RELIABILITY OF REPAIRABLE MULTICOMPONENT REDUNDANT SYSTEM

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Abstract This investigation deals with transient analysis of cold standby system with n units. Chapman-Kolmogorov equations are developed for repair facility with two repairmen and solved by using matrix technique. Availability and reliability factors have been obtained with probability of the system. A particular case has also been discussed.

Key Words Transient, Standby, Availability, Reliability

INTRODUCTION

In the modern age of technology, a system is of no use if it is not reliable. However, the importance and effect of reliability could not be realized without mathematical modelling. The mathematical theory of reliability has grown out of the demands of modern technology. In general, the reliability can be defined as the freedom from failure of a component or system while maintaining a specific problem such as inventory, industrial or machine repair problem. Reliability technique is of immense importance for performance evaluation and accurate measurement of computer, communication, manufacturing systems, planning and designing of the components in electronic equipments etc. Stochastic models are becoming increasingly important for understanding or making a performance evaluation of queueing problems of real life situations. The redundant system unreliability has been developed by several researchers. Some of them are Harvill and Pipes [1] and Gopalan and Natesan [2]. These researchers have developed the techniques for stochastic behavior of one server with n units in the system.

In this paper, we consider cost analysis for such system with one additional repairman. The arrival (failure) rate and service rate are independently negative exponentially distributed with FCFS discipline. We find the expressions for steady state characteristics including stationary availability and interval reliability.

THE MATHEMATICAL MODEL

In this paper, we consider the system with n
dissimilar units; in which one channel is in operating condition and others are as standby Multi server facility always available with the system if more than one failed machine enter in the service system. The life times and repair times of all the units are independently distributed negative exponential variates.

The main notations used in the investigation are as follows:

- $l_i$: Life time rate of units, $i=1,2,..., n$
- $m_i$: Repair time rate of units, $i=1,2,..., n$
- $X(t)$: Number of the operating units at time $t$. $X(t) = i$ if the $i$th unit is operating, given that initially unit $j$ was operative.
- $P_i(t)$: Probability that the $i$th unit is operating, given that initially unit $j$ was operative.
- $f(t)$: Laplace transform of the function $f(t)$
- $M + S$: 
- $M_i = \sum_{j=1}^{i} m_j + \sum_{i=1}^{n} m_i$
- $S_i = \sum_{j=1}^{i} l_j + \sum_{i=1}^{n} l_i$

### THE BALANCE EQUATIONS AND THE SOLUTION METHOD

We consider $[X(t), t \in [0, \infty))$ as a birth and death Markov process with state space. The balance equations in set of Chapman-Kalmogorov processes for $P_i(t)$ and $Q_i(t)$ are as follows:

\[
\frac{d}{dt} P_i(t) = -(l_i + m_{i-1}) P_i(t) + m_{i-1} P_{i-1}(t) + l_{i+1} P_{i+1}(t) \quad i = 0, 1, 2, ..., n
\]

\[
\frac{d}{dt} P_0(t) = -m_0 P_0(t) + l_1 P_1(t)
\]

\[
\frac{d}{dt} Q_i(t) = -(l_{i+1} + m_i) Q_i(t) + m_i Q_{i-1}(t) + l_{i-1} Q_{i+1}(t) \quad i = 1, 2, ..., n
\]

\[
\frac{d}{dt} Q_n(t) = -(l_n + m_n) Q_n(t) + m_n Q_{n-1}(t)
\]

For stochastic process $\{X(t), t \in [0, \infty)\}$, the stationary distribution $\{P_i\}$ is the limiting solution of (1) as $t \to \infty$ with derivatives being replaced by zero.

\[
m_0 P_0 + l_1 P_1 = 0
\]

\[
(l_i + m_{i-1}) P_i + m_{i-1} P_{i-1} + l_{i+1} P_{i+1} = 0
\]

\[
(l_i + 2m_i) P_i + 2m_{i-1} P_{i-1} + l_{i+1} P_{i+1} = 0
\]

\[
l_n P_n + 2m_n P_{n-1} = 0
\]

\[
l_i P_i = m_i P_{i-1} \quad i = 0, 1, 2, ..., n
\]

Thus

\[
P_i = P_j P \left( \frac{m_j}{l_j} \right)
\]

where

\[
P_0 = \left[ 1 - \sum_{i=1}^{n} \left( \frac{m_i}{l_i} \right) \right]^{-1}
\]

We consider pointwise system availability to be

\[
A(t) = 1 - P_0(t)
\]

and Reliability of the system would be

\[
R(t) = \sum_{i=1}^{n} q_i(t)
\]

Mean time to system failure is

\[
E(T) = \int_0^\infty R(x) dx = \sum_{i=1}^{n} \int_0^\infty q_i(x) dx = \sum_{i=1}^{n} q_i(0)
\]

The probability of busy repairman at $t$ is

\[
B(t) = 1 - P_n(t)
\]

The expected fraction of time for under repair system is

\[
\lim_{t \to \infty} B(t) = 1 - P_n
\]
After taking Laplace transformation of Equations 1 and 2 with matrix notation, we get

\[ A(s) = \begin{pmatrix} A_i(s) & \cdots & A_n(s) \end{pmatrix} \]

and

\[ B(s) = \begin{pmatrix} B_i(s) & \cdots & B_n(s) \end{pmatrix} \]

Using Cramer’s rule, equations (11) and (12) give

\[ p_i(s) = \hat{U}_A(s) \hat{U} \hat{U}_A(s) \hat{U}_0 \times i \times n \quad (14.1) \]

and

\[ q_j(s) = \hat{U}_B(s) \hat{U} \hat{U}_B(s) \hat{U}_1 \times i \times n \quad (14.2) \]

We can find \( A_i(s) \) and \( B_i(s) \) from \( A(s) \) and \( B(s) \) by replacing ith column by unit vector in RII's of 11 and 12.

Now we apply elementary row and column transformations on \( \hat{U}_A(s) \hat{U} \) and \( \hat{U}_B(s) \hat{U} \) we get

\[ \hat{U}_A(s) \hat{U} = s \hat{U} f(s) \hat{U} \quad (15.1) \]

and

\[ \hat{U}_B(s) \hat{U} = \hat{U} y(s) \hat{U} \quad (15.2) \]

By using the eigen values of tridiagonal positive definite matrices from the method developed by Gregory and Young [5] where the zeros of polynomials \( \hat{U} f(s) \hat{U} \) and \( \hat{U} y(s) \hat{U} \) are the negatives of eigen values of matrices \( a(0) \) and \( b(0) \).

\[ \hat{U} f(n) \hat{U} \text{ and } \hat{U} y(n) \hat{U} \text{ at } f_1, f_2, \ldots, f_n \text{ and } y_1, y_2, \ldots, y_n \text{ are the zeros of } \hat{U} f(n) \hat{U} \text{ and } \hat{U} y(n) \hat{U} \]

The partial fractions of 17.1 and 17.2 give

\[ p_j(s) = a_{0j} + \sum_{k=1}^{n} a_{kj} / (s - f_k) \quad 0 \times i \times n \quad (18.1) \]

and

\[ q_j(s) = \sum_{k=1}^{n} b_{kj} / (s - y_k) \quad 1 \times i \times n \quad (18.2) \]

with

\[ a_{0j} = \hat{U} A_i(0) \hat{U} P_{j=1}^{n} f_k \quad (19.1) \]
\[ a_{ki} = \hat{U}_{A_i}(f_k)\hat{U} f_k^n P (f_k - f_j) \]  
\[ b_{ki} = \hat{U}_{B_i}(y_k)\hat{U} y_k^n P (y_k - y_j) \]  

Inverse Laplace transforms of 18.1 and 18.2 give

\[ p_1(t) = a_{bi} + \sum_{k=1}^{n} a_{ki} e^{f_1 t} \quad 0 \times i \times n \]  
\[ q_i(t) = \sum_{k=1}^{n} b_{ki} e^{y} \quad 1 \times i \times n \]

With the help of these terms we can find steady state characteristics and other desired transient indices of the system.

**A SPECIAL CASE**

For \( n = 3 \), the system becomes

\[ \hat{U}_{A(s)} \hat{U} = \begin{bmatrix} s + m_1 & -1 & 0 & 0 \\ -m_1 & s + l_1 + m_1 & -l_2 & 0 \\ 0 & -m_2 & l_2 + 2m & -l_3 \\ 0 & 0 & -2m & l_3 \end{bmatrix} \]  
\[ \hat{U}_{B(s)} \hat{U} = \begin{bmatrix} s + l_1 + m_1 & -l_2 & 0 \\ -m_1 & s + l_1 + 2m & -l_3 \\ 0 & -2m & s + l_3 \end{bmatrix} \]  

and

\[ \hat{U}_{f(s)} \hat{U} = \begin{bmatrix} s + l_1 + m_1 & -\alpha & \alpha \beta & 0 \\ -\alpha & s + l_1 + m_1 & \alpha \beta & 0 \\ 0 & -\alpha \beta & \alpha \beta & s + l_3 + 2m \end{bmatrix} \]  
\[ \hat{U}_{y(s)} \hat{U} = \begin{bmatrix} s + l_1 & -\alpha & \alpha \beta & 0 \\ -\alpha & s + l_1 + m_1 & \alpha \beta & 0 \\ 0 & -\alpha \beta & \alpha \beta & s + l_3 + 2m \end{bmatrix} \]

We can calculate the factors of 22.1 and 22.2 as \( f_m \) and \( y_m \) (m = 1, 2, 3) with suitable techniques. These factors will be helpful to find

\[ p_1(t) = \frac{m_1 l_1}{f f_1} + \frac{f_1 m_1}{f_1 (f_1, f_2, f_3)} e^{f_1 t} + \]  
\[ \frac{f_1 m_1}{f_1 (f_2, f_3)} e^{f_2 t} + \frac{f_1 m_1}{f_1 (f_3, f_1)} e^{f_3 t} \]  

\[ q_1(t) = \frac{m_1 l_1}{f f_1} + \frac{f_1 m_1}{f_1 (f_1, f_2, f_3)} e^{f_1 t} + \]  
\[ \frac{f_1 m_1}{f_1 (f_2, f_3)} e^{f_2 t} + \frac{f_1 m_1}{f_1 (f_3, f_1)} e^{f_3 t} \]  

This method can be applied for all multiple unit redundant system with \( n \hat{U} 3 \). We can find various characteristics given in Equations 6-10 by using Equations 23-29.

**ILLUSTRATION**

For numerical illustration purpose, we consider

\[ p_1(t) = \frac{m_1 l_1}{f f_1} + \frac{f_1 m_1}{f_1 (f_1, f_2, f_3)} e^{f_1 t} + \]  
\[ \frac{f_1 m_1}{f_1 (f_2, f_3)} e^{f_2 t} + \frac{f_1 m_1}{f_1 (f_3, f_1)} e^{f_3 t} \]  

\[ q_1(t) = \frac{m_1 l_1}{f f_1} + \frac{f_1 m_1}{f_1 (f_1, f_2, f_3)} e^{f_1 t} + \]  
\[ \frac{f_1 m_1}{f_1 (f_2, f_3)} e^{f_2 t} + \frac{f_1 m_1}{f_1 (f_3, f_1)} e^{f_3 t} \]  

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The system characteristics are

\[ q_1(t) = 0.7143 + 0.4 e^{-0.01443t} - 0.1176 e^{-0.04905t} \]

\[ R(t) = 1.1113 e^{-0.00448t} - 0.1113 e^{-0.04472t} \]

\[ B(t) = 0.59143 - 0.36072 e^{-0.01443t} - 0.2343 e^{-0.04905t} \]

For similar units case we take

\[ l_1 = l_2 = 1/80 \text{ and } m_1 = m_2 = 1/50 \]

Now we have

\[ f_1 = -0.0167, \quad f_2 = -0.0483, \quad y_1 = -0.00389, \quad y_2 = -0.0411 \]

So that

\[ p_0(t) = 0.1975 - 0.3019 e^{-0.0167t} + 0.1046 e^{-0.0483t} \]

\[ p_1(t) = 0.3086 - 0.0779 e^{-0.0167t} + 0.2288 e^{-0.0483t} \]

\[ p_2(t) = 0.4938 - 0.3757 e^{-0.0167t} + 0.1287 e^{-0.0483t} \]

\[ q_1(t) = 0.3359 (e^{-0.00389t} - e^{-0.0411t}) \]

\[ q_2(t) = 0.7689 e^{-0.00389t} + 0.2318 e^{-0.0411t} \]

Hence the system characteristics are

\[ A(t) = 0.8025 + 0.3019 e^{-0.0167t} - 0.1046 e^{-0.0483t} \]

\[ R(t) = 1.1048 e^{-0.00389t} - 0.1041 e^{-0.0411t} \]

\[ B(t) = 0.5062 - 0.3757 e^{-0.0167t} - 0.1287 e^{-0.0483t} \]

The reliability and availability curves for the similar and dissimilar units system are shown in Figures 1 and 2 respectively. It can be noticed that reliability and availability decrease as time increases. The transient state probabilities by varying time for the case of dissimilar and similar units are summarized in Tables 1 and 2 respectively.

**CONCLUSION**

In this paper, the transient analysis of cold standby system with n dissimilar units is discussed. The reliability and availability indices for redundant system are obtained by solving balance equations. We have also illustrated numerically the tactability of the approach by taking n = 2 for similar as well as dissimilar units.

6. REFERENCES

For Similar Units

For Dissimilar Units

Figure 1. Reliability for two units system.

For Similar Units

For Dissimilar Units

Figure 2. Availability for two units system.

Table 1. Transition Probabilities for the System with Similar Units.

<table>
<thead>
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<th>$p_1(t)$</th>
<th>$p_2(t)$</th>
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<td>10</td>
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<td>0.6964</td>
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<td>0.2956</td>
<td>0.6443</td>
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</table>

Table 2. Transition Probabilities for the System with Dissimilar Units.

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<th>$p_1(t)$</th>
<th>$p_2(t)$</th>
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