SOLUTION OF WAVE EQUATIONS NEAR SEAWALLS BY FINITE ELEMENT METHOD

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(Received: October 1, 2006 – Accepted in Revised Form: September 13, 2007)

Abstract A 2D finite element model for the solution of wave equations is developed. The fluid is considered as incompressible and irrotational. This is a difficult mathematical problem to solve numerically as well as analytically because the condition of the dynamic boundary (Bernoulli’s equation) on the free surface is not fixed and varies with time. The finite element technique is applied to solve nonlinear wave equations. The finite element model includes the conventional method based on a variational principle. This model minimizes the relevant function of the problem. After calculating two independent variables (i.e. $\phi$ and $\eta$) the pressure, forces and moments acting on sea-walls can be computed. These values are compared with existing experimental and theoretical outputs. The standing wave behavior is well described by the model, e.g. we can get the envelope of breaking waves in curve designs, which are developed for non-breaking waves. Also we can estimate the effective depth of a certain wave. Therefore the model can be used to propose some design curves.

Keywords Finite Element Method, Wave Equation, Seawalls, Numerical Method

1. INTRODUCTION

One of the main problems in the analysis of wave effects on marine structures is to approximate the forces acting on sea-walls. Unfortunately, there is no explicit formula for calculating these forces and their momentums.

A good understanding of the behavior of an offshore structure depends on a good understanding of the surrounding wave fields and relevant forces. In other words, the main step in loading such structures is to solve the wave field around them.

The literature review on this subject suggests that a number of wave crests are parallel to the wall so that the reflection effect can be ignored [1]. Therefore, it will be enough to analyze only the
effect of a standing wave on the wall. It is supposed that the maximum pressure load on the wall occurs in such case; however this has been shown that the ultimate pressure belongs to the waves that strikes the wall obliquely and then reflects back.

One of the most important problems in fluid mechanics is the analysis of nonlinear behavior of a fluid with a free surface. Evidently, such problems are numerically-analytically troublesome. This difficulty is caused by dynamic boundary condition (Bernoulli equation) at free-surface, where the location of free surfaces is time dependent and may or may not be initially determined.

Although several investigations are conducted with this theory never the less, there are many problems left for the future. In this paper a finite element model is developed to solve the wave equations near seawalls. This includes the conventional methods based on variational principles.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Assuming the flow is incompressible and irrotational, the Laplace equation defines the problem over the flow domain:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]  

(1)

With four boundary conditions:

\[
\frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = -h
\]  

(2)

\[
\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = \eta
\]  

(3)

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} \right) + g\eta = 0 \quad \text{at} \quad y = \eta
\]  

(4)

\[
\phi(x,y,t) = \phi(x-ct,y)
\]  

(4a)

\[
\frac{\partial \phi}{\partial x} = 0 \quad \text{at} \quad x = 0
\]  

(5)

Where X and Y are horizontal and vertical coordinate, \( h \) is the mean water level and \( \eta(x,t) \) states the location of free surface above the still water level. Equation 2 sets the horizontal velocity component to be zero at bottom. Equations 3 and 4 state the kinematic and dynamic boundary conditions at free surface, respectively. Equation 3 also indicates that the fluid particles just in contact with free surface will remain in contact with it. The dynamic boundary condition states the iso-pressure free surface (which is set to be zero in Bernoulli equation). This equation stands where the surface tension is neglected. The velocities at vertical boundaries are set to be zero in Equation 5.

Now what is left is to find the potential function \( \phi \) and to compute the desired forces and moments. Figure 1 shows the geometry, the boundary conditions and also the direction of the wave.

2.1. Literature Review of Existing Methods

The problem of a standing wave with zero contact angle is analyzed by Tadjbaksh, et al [2]. Goda [3] has analyzed the same problem using fourth order approximation. He studied the pressure of standing waves in more details. Sainflous, et al [4] found that in the case of high amplitude waves where the water is deep enough, for most cases the maximum pressure occurs near the wave crest, not just at maximum.

Tsuchiya, et al [5] made a comparison between various analytical methods and experimental data and observed that the first and second order theories stand over limited ranges while the third and fourth order theories agree with a large domain of experiments.

Nagai [1] made a comparison between experimental data and irrational theories and
developed relations for maximum applicable pressure. In case of inclining reflected waves, the intersecting waves are a kind of short-crested waves in which the top view of intersections has regularly, repeated diamond-like appearance.

The common theoretical method for the solution of short-crested waves is the Stokes theory, which is applicable for deep waters. Hsu, et al [6], developed the third order method. Roberts [7] and Roberts, et al [8] studied the problems in more details. Using Fourier expansion they developed numerical approximation to higher orders.


2.2. Finite Element Model This model was first used by Washizu [16] for slashing problem in wave tanks. The model used the variational technique to minimize the relevant function of the problem.

This kind of problem can be defined by the following equation:

$$\int_{\Omega} G_j(u) \, d\Omega + \int_{\Gamma} g_j(u) \, d\Gamma = 0$$  \hspace{1cm} (6)

In which $G_j$ and $g_j$ are the specific function that can be integrated. The Equation 6 can be changed in the form:

$$\int_{\Omega} G_j(u) \, d\Omega + \int_{\Gamma} g_j(u) \, d\Gamma = \sum_{e=1}^{m} \left( \int_{\Omega^e} G_j(u) \, d\Omega + \int_{\Gamma^e} g_j(u) \, d\Gamma \right)$$  \hspace{1cm} (7)

In which $\Omega$ is the domain element and $\Gamma$ is the boundary.

For the application of finite element method to solve this kind of problem, first the variational method is used in the following equation:

$$\Pi = \int_{\Omega} F \left( u, \frac{\partial u}{\partial x}, \ldots \right) \, d\Omega + \int_{\Gamma} E \left( u, \frac{\partial u}{\partial x}, \ldots \right) \, d\Gamma$$  \hspace{1cm} (8)

In which $u$ is the unknown function, $E$ and $F$ are specific operators.

Luck [17] has stated this variational principle as below:

$$x = I_1 + I_2 + I_3 + I_4$$  \hspace{1cm} (9)

$$I_1 = \frac{1}{2} \int_{\Omega} \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \, dx \, dy$$  \hspace{1cm} (10)

$$I_2 = \frac{1}{2} \int_0^{\infty} g \eta^2 \, dx$$  \hspace{1cm} (11)

$$I_3 = \int_0^{\infty} \frac{\partial \phi}{\partial t} \eta \, dx$$  \hspace{1cm} (12)

$$I_4 = -\int_0^{\infty} \frac{\partial \eta}{\partial t} \phi \, dx$$  \hspace{1cm} (13)

In which $\phi$ and $\eta$ are independent variables of variational expression. Note that the fluid volume $V$ acts as a function of $\eta$. Also $\frac{\partial \phi}{\partial t}$ and $\frac{\partial \eta}{\partial t}$ are assumed to be constant during the time increment $\Delta t$; so their first order variations are neglected in each element as shown in Figure 2, the parameter $\phi$ is specified as a linear function of $x$ and $y$:

$$\phi = a + bx + cy$$  \hspace{1cm} (14)

The function $\eta$ on the boundary is defined as a

Figure 2. The finite-element mesh for the problem.
product of isoperimetric functions as below:
\[
\eta = \frac{1}{2}(1-r)\eta_1 + \frac{1}{2}(1+r)\eta_k
\]  
(15)

After each time increment, the unknown values of the problem are considered to have the form of:
\[
\phi = \phi_0 + \Delta \phi, \quad \eta = \eta_0 + \Delta \eta \quad \text{at} \quad t = t_0 + \Delta t
\]  
(16)

Where \( \phi_0 \) and \( \eta_0 \) are the values obtained in previous step. Considering Figure 3 and regarding back to Equation 14, for the time \( t_0 \) it can be written that:
\[
\frac{\partial \phi}{\partial x} = A_0 T \phi = b = \text{cnst} \]  
(17)
\[
\frac{\partial \phi}{\partial y} = B_0 T \phi = c = \text{cnst} \]  
(18)
\[
B_0 T = \frac{1}{2\Delta_0} \left[ X_k - X_j, X_i - X_k, X_j - X_i \right]
\]  
(19)
\[
\phi_0 T = \left[ \phi_{i,j,k} \phi_0,j \phi_0,i \right]
\]  
(20)

After each tie increment, the area of the triangle \( p_i p_j p_k \) is:
\[
\Delta = \Delta_0 + P^T \Delta \eta
\]  
(21)

Where \( \Delta \eta \) represents the vector of unknowns and \( \Delta_0 \) and \( P \) contains the known values of pervious step:
\[
P^T = \frac{1}{2} \left[ X_K - X_j, X_i - X_K, X_j - X_i \right]
\]  
(22)
\[
\Delta \eta^T = \left[ \Delta \eta_{i,0}, \Delta \eta_{j,0}, \Delta \eta_{k,0} \right]
\]  
(23)

Also:
\[
\frac{\partial \phi}{\partial x} = A_0 T \left\{ \phi_0 + \Delta \phi \right\} + \Delta \eta^T A_1 T \phi_0
\]  
(24)
\[
\frac{\partial \phi}{\partial y} = (A_0 + A_1 \Delta \eta)^T \left\{ \phi_0 + \Delta \phi \right\}
\]  
(25)
\[
A_i = \frac{1}{(2\Delta_0)^2}
\]  
(26)
\[
\left[ (y_j - y_k)(x_k - x_j) \quad 0 \quad 2\Delta_0 + (y_j - y_k)(x_j - x_i) \quad 2\Delta_0 + (y_j - y_k)(x_j - x_i) \quad -2\Delta_0 + (y_i - y_j)(x_j - x_i) \quad 0 \quad (y_i - y_j)(x_j - x_i) \quad 2\Delta_0 + (y_j - y_k)(x_j - x_i) \quad 0 \quad (y_i - y_j)(x_j - x_i) \right]
\]  
(27)
\[
\frac{\partial \phi}{\partial y} = B_0 T \left\{ \phi_0 + \Delta \phi \right\} + \Delta \eta^T B_1 T \phi_0
\]  
(28)
\[
B_1 = \frac{1}{(2\Delta_0)^2}
\]  
(29)
\[
\left[ (x_k - x_j)^2 \quad (x_k - x_j)(x_j - x_i) \quad (x_k - x_j)(x_j - x_i) \quad 0 \quad (x_i - x_k)(x_j - x_i) \quad (x_k - x_j)(x_j - x_i) \quad 0 \quad (x_j - x_i)^2 \right]
\]  
(30)

Now, substituting each of variables in the variational expression, leads to:
\[
I_1 = \frac{1}{2} \int \int [b^2 + c^2] dx dy
\]  
(30)
\[ I_1 = \frac{1}{2} \sum_{e} (\phi_0 + \Delta \phi)^T \{ [A_0 + A_1 \Delta \eta][A_0^T + \Delta \eta^T A_1^T] + [B_0 + B_1 \Delta \eta][B_0^T + \Delta \eta^T B_1^T] \} \phi_0 + \Delta \phi \{ A_0 + P^T \Delta \eta \} \]  
\[ (31) \]

Also Equation 11 can be computed as bellow:

\[ I_2 = \frac{1}{2} \sum_{se} (\eta)^T F(\eta) \]  
\[ (32) \]

\[ I_2 = \frac{1}{2} \sum_{se} (\eta_0 + \Delta \eta)^T F(\eta_0 + \Delta \eta) \]  
\[ (33) \]

\[ F = \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]  
\[ (34) \]

Where \( l \) represents the horizontal length of the element on the boundary \( S_1 \), for example \( l = x_i - x_k \) in Figure 4. Since the isoperimetric functions are used for all variables over the boundary, for the variational term \( I_3 \) it can be shown that:

\[ \frac{\partial \phi}{\partial t} = \frac{1}{2} (1-r) (\frac{\partial \phi}{\partial t})_i + \frac{1}{2} (1+r) (\frac{\partial \phi}{\partial t})_k \]  
\[ (35) \]

\[ I_3 = \sum_{se} \int_{-1}^{+1} \{ \eta \}^T \left[ \begin{array}{c} \frac{1-r}{2} \\ \frac{1+r}{2} \end{array} \right] \left[ \begin{array}{c} \frac{1-r}{2} \\ \frac{1+r}{2} \end{array} \right] \{ \frac{\partial \phi}{\partial t} \} \]  
\[ (36) \]

\[ I_3 = \sum_{se} \{ \eta \}^T F \left[ \frac{\partial \phi}{\partial t} \right] \]  
\[ (37) \]

\[ I_3 = \sum_{se} \{ \eta_0 + \Delta \eta \}^T F \left[ \frac{\partial \phi}{\partial t} \right] \]  
\[ (38) \]

Also the parameters \( \frac{\partial \eta}{\partial t} \) and \( \phi \) over the boundary may be shown as:

\[ \frac{\partial \eta}{\partial t} = \frac{1}{2} (1-r) (\frac{\partial \eta}{\partial t})_i + \frac{1}{2} (1+r) (\frac{\partial \eta}{\partial t})_k \]  
\[ (39) \]

\[ \phi = \frac{1}{2} (1-r) \phi_i + \frac{1}{2} (1+r) \phi_k \]  
\[ (40) \]

Then:

\[ I_4 = -\sum_{se} \int_{-1}^{+1} \{ \phi \}^T \left[ \begin{array}{c} \frac{1-r}{2} \\ \frac{1+r}{2} \end{array} \right] \left[ \begin{array}{c} \frac{1-r}{2} \\ \frac{1+r}{2} \end{array} \right] \{ \frac{\partial \eta}{\partial t} \} \]  
\[ (41) \]

\[ I_4 = \sum_{se} \{ \phi_0 + \Delta \phi \}^T F \left[ \frac{\partial \eta}{\partial t} \right] \]  
\[ (42) \]

\[ I_4 = \sum_{se} \{ \phi_0 + \Delta \phi \}^T F \left[ \frac{\partial \phi}{\partial t} \right] \]  
\[ (43) \]

Finally, the variational expression has the form of:

\[ x = \frac{1}{2} \sum_{e} (\phi_0 + \Delta \phi)^T \{ [A_0 + A_1 \Delta \eta][A_0^T + \Delta \eta^T A_1^T] + [B_0 + B_1 \Delta \eta][B_0^T + \Delta \eta^T B_1^T] \} \phi_0 + \Delta \phi \} \{ A_0 + P^T \Delta \eta \} + \frac{1}{2} g \sum_{se} (\eta_0 + \Delta \eta)^T F \{ \eta_0 + \Delta \eta \} + \sum_{se} (\eta_0 + \Delta \eta)^T F \left[ \frac{\partial \phi}{\partial t} \right] \]

\[ - \sum_{se} \{ \phi_0 + \Delta \phi \}^T F \left[ \frac{\partial \phi}{\partial t} \right] \]  
\[ (44) \]

Having \( \Delta \phi \) and \( \Delta \eta \) and the variables, the variation of above expression with respect to the variables gives:

\[ \delta \chi = \frac{1}{2} \sum_{e} \delta \Delta \phi^T \{ (A_0 A_0^T + B_0 B_0^T) \phi_0 A_0 + \}

\[ (A_0^T \phi_0 I + A_0 \phi_0^T) A_1 + (B_0^T \phi_0 I + B_0 \phi_0^T) B_1 \} \Delta \phi \delta \eta \]  

\[ (45) \]

**Figure 4.** Variation of \( \eta \) with respect to \( x \).
\[ + (A_0 A_0^T + B_0 B_0^T) \Delta \phi \delta_0 + \\
\frac{1}{2} \sum \Delta \eta T P \phi_0 (A_0 A_0^T + B_0 B_0^T) \phi_0 \\
2 \delta_0^T (A_0^T A_1 + B_0^T B_1) \Delta \eta + \\
2 \phi_0^T (A_0^T A_0 + B_0^T B_0) \Delta \phi \\
+ g \sum \delta \Delta \eta^T (F \eta_0 + F \Delta \eta) + \\
\sum \delta \Delta \eta^T F \left[ \frac{\partial \phi}{\partial t} \right] - \sum \delta \Delta \phi^T F \left[ \frac{\partial \eta}{\partial t} \right] \]  

In which I indicates the unique matrix. Now Equation 45 leaves a system of linear equations which are really the stability conditions for \( x \) with respect to \( \Delta \phi \) and \( \Delta \eta \), i.e.

\[ \begin{align*}
\frac{\delta x}{\delta \Delta \phi} &= 0 \\
\frac{\delta x}{\delta \Delta \eta} &= 0
\end{align*} \]  

Also \( \frac{\partial \phi}{\partial t} \) and \( \frac{\partial \eta}{\partial t} \) can be assumed to vary as linear functions of time during the time step \( \Delta t \). Therefore, where the average slope of beginning and end points of each time step is defined as \( \frac{\Delta \phi}{\Delta t} \), it can be shown that:

\[ \frac{\partial \phi}{\partial t} = \frac{2}{\Delta t} \Delta \phi - \left( \frac{\partial \phi}{\partial t} \right)_0 \]  

Replacing these equations in 45 and satisfying the stability conditions 46 and 47, leads to:

\[ [K] \{ \Delta u \} = [R] \]  

\[ \begin{bmatrix} G_{11} & G_{12} \\
G_{21} & G_{22} \end{bmatrix} [\Delta \phi] = [R_1] \]  

In which:

\[ G_{11} = (A_0 A_0^T + B_0 B_0^T) \Delta_0 \\
G_{12} = (A_0^T \phi_0 I + A_0 \phi_0^T) A_1 + (B_0^T \phi_0 I + B_0 \phi_0^T) B_1 \Delta_0 \\
G_{21} = P \phi_0^T (A_0^T A_0 + B_0^T B_0) \phi_0 + \frac{2}{\Delta t} F \\
G_{22} = P \phi_0^T (A_0^T A_1 + B_0^T B_1) + g F \\
R_1 = -(A_0 A_0^T + B_0 B_0^T) \phi_0 \Delta_0 - F \left[ \frac{\partial \eta}{\partial t} \right]_0 \\
R_2 = -\frac{1}{2} P \phi_0^T (A_0 A_0^T + B_0 B_0^T) \phi_0 - g F \phi_0 \left[ \frac{\partial \phi}{\partial t} \right]_0
\]

Solving Equation 51 and replacing \( \Delta \phi \) and \( \Delta \eta \) in 17, the values of \( \phi \) and \( \eta \) can be obtained for that time step. The same Procedure may be repeated for achieving the dynamic response of the fluid.

### 2.3. Initial Values

The statement 52 includes Figure 5. The situation of open boundary.

![Figure 5](image-url)
the initial values of $\phi_0$, $\eta_0$, $(\frac{\partial \phi}{\partial t})_0$ and $(\frac{\partial \eta}{\partial t})_0$. To define the initial values for the program, it is assumed that the wave comes near the wall as shown in Figure 5. It is assumed that at the spacing $b$ from the wall, the disturbance caused by the wave is eliminated and the wave has a fixed shape. Therefore it can be assumed that in each time step the values of $\phi$ and $\eta$ over the boundary $S_4$ are constant $\Delta \phi$ and $\Delta \eta$ are zero.

At the first step of program execution, the values of $\phi_0$ and $\eta_0$ are considered to be zero, except for the boundary $S_4$, on which the above variables are constant for each time step. To evaluate these values, the specifications of a short-crested wave are considered. As it is shown in Figure 5, just at the beginning of execution, $\phi_0$ and $\eta_0$ have the values of:

$$\phi_0 = \begin{bmatrix} 0 \\ \phi_n \\ \phi_{n+1} \\ \phi_{n+2} \\ \ldots \end{bmatrix} \quad \text{at } t = 0$$

(53)

$$\eta_0 = \begin{bmatrix} 0 \\ \eta_n \\ \eta_{n+1} \\ \eta_{n+2} \\ \ldots \end{bmatrix} \quad \text{at } t = 0$$

(54)

And in a desirable time $t_0$ followed by $t_0 + \Delta t$:

$$\phi_0^{t_0 + \Delta t} = \begin{bmatrix} \ldots \\ \phi_0^{t_0 + \Delta t} \\ \phi_{n+1}^{t_0 + \Delta t} \\ \phi_{n+2}^{t_0 + \Delta t} \\ \ldots \end{bmatrix} \quad \text{at } t = t_0 + \Delta t$$

(57)

$$\eta_0^{t_0 + \Delta t} = \begin{bmatrix} \ldots \\ \eta_0^{t_0 + \Delta t} \\ \eta_{n+1}^{t_0 + \Delta t} \\ \eta_{n+2}^{t_0 + \Delta t} \\ \ldots \end{bmatrix} \quad \text{at } t = t_0 + \Delta t$$

(58)

Where dots indicate nonzero values.

Specifying the values of $\phi_0$ and $\eta_0$ for each time step, the relevant values of $(\frac{\partial \phi}{\partial t})_0$ and $(\frac{\partial \eta}{\partial t})_0$ may be simply extracted from dynamic/kinematic boundary conditions. Considering the Bernoulli equation we have:

$$\frac{\partial \phi}{\partial t} = \frac{-1}{2} \left[ (\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 \right] - gy$$

(59)

$$\frac{\partial \phi}{\partial t} = \frac{-1}{2} \left[ \frac{b^2}{(A_0 T \phi_0)(\phi_0 T A_0)} + \frac{c^2}{(B_0 T \phi_0)(\phi_0 T B_0)} \right] \quad \text{at } t = t_0$$

(60)

In which:

$$\phi_0 = \phi_{0 \text{element}}, \quad \eta_0 = \eta_{0 \text{element}}$$

Applying the kinematic boundary condition together with Figure 4, it can be concluded that:

$$\frac{\partial \eta}{\partial x} = \frac{\eta_i - \eta_j}{TE} = K$$

(61)

Therefore for each boundary piece:

$$\frac{\partial \eta}{\partial x} = bK - c = \text{const}$$

(62)
Also as it was mentioned, $\Delta \phi$ and $\Delta \eta$ give zero values over that boundary. To employ this condition, the program considers some zero elements in $\{\Delta u\}$ and in the force vector. On the other hand, in the matrix $[K]$, the relevant values for the nodes locating on the boundary are initially set to zero, then the diagonal elements give the value of unity.

### 3. ITERATION PROCESS

Figure 6 shows the process of iterations. The superscripts indicate that the relevant values belong to the boundary $S_4$.

Specifying $\{u_o\} = \begin{bmatrix} \phi_0 \\ \eta_0 \end{bmatrix}$, the stiffness matrix and the force vector can be computed using 52, after which $\{u_1\}$ may be computed with $\{u_o\}$. Owen, et al [18] have shown that the solution converges when the following condition is satisfied:

$$RCON \leq \frac{\sum_{i=1}^{N} (u_i(t')^2 - u_i(t)^2)}{\sum_{i=1}^{N} (u_i(l)^2)} \times 100$$  \hspace{1cm} (63)

In application, the parameter $RCON$ usually gives the values of unity. After the solution converge for a certain time step, the program considers $\{u_r\}$, which is the solution of that step, as the initial value for the next step.

#### 3.1. Forces and Moments

After the velocity potentials are computed for nodal points, the pressure may be computed for the nodes locating in contact with the wall, applying the Bernoulli equation as follows:

$$p = -gz - \frac{\partial \phi}{\partial t} - \frac{1}{2} \left( \frac{\partial \phi}{\partial x}^2 + \frac{\partial \phi}{\partial y}^2 \right)$$  \hspace{1cm} (64)

$$p = -gz - \frac{2}{\Delta t} \Delta \phi - (\frac{\partial \phi}{\partial x} \phi_0)$$

$$\frac{1}{2} \left( \frac{b^2}{(A_0 T \phi_0)T_A + (B_0 T \phi_0)T_B} \right)$$  \hspace{1cm} (65)

Finally, the forces and moments can be computed integration:

$$F = \int_{-h}^{h} p(x, y, t) dy$$  \hspace{1cm} (66)

$$M = \int_{-h}^{h} (h + y)p(x, y, t) dy$$  \hspace{1cm} (67)

### 4. DISCUSSION AND EVALUATION OF RESULTS

In this section, several examples are solved by the program. The parameters required before the execution are:

- $a =$ The wave amplitude (m)
- $h =$ The depth of still water (m)
- $T =$ The wave period (s)
- $DT =$ The time increment (s)
- $TE =$ The base length of rectangular element (m)

#### 4.1. The Effect of Wave Amplitude

As the first set of examples, the values of forces and moments on the wall are computed for 12 distinct cases, in which $h = 5$, $T = 8$, $DT = 0.1$ and $TE = 1$. A summary of outputs are compared with those of Sainflou's Formula modified by Miche-
Rundgren (MR, S, [4]) (Table 1).

The variation of forces and moments with time is shown in Figures 7 to 9.

Finally, outputs are compared with the existing data in Figures 10 and 11. The results show more agreement with (MR and S) in comparison with the outputs of Nagai. Note that (MR and S) have used higher order methods.

The following results can be obtained from the previous figures:

- As a confirmation of the program, the lower the wave amplitude, the lower differences observed between the forces (developed by the wave) and relevant hydrostatic values, as well as moments and hydrostatic ones.
- The maximum values show much variance with respect to the hydrostatic one. This is due to the first and second powers of depth, which are arisen during integration of forces and moments respectively.
- The force and moments increase due to an increase in wave amplitude.

4.2. The Effect of Wave Period

The Parameters assumed for the second set of examples are: \( h = 15 \), \( a = 2 \), \( DT = 0.1 \), \( TE = 1 \) and \( H/h = 0.267 \). A summary of outputs are illustrated in Table 2.

Nagai’s outputs are approximate results, which are attained comparing the experimental data with the nearest relevant theoretical ones. The results obtained near the wave-breaking zone are not valid. Also the outputs of the program at such zones show relatively high peaks as shown in Figures 12 and 13 for \( T = 3 \).

Therefore performing enough examples, a design curve may be extracted as the envelope of this breaking zone (which occurs about \( T = 4 \) in this example). After the breaking-zone the outputs show more reliability and well agreement with (MR and S) results (with about 0.5 % error), while Nagai’s results show about 20 % deviation with this outputs (Figure 5). The figure states; as the period increases, the forces increase, directly.

4.3. The Sensitivity of Outputs to Time Increment

A sensitivity analysis is carried out to show the effect of time increments on the maximum forces and moments. For all of 15 examples here, \( h = 5 \), \( T = 1 \), \( a = 1 \) and \( TE = 1 \). A summary of outputs is presented in Table 3. To attain the generality of the problem, another set of examples is solved assuming \( h = 5 \), \( T = 2 \), \( a = 1.25 \) and \( TE = 1 \).

From these two examples, the following results can be obtained:

<table>
<thead>
<tr>
<th>( a_i )</th>
<th>( \frac{H_i}{h_i} )</th>
<th>( \frac{H_i}{2gT^2} )</th>
<th>( F_{total} )</th>
<th>( M_{total} )</th>
<th>( F_{total} )</th>
<th>( M_{total} )</th>
<th>( F_{total} )</th>
<th>( M_{total} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.02</td>
<td>1.59 E-4</td>
<td>127.5</td>
<td>216.4</td>
<td>127.7</td>
<td>213.0</td>
<td>216.4</td>
<td>213.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.04</td>
<td>3.18 E-4</td>
<td>132.5</td>
<td>228.7</td>
<td>133.1</td>
<td>224.2</td>
<td>133.1</td>
<td>224.2</td>
</tr>
<tr>
<td>0.2</td>
<td>0.08</td>
<td>6.37 E-4</td>
<td>140.1</td>
<td>253.2</td>
<td>144.4</td>
<td>242.0</td>
<td>144.4</td>
<td>242.0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.12</td>
<td>9.56 E-4</td>
<td>149.4</td>
<td>276.2</td>
<td>156.3</td>
<td>264.3</td>
<td>156.3</td>
<td>264.3</td>
</tr>
<tr>
<td>0.4</td>
<td>0.16</td>
<td>1.27 E-3</td>
<td>157.5</td>
<td>299.7</td>
<td>163.6</td>
<td>290.1</td>
<td>163.6</td>
<td>290.1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.20</td>
<td>1.59 E-3</td>
<td>172.2</td>
<td>357.3</td>
<td>182.1</td>
<td>320.0</td>
<td>182.1</td>
<td>320.0</td>
</tr>
<tr>
<td>0.6</td>
<td>0.24</td>
<td>1.91 E-3</td>
<td>185.2</td>
<td>394.0</td>
<td>196.0</td>
<td>354.4</td>
<td>196.0</td>
<td>354.4</td>
</tr>
<tr>
<td>0.7</td>
<td>0.28</td>
<td>2.23 E-3</td>
<td>97.72</td>
<td>432.0</td>
<td>210.5</td>
<td>395.3</td>
<td>210.5</td>
<td>395.3</td>
</tr>
<tr>
<td>0.8</td>
<td>0.32</td>
<td>2.55 E-3</td>
<td>213.1</td>
<td>465.1</td>
<td>220.8</td>
<td>442.9</td>
<td>220.8</td>
<td>442.9</td>
</tr>
<tr>
<td>0.9</td>
<td>0.36</td>
<td>2.87 E-3</td>
<td>225.2</td>
<td>534.9</td>
<td>241.7</td>
<td>498.5</td>
<td>241.7</td>
<td>498.5</td>
</tr>
<tr>
<td>1.0</td>
<td>0.40</td>
<td>3.18 E-3</td>
<td>240.2</td>
<td>591.3</td>
<td>258.3</td>
<td>568.5</td>
<td>258.3</td>
<td>568.5</td>
</tr>
<tr>
<td>1.1</td>
<td>0.44</td>
<td>3.51 E-3</td>
<td>254.8</td>
<td>651.3</td>
<td>275.6</td>
<td>643.3</td>
<td>275.6</td>
<td>643.3</td>
</tr>
</tbody>
</table>
The time increment should be chosen such that the value $T/\Delta t$ falls below 5.

The moments are much sensitive than forces.

### 4.4. The Effect of Water Depth

Here a set of examples is solved for various depth conditions in which $T = 5$, $a = 5$ and $DR = 0.1$. Let’s define the relative force and moment as:

$$F_{rel} = \frac{F_{wave}}{\gamma h^2}$$  \hspace{1cm} (68)
### TABLE 2. The Force and Moments for Various Wave Periods.

<table>
<thead>
<tr>
<th>T</th>
<th>( \frac{H_i}{gT^2} )</th>
<th>( \frac{h}{L} )</th>
<th>( F_{\text{total}} )</th>
<th>( M_{\text{total}} )</th>
<th>( F_{\text{wave}} )</th>
<th>( M_{\text{wave}} )</th>
<th>( F_{\text{total}} )</th>
<th>( M_{\text{total}} )</th>
<th>( F_{\text{total}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.045</td>
<td>1.07</td>
<td>1554</td>
<td>13470</td>
<td>451.6</td>
<td>7958</td>
<td>-</td>
<td>-</td>
<td>1111.8</td>
</tr>
<tr>
<td>4</td>
<td>0.0255</td>
<td>0.601</td>
<td>1352</td>
<td>9030</td>
<td>250.1</td>
<td>3518</td>
<td>-</td>
<td>-</td>
<td>1179.8</td>
</tr>
<tr>
<td>5</td>
<td>0.0163</td>
<td>0.390</td>
<td>1372</td>
<td>8703</td>
<td>269.5</td>
<td>3191</td>
<td>-</td>
<td>-</td>
<td>1263.1</td>
</tr>
<tr>
<td>6</td>
<td>0.113</td>
<td>0.282</td>
<td>1420</td>
<td>8883</td>
<td>317.7</td>
<td>3371</td>
<td>1395</td>
<td>8882.1</td>
<td>1493.6</td>
</tr>
<tr>
<td>7</td>
<td>0.0083</td>
<td>0.222</td>
<td>1464</td>
<td>9094</td>
<td>365.6</td>
<td>3581</td>
<td>1462.3</td>
<td>9091.5</td>
<td>1553.9</td>
</tr>
<tr>
<td>8</td>
<td>0.0064</td>
<td>0.183</td>
<td>1500</td>
<td>9283</td>
<td>402.5</td>
<td>3770</td>
<td>1487.7</td>
<td>9282.0</td>
<td>1598.6</td>
</tr>
<tr>
<td>9</td>
<td>0.0050</td>
<td>0.157</td>
<td>1530</td>
<td>9461</td>
<td>432.0</td>
<td>3949</td>
<td>1529.4</td>
<td>9459.5</td>
<td>1631.5</td>
</tr>
<tr>
<td>10</td>
<td>0.0041</td>
<td>0.137</td>
<td>1557</td>
<td>9627</td>
<td>460.0</td>
<td>4115</td>
<td>1555.2</td>
<td>9625.2</td>
<td>1656.1</td>
</tr>
</tbody>
</table>

**Figure 12.** The variation of force with time for different periods.

**Figure 13.** The variation of moment with time for different periods.
A summary of outputs for forces and moments is illustrated in Figures 14 to 18. The distribution of wave pressure for different depths is shown in Figures 19 and 20. Values of forces and moments for different depth conditions are presented in Table 4.

These figures show that, as the water depth increases, the forces and moments increase as well. While the values of relative force/moment decrease. Figures 18 to 20 also indicate that the wave pressures change adversely as the water depth varies (note that the resultant force increases as the water depth increases).

**4.5. Design Curves (for $H_i/h_i = 0.4$)** Here a set of examples is solved in order to acquire a dimensionless design curve such a design curve

\[
M_{\text{rel}} = M_{\text{wave}} / \gamma h^3
\]  

(69)

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$\frac{T}{\Delta t}$</th>
<th>Force Error</th>
<th>Moment Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.67</td>
<td>28.68</td>
<td>43.64</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
<td>0.1254</td>
<td>0.2159</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>0.1254</td>
<td>0.2159</td>
</tr>
<tr>
<td>0.9</td>
<td>5.56</td>
<td>9.067</td>
<td>15.08</td>
</tr>
<tr>
<td>0.8</td>
<td>6.25</td>
<td>0.1254</td>
<td>0.2159</td>
</tr>
<tr>
<td>0.7</td>
<td>7.14</td>
<td>1.395</td>
<td>2.394</td>
</tr>
<tr>
<td>0.6</td>
<td>8.33</td>
<td>1.395</td>
<td>2.394</td>
</tr>
<tr>
<td>0.5</td>
<td>10</td>
<td>0.1254</td>
<td>0.2159</td>
</tr>
<tr>
<td>0.4</td>
<td>12.5</td>
<td>0.1254</td>
<td>0.2159</td>
</tr>
<tr>
<td>0.3</td>
<td>16.67</td>
<td>1.183</td>
<td>2.031</td>
</tr>
<tr>
<td>0.2</td>
<td>25</td>
<td>0.1254</td>
<td>0.2159</td>
</tr>
<tr>
<td>0.1</td>
<td>50</td>
<td>0.1254</td>
<td>0.2159</td>
</tr>
<tr>
<td>0.08</td>
<td>62.5</td>
<td>0.06328</td>
<td>0.1092</td>
</tr>
<tr>
<td>0.06</td>
<td>83.33</td>
<td>0.04128</td>
<td>0.7111</td>
</tr>
<tr>
<td>0.05</td>
<td>100</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
The parameters assumed here are: h = 4, a = 0.8, DT = 0.1 and TE = 1.

5. SUMMARY AND CONCLUSION

The outputs obtained from the model shows that the effects of wave amplitude in the present study are in agreement with the wave behavior. When the wave amplitude is relatively small, as the force history passes the hydrostatic point, no distinct local maximum/minimum point can be observed. In case of higher amplitudes, however, a distinct local maximum/minimum point can be observed just when the curve passes the hydrostatic Point. Therefore the theories of limited amplitudes, rule such problems where the profiles come out the sinusoidal shapes. In some cases a small depression may be developed at the wave crest, as well as a small knob at the perigee. The outputs (forces and moments) show about 6 % difference with Nagai's experimental data [1].

The results obtained for the effect of period agrees with data obtained by Nagai [1], with about 4 percent difference. The breaking-zone and the envelope of this zone can simply be discretized in the figures.

The moment's sensitivity to time increments is more than the relevant forces. Where Δt < 0.2 s, the computational error is negligible.
TABLE 4. Values of Forces and Moments for Different Depth Conditions.

<table>
<thead>
<tr>
<th>Depth</th>
<th>$F_{\text{total}}$</th>
<th>$M_{\text{total}}$</th>
<th>$F_{\text{wave}}$</th>
<th>$M_{\text{wave}}$</th>
<th>$F_{\text{rel}}$</th>
<th>$M_{\text{rel}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>154.30</td>
<td>319.68</td>
<td>75.898</td>
<td>215.14</td>
<td>0.48404</td>
<td>0.34302</td>
</tr>
<tr>
<td>5</td>
<td>200.99</td>
<td>468.71</td>
<td>78.487</td>
<td>264.54</td>
<td>0.32036</td>
<td>0.21595</td>
</tr>
<tr>
<td>6</td>
<td>259.50</td>
<td>682.26</td>
<td>83.100</td>
<td>329.46</td>
<td>0.23554</td>
<td>0.15564</td>
</tr>
<tr>
<td>7</td>
<td>327.55</td>
<td>961.90</td>
<td>87.449</td>
<td>401.66</td>
<td>0.18211</td>
<td>0.11949</td>
</tr>
<tr>
<td>8</td>
<td>405.12</td>
<td>1317.2</td>
<td>91.522</td>
<td>480.90</td>
<td>0.14592</td>
<td>0.09584</td>
</tr>
<tr>
<td>9</td>
<td>429.07</td>
<td>1756.6</td>
<td>95.166</td>
<td>656.91</td>
<td>0.11989</td>
<td>0.07921</td>
</tr>
<tr>
<td>10</td>
<td>588.15</td>
<td>2287.7</td>
<td>98.153</td>
<td>654.34</td>
<td>0.10016</td>
<td>0.06677</td>
</tr>
<tr>
<td>11</td>
<td>693.45</td>
<td>2919.5</td>
<td>100.55</td>
<td>755.12</td>
<td>0.08480</td>
<td>0.05716</td>
</tr>
</tbody>
</table>

Figure 21. Design curves for $H_i/h_i = 0.4$.

TABLE 5. Dimensionless Values of Forces and Moments for $H_i/h_i = 0.4$.

<table>
<thead>
<tr>
<th>PERIOD</th>
<th>$H_i/gT^2$</th>
<th>MAX RTL-FRC</th>
<th>MAX RTL-MOM</th>
<th>MIN RTL-FRC</th>
<th>MIN RTL-MOM</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.010194</td>
<td>0.28894</td>
<td>0.19971</td>
<td>-0.18326</td>
<td>-0.08737</td>
</tr>
<tr>
<td>5</td>
<td>0.006524</td>
<td>0.34051</td>
<td>0.22391</td>
<td>-0.22265</td>
<td>-0.10056</td>
</tr>
<tr>
<td>6</td>
<td>0.004531</td>
<td>0.38908</td>
<td>0.25098</td>
<td>-0.25411</td>
<td>-0.11090</td>
</tr>
<tr>
<td>7</td>
<td>0.003329</td>
<td>0.44310</td>
<td>0.28508</td>
<td>-0.28367</td>
<td>-0.12035</td>
</tr>
<tr>
<td>8</td>
<td>0.002548</td>
<td>0.50466</td>
<td>0.32697</td>
<td>-0.31287</td>
<td>-0.12926</td>
</tr>
<tr>
<td>9</td>
<td>0.002014</td>
<td>0.57594</td>
<td>0.37827</td>
<td>-0.34232</td>
<td>-0.13765</td>
</tr>
<tr>
<td>10</td>
<td>0.001631</td>
<td>0.65839</td>
<td>0.44054</td>
<td>-0.37173</td>
<td>-0.14535</td>
</tr>
</tbody>
</table>
The outputs of the last set of examples indicate that as the water depth increases, the effect of wave on the lower part of the wall decreases and by comparing the experimental data, in a short, the Program or the numerical model can be used to study the effect of wave under various conditions.

**Appendix I.** The flow chart of the finite element program is shown in Figure 22.

6. REFERENCES


![Figure 22](image-url)


