit' (or forward difference method) since the strain increment is completely determined from the existing conditions at time \( t_n \). On the other hand, taking \( \theta = 1 \) gives a 'fully implicit' (or backward difference) scheme with strain increment being determined from the strain rate corresponding to the end of the time interval. The case \( \theta = \frac{1}{2} \) results in the so-called 'implicit trapezoidal' scheme which is also known generally as the Crank-Nicolson rule in the context of linear equation. To define \( \varepsilon^{vp}_{n+1} \) in Equation (15), the limited Taylor series expansion can be used [16]:

\[ \varepsilon^{vp}_{n+1} = \varepsilon^{vp}_n + \mathbf{H}^{n} : \Delta \sigma^n, \]
\[ (16) \]

where,

\[ \mathbf{H}^{n} = \frac{\partial \mathbf{F}}{\partial \sigma} = \mathbf{H}, \]
\[ (17) \]

where \( \Delta \sigma^n \) is the stress change occurring in the time interval \( \Delta t_n = t_{n+1} - t_n \).

Thus Equation (15) can be written as:

\[ \Delta \varepsilon^{vp}_n = \varepsilon^{vp}_n \Delta t_n + \mathbf{C}^{n} : \Delta \sigma^n, \]
\[ (18) \]

where,

\[ \mathbf{C}^{n} = \theta \Delta t_n \mathbf{H}, \]
\[ (19) \]

#### 4. 2. Evaluation of Matrix \( \mathbf{H} \)

To use the fully implicit or semi-implicit (trapezoidal) time stepping scheme, the matrix \( \mathbf{C} \) is required which in turn can be expressed in terms of \( \mathbf{H} \) as indicated in Equation (17). Matrix \( \mathbf{H} \) has to be explicitly determined for the yield surface assumed for material behavior. From Equations (12) and (17) we have:

\[ \Delta \varepsilon^{vp}_n = \varepsilon^{vp}_n \Delta t_n + \mathbf{C}^{n} : \Delta \sigma^n, \]
\[ (18) \]

where,

\[ \mathbf{C}^{n} = \theta \Delta t_n \mathbf{H}, \]
\[ (19) \]

#### 5. COMPUTATIONAL PROCEDURE

The essential steps in solving process can be summarized here. The solution begins from the known initial conditions at \( t = 0 \), which are the static elastic situation. At this stage \( d^n, F^0, \varepsilon^0 \) and \( \sigma^0 \) are known and \( \varepsilon^{vp}_0 = 0 \). The time marching scheme described in the previous section is then employed to advance the solution by one time stepping at a time. The solution sequence can be observed in Table 2.
TABLE 2. The solution sequence adopted in the current study.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
</table>
| 1    | Suppose at time $t = t_n$ we have an equilibrium situation and $\bar{B}^n$, $\bar{σ}^n$, $\bar{ε}^n$, $\bar{ε}_{vp}^n$, $\bar{f}^n$ are known. The following quantities are calculated: $\bar{B}^n = \bar{B}_0 + \bar{B}_{nt}$, $\bar{σ}^n = (\bar{B}^{-1} + \bar{C}^n)^{-1}$, $\bar{K}^n = \int B^T : \bar{σ}^n B \, dΩ$, $\bar{ε}_{vp}^n = γ < φ > \frac{∂^n}{∂σ}$.
| 2    | Compute the displacement increments $\bar{d}^n$: $\bar{d}^n = \bar{K}^{-1} : \bar{V}^n$, $\bar{V}^n = \int B^T : \bar{B}^n dΩ$. Calculate the stress increment $\bar{σ}^n$: $\bar{σ}^n = \bar{B}^n (\bar{B}^n \bar{d}^n - \bar{ε}_{vp}^n \bar{α} t_n)$. |
| 3    | Determine the total displacements and stresses: $\bar{d}^{n+1} = \bar{d}^n + \bar{d}^n$, $\bar{σ}^{n+1} = \bar{σ}^n + \bar{σ}^n$. |
| 4    | Calculate the viscoplastic strain rate: $\bar{ε}_{vp}^{n+1} = γ < φ > \frac{∂^{n+1}}{∂σ}$. |
| 5    | Apply the equilibrium correction: First calculate $\bar{B}^{n+1}$ using displacements $\bar{d}^{n+1}$ and substitute stresses $\bar{σ}^{n+1}$ into equilibrium forces $\bar{F}^{n+1}$: $\bar{F}^{n+1} = \int B^T : B^{n+1} \bar{σ}^{n+1} dΩ + f^{n+1}$, add these to the vector of incremental pseudo loads for use in the next time step: $\bar{V}^{n+1} = \int B^T : B^{n+1} \bar{σ}^{n+1} \bar{α} t_{n+1} dΩ$. |
| 6    | Check to see if the viscoplastic strain rate $\bar{ε}_{vp}^{n+1}$ is small enough at each Gaussian integrating point throughout the structure (i.e. to within a specified tolerance), if so, steady state conditions are achieved and the solution is either terminated or the next load increment is applied. If $\bar{ε}_{vp}^{n+1}$ is non-zero, return to Step 1 and repeat the entire procedure for the next time step. |
| 7    | Suppose at time $t = t_n$ we have an equilibrium situation and $\bar{B}^n$, $\bar{σ}^n$, $\bar{ε}^n$, $\bar{ε}_{vp}^n$, $\bar{f}^n$ are known. The following quantities are calculated: $\bar{B}^n = \bar{B}_0 + \bar{B}_{nt}$, $\bar{σ}^n = (\bar{B}^{-1} + \bar{C}^n)^{-1}$, $\bar{K}^n = \int B^T : \bar{σ}^n B \, dΩ$, $\bar{ε}_{vp}^n = γ < φ > \frac{∂^n}{∂σ}$. |

6. RESULTS AND DISCUSSIONS

In this part an internally pressurized elastic-viscoplastic thick walled cylinder as illustrated in Figure 3 is investigated. The mechanical properties are assumed as follows: Young modulus of elasticity ($E = 21000 \, dN/mm^2$), Poisson’ ratio ($γ = 0.3$), yield stress ($σ_0 = 24.0 \, dN/mm^2$), plastic modulus ($H = 0.0 \, dN/mm^2$), fluidity parameter ($γ = 0.001/\text{day}$), inner radius of the cylinder ($a = 100 \, mm$) and outer radius of the cylinder ($b = 200 \, mm$), and the flow function $Φ(F) = F$. In order to verify the developed computer code, the explicit and implicit trapezoidal and the fully implicit time stepping schemes along with the von Mises criterion and associated flow rule are used. An explicit time stepping algorithm ($Θ = 0$) is initially employed and the variation of radial displacement at the inner surface with time can be observed in the curves of Figure 4. In the first one, the load increment $P = 14 \, dN/mm^2$ is applied in one load step for two time steps. As it is seen, the reduction of accuracy is apparent with the larger time stepping which overestimating the viscoplastic strain rates. In the second one, the load increment $P = 14 \, dN/mm^2$ is applied in two load steps, i.e. $12 \, dN/mm^2$ and $2 \, dN/mm^2$ for $τ = 0.01$. As it is seen the convergence of two curves is acceptable (see Figure 4).

The initial time stepping was chosen as 0.1 days and the steady state convergence tolerance parameter was taken as 0.1 % as in Ref. [16]. The problem was then resolved, using the implicit trapezoidal time stepping scheme ($Θ = 0.5$) and the full implicit or backward difference scheme ($Θ = 1$). Good agreements between the three time integration schemes are evident in Figure 5.

Figure 6 shows the steady state hoop stress distributions for the time integration schemes $Θ = 0$ and $Θ = 1$. The results presented in Figures 4-6 are quite close to those of Ref. [16]. In the following, different investigations and comparisons are presented including comparison between the explicit, implicit trapezoidal and fully implicit time stepping schemes for both independent and dependent criteria to hydrostatic pressure. It is noted that in Ref. [16] only the solution for von Mises criteria was presented, however, with the current calculation approach, the second differentiation of a yield criterion the solution for any yield criterion can be achieved.
In the following, different investigations and comparisons are presented including comparison between the explicit, implicit trapezoidal and fully implicit time stepping schemes for both independent and dependent criteria to hydrostatic pressure. It is noted that in Ref. [16] only the solution for von Mises criteria was presented, however, with the current approach of calculating the second differentiation of a yield criterion the solution for any yield criterion can be achieved. Figures 7-8 show the variation of circumferential strain of the inner surface \( \frac{\tau}{R} \leq 1 \) with time and also the steady state circumferential stress distributions for \( 1 \leq \frac{\tau}{R} \leq 2 \) for explicit, implicit trapezoidal and fully implicit time stepping schemes while the von Mises criterion, perfect-plastic material...
and linear isotropic hardening behavior, \( H' = \frac{E}{10} \), are used and the geometry of the cylinder is assumed to be \( \frac{b}{a} = 2 \).

Figure 7 shows that with increasing time and development of plastic zone the differences between the results of three time stepping schemes increase. Moreover, the results of the trapezoidal implicit and fully implicit are more close to each other than those of explicit schemes. Furthermore, the circumferential strains predicted by the implicit time stepping scheme are larger than those predicted by the others. In addition, the steady hoop stresses predicted by the fully implicit time stepping scheme are smaller than those of the others, Figure 8.

The results show that the hardening increases the difference between three time stepping schemes. Figures 9-14 show the same parameters based on Tresca, Mohr-Coulomb and Drucker-Prager criteria, respectively. In using Mohr-Coulomb criterion, it is assumed that \( \tan \Phi = 0.879 \). The linear isotropic hardening is assumed and the hardening module is taken as \( H' = E/20 \) and \( H' = E/10\sqrt{3} \) for Mohr-Coulomb and Drucker-Prager criteria, respectively. The results show the same trend as explained for Figures 7 and 8, previously. Moreover, it is seen that the difference between the three time stepping schemes in Mohr-Coulomb and Drucker-Prager criteria are larger than those of Tresca and von-Mises criteria, respectively. In other words, when the yielding criteria are dependent on the hydrostatic pressure, the difference between the results obtained from the explicit, implicit trapezoidal and fully implicit time stepping schemes become more significant than those of the hydrostatic pressure independent yielding criterion. Furthermore, for nonlinear isotropic hardening case this difference increases compared to perfect-plastic one.

The results obtained by the explicit, implicit trapezoidal and fully implicit time stepping schemes in steady state condition and experimental results are compared in Figure 15. This figure demonstrates the internal pressure in overstrain of 100% \( (E_{\sigma}/\sigma_y = 1) \) at external surface of the vessel with respect to the variation of ratio of external radius/internal radius, \( (b/a) \), for perfect-plastic material and von Mises criterion. The figure shows that using the fully implicit time stepping scheme could predict the experimental results more accurately. It is also observed that the difference between the numerical simulations and experimental data is maximum in \( b/a = 1.6 \) and is minimum in \( b/a = 2.4 \).
7. CONCLUSIONS

The trapezoidal implicit and fully implicit time stepping schemes and also explicit time stepping schemes for different yield criteria along with the derivation of the second differentiation of the yield surface were studied. The results can be summarized as follows:

1- Applying more loads and development of plastic zone make the difference between the results more apparent.

2- The hydrostatic pressure dependent yield criteria were too sensitive in using time stepping schemes.

3- Implementing fully implicit time stepping scheme in rate-dependant plasticity predicts the experimental results more accurately than the other schemes.
8. REFERENCES


APPENDIX I

The values of $C_1$, $C_2$ and $C_3$ are as follows:

$$
C_1 = \frac{\partial \Phi}{\partial \epsilon} \\
C_2 = \frac{\partial \Phi}{\partial \epsilon} - \frac{\tan \beta}{{\cal I}_2} \frac{\partial \Phi}{\partial \beta} \\
C_3 = -\sqrt{3} \frac{1}{2 \cos \beta} \frac{\partial \Phi}{\partial \beta}
$$

(A1.1)

These values are obtained for four famous criteria in Table A1.

The vectors $\frac{\partial \Phi}{\partial \epsilon}$ and $\frac{\partial \Phi}{\partial \beta}$ are independent from type of criterion and are as follows:

$$
\frac{\partial \Phi}{\partial \epsilon} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
$$

(A1.2)

$$
\frac{\partial \Phi}{\partial \beta} = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z
\end{bmatrix}
$$

where:

$$
\sigma_x = \frac{\partial \sigma_x}{\partial \tau_{yy}} + \frac{\partial \sigma_z}{\partial \tau_{yx}} + \frac{\partial \sigma_y}{\partial \tau_{xy}} \\
\sigma_y = \frac{\partial \sigma_y}{\partial \tau_{yy}} + \frac{\partial \sigma_z}{\partial \tau_{yx}} + \frac{\partial \sigma_y}{\partial \tau_{xy}} \\
\sigma_z = \frac{\partial \sigma_z}{\partial \tau_{yy}} + \frac{\partial \sigma_z}{\partial \tau_{yx}} + \frac{\partial \sigma_y}{\partial \tau_{xy}}
$$
APPENDIX II

The vectors \( \vec{c}_1, \vec{c}_2\) and \( \vec{c}_3\) are as follows:

\[
\vec{c}_1 = \frac{\delta f}{\delta l_z}\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T,
\]

\[
\vec{c}_2 = \frac{\delta f}{\delta l_z} \begin{bmatrix} \sigma_x^2 - \tau_{yx} + \frac{\delta \sigma_x}{\delta l_z} \frac{\delta \tau_{yx}}{\delta l_z} \\
\sigma_x \sigma_y - \tau_{yx} + \frac{\delta \sigma_x}{\delta l_z} \frac{\delta \tau_{yx}}{\delta l_z} \\
\sigma_y \sigma_z - \tau_{zx} + \frac{\delta \sigma_y}{\delta l_z} \frac{\delta \tau_{zx}}{\delta l_z} \\
\sigma_z \sigma_y - \tau_{yz} + \frac{\delta \sigma_z}{\delta l_z} \frac{\delta \tau_{yz}}{\delta l_z} \\
\end{bmatrix} + \frac{\delta C_3}{\delta l_z}
\]

\[
\vec{c}_3 = \frac{\delta f}{\delta l_z} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T (A2.1)
\]

The matrices \( \vec{c}_1, \vec{c}_2\), and \( \vec{c}_3\) are as in (A 2.2):

\[
\frac{\delta^2 f}{\delta l_z^2} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\frac{\delta^2 f}{\delta l_z^2} = \begin{bmatrix}
2 & -1 & 0 & 0 & 0
2 & -1 & 0 & 0 & 0
0 & 2 & 0 & 0 & 0
0 & 2 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\frac{\delta^3 f}{\delta l_z^3} = \begin{bmatrix}
\sigma_x & \sigma_y & \sigma_z & 2 \tau_{xy} & \tau_{zx}
\sigma_y & \sigma_z & \sigma_x & 2 \tau_{zx} & \tau_{xy}
\sigma_z & \sigma_x & \sigma_y & 2 \tau_{xy} & \tau_{zx}
\end{bmatrix}
\]

APPENDIX III

Vectors \( \vec{e}, \vec{f}, \vec{g}\) along with the matrices \( \vec{M}, \vec{N}\) are presented in the (A 3.1):

\[
\vec{e} = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & 2 \tau_{xy} & 2 \tau_{zx} \end{bmatrix},
\]

\[
\vec{f} = \begin{bmatrix} \sigma_x \sigma_x - \tau_{zx} + \frac{\delta \sigma_x}{\delta l_z} \frac{\delta \tau_{zx}}{\delta l_z} \\
\sigma_y \sigma_y - \tau_{xy} + \frac{\delta \sigma_y}{\delta l_z} \frac{\delta \tau_{xy}}{\delta l_z} \\
\sigma_z \sigma_z - \tau_{yz} + \frac{\delta \sigma_z}{\delta l_z} \frac{\delta \tau_{yz}}{\delta l_z} \\
\end{bmatrix},
\]

\[
\vec{g} = \begin{bmatrix} \frac{1}{2} & -1 & -1 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 \\
\end{bmatrix} (A3.1)
\]

\[
\vec{M} = \begin{bmatrix}
\sigma_x & \sigma_y & \sigma_z & 2 \tau_{xy} & \tau_{zx}
\sigma_y & \sigma_z & \sigma_x & 2 \tau_{zx} & \tau_{xy}
\sigma_z & \sigma_x & \sigma_y & 2 \tau_{xy} & \tau_{zx}
\end{bmatrix}
\]

\[
\vec{N} = \begin{bmatrix}
\sigma_x & \sigma_y & \sigma_z & -3 \tau_{yx} & 3 \tau_{xy}
\sigma_y & \sigma_z & \sigma_x & -3 \tau_{zx} & 3 \tau_{zx}
\sigma_z & \sigma_x & \sigma_y & -3 \tau_{xy} & 3 \tau_{xy}
\end{bmatrix}
\]
<table>
<thead>
<tr>
<th>Yield criterion</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tresca</td>
<td>0</td>
<td>$\frac{\cos \theta}{(\eta \tan \theta \cos \theta)}$</td>
<td>$\frac{\sqrt{3} \sin \theta}{\eta \cos \theta}$</td>
</tr>
<tr>
<td>von Mises</td>
<td>0</td>
<td>$\frac{\sqrt{3} \sin \theta}{\eta \cos \theta}$</td>
<td>0</td>
</tr>
<tr>
<td>Mohr-Coulomb</td>
<td>$\frac{1}{2} \sin \phi \frac{\cos \theta}{2(\eta \tan \theta \cos \theta)^2}$</td>
<td>$[(1 + \tan \theta \tan 3\theta) + \frac{\sin \phi}{\sqrt{3}}]$</td>
<td>$\frac{1}{2\eta \cos \theta} (\sqrt{3} \sin \theta + \cos \theta \sin \phi)$</td>
</tr>
<tr>
<td>Drucker-Prager</td>
<td>0</td>
<td>$\frac{1-\tan^2 \theta}{2(\eta \tan \theta \cos \theta)^2}$</td>
<td>0</td>
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</tbody>
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<td>Tresca</td>
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<td>$-C_2 \tan \theta + \frac{1}{(\eta \tan \theta \cos \theta)^2} \left[ \frac{\tan 3\theta}{\cos \theta} + \frac{3 \sin \theta}{\cos^2 \theta} \right]$</td>
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<td>$\frac{\cos \theta}{2\eta \cos \theta} \left[ \sqrt{3} (1 + 3 \tan \theta \tan 3\theta) + \sin \phi (3 \tan 3\theta - \tan \theta) \right]$</td>
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A General Solution for Implicit Time Stepping Scheme in Rate-dependant Plasticity

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