NONRESONANT EXCITATION OF THE FORCED DUFFING EQUATION

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Abstract We investigate the hard nonresonant excitation of the forced Duffing equation with a positive damping parameter $\epsilon$. Using the symbolic manipulation system MACSYMA, a computer algebra system, we derive the two term perturbation expansion by the method of multiple time scales. The resulting approximate solution is valid for small values of the coefficient $\epsilon$. As the damping parameter $\epsilon$ increases, the accuracy of this solution degrades. In order to obtain an improved approximate solution to the given time dependent initial value problem, a hybrid perturbation-Galerkin method is applied to the perturbation solution. The hybrid method is based on Galerkin’s method for determining an approximate solution to a differential equation using the perturbation solutions as trial functions. This hybrid method has the potential of overcoming some of the drawbacks of the perturbation method and the Galerkin method when they are applied separately, while combining some of the good features of both. We compare these two solutions for various values of $\epsilon$ and $\Omega$ (the frequency of the external force) and demonstrate the effectiveness of the hybrid method. Both the perturbation and hybrid solutions are also compared to a fourth order Runge-Kutta solution of the Duffing equation. For small values of $\epsilon$, the hybrid solution is very close to the numerical solution for most values of $\Omega$ while the perturbation solution slightly overestimates the numerical solution. For larger values of $\epsilon$, the perturbation solution deviates from the numerical solution very rapidly while the hybrid method remains close to the numerical model.

Key Words Nonresonant, Excitation, Duffing Equation, Damping Parameter, Perturbation

INTRODUCTION Perturbation solutions have been used successfully in a variety of differential equation type problems ([11], [13], [14], [5], [10], [11], and [13]). In constructing the perturbation solution, the usefulness of computer algebra systems have
been realized and demonstrated by many investigators (see, e.g., [2] and [6]). The complexity of the perturbation solution increases as more terms are required. Although computer algebra systems alleviate some of the complications, it is still impractical to compute the higher order perturbation terms beyond a level. Unfortunately, in some applications, a large number of perturbation terms are required in order to obtain a reasonable approximation to the problem's solution. In such cases, one should try to make as much use as possible of information contained in the few lower order terms. The hybrid perturbation Galerkin method (which we will describe below) appears to increase the power and usefulness of the perturbation solution to a given problem [2], [7], [8].

The hybrid perturbation-Galerkin method is a two step technique, based on Galerkin's method for determining an approximate solution to a differential equation. In step one of the method, a formal perturbation solution of the problem $L(u, \epsilon) = 0$ is constructed, say of the form $u = \sum_{k=0}^{N-1} \epsilon^k u_k + O(\epsilon^N)$ which is formally valid as $\epsilon \to 0$. In step two of the method, a improved approximate solution $\tilde{u}$ is sought in the form $\tilde{u} = \sum_{k=0}^{N-1} \lambda_k u_k$. The new "amplitudes" are obtained by requiring that $L(\tilde{u}, \epsilon)$ is orthogonal to each of the coordinate function $u_j$, i.e.

$$\int L(\tilde{u}, \epsilon) u_j(\epsilon) \, dt = 0, \quad j = 0, 1, \ldots, N-1.$$

Equation 1 is a system of $N$ equations for the $N$ unknown coefficients $\{\lambda_k\}$.

In this paper, we study the hard nonresonant excitation of the forced Duffing equations

$$\ddot{u} + 2\epsilon \dot{u}^3 + \omega_0^2 u + \epsilon \omega_3 u^3 = F \sin \Omega t$$

$$u = 0 \text{ and } \dot{u} = 0 \text{ when } t = 0.$$  

In this equation $\ddot{u}$ and $\dot{u}$ are the first and second derivatives with respect to time, $t$, the parameters $\mu$, $\omega_0$, $\alpha$, $F$, and $\Omega$ are treated as fixed constants, while $\epsilon > 0$ is a positive damping parameter. We first develop a perturbation solution to the initial value problem (2) using the method of multiple time scales. The resulting solution is valid for small values of the perturbation parameter $\epsilon$. As $\epsilon$ increases, the accuracy of the perturbation solution will degrade. In order to improve this solution, we will apply the hybrid perturbation-Galerkin method just outlined.

**HARD NORESONANT EXCITATION**

For this case we assume $F = O(1)$ in Equation 2 and use the multiple time scales method [11] to construct a two term perturbation expansion of $u(t)$. We first define

$$T_0 = t \quad \text{and} \quad T_1 = \epsilon t,$$

and let

$$u(t) = u(T_0, T_1) = u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + O(\epsilon^2)$$  

Then

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + O(\epsilon^2),$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_0 \partial T_1} + O(\epsilon^2)$$

Using (3) - (5), Equation 2 becomes

$$(D_0^3 + 2\epsilon D_0 D_1 + \ldots) u + 2\epsilon^1 (D_0 + \epsilon D_1 + \ldots) u + \omega_0^2 u + \epsilon \omega_3 u^3 = F \sin \Omega t$$

where $D_j = \frac{\partial^j}{\partial T_j^j}$. Substituting (4) for $u$ gives

$$D_0^3 u_0 + \epsilon D_0^2 u_1 + 2\epsilon D_0 D_1 u_0 + 2\epsilon^1 D_1 u_0 + \omega_0^2 u_0 + \epsilon \omega_3 u_0^3 + O(\epsilon^2) = F \sin \Omega t$$

Equating the coefficients of $\epsilon^0$ and $\epsilon^1$ respectively results in the following equations

$$D_0^3 u_0 + \omega_0^2 u_0 = F \sin \Omega T_0$$

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\[ D_0^2 u_t + \alpha^2 u_t = -2 D_0 D_j u_{ij} - 2 \mu H_0 u_t - \alpha u_t^3 \]  \hspace{1cm} (9)

From (8) the solution \( u_t \) is given by

\[ u_t = C(T_0) \cos (\omega T_0 + \Phi(T_0)) + \frac{F}{\alpha^2 - \Omega^2} \sin \omega T_0 \]  \hspace{1cm} (10)

where \( C \) and \( \Phi \) are functions of \( T_0 \) and will be determined by requiring the elimination of the secular terms in higher order terms. Substituting (10) for \( u_t \) into (9), we obtain the following differential equation for \( u_t \):

\[
D_0^2 u_t + \alpha^2 u_t = 2 \omega_0 \left( \frac{dC}{dT_0} + \mu C \right) \sin (\omega T_0 + \Phi) \\
+ 2C \left( \omega_0 \frac{d\Phi}{dT_0} - 3 \alpha \frac{\alpha C^2 - C}{8} \right) \sin (\omega T_0 + \Phi) \\
- 3 \alpha \tilde{F} \left( 2 \tilde{F}^2 + \tilde{C}^2 \right) \sin \Omega T_0 - 4 \mu \Omega \tilde{F} \cos \Omega T_0 \\
+ 2 \alpha \tilde{F}^2 \sin 3 \Omega T_0 - 3 \alpha C \tilde{F}^2 \sin 2 \Omega T_0 \cos (2(\omega T_0 + \Phi)) \\
+ 6 \alpha \tilde{C} \tilde{F}^2 \cos 2 \Omega T_0 \cos (\omega T_0 + \Phi) \\
- \frac{\alpha C^3}{4} \cos (3(\omega T_0 + \Phi)) \]  \hspace{1cm} (11)

where 
\[
\tilde{F} = \frac{1}{2} \frac{F}{\alpha^2 - \Omega^2}
\]

To eliminate the secular terms, we set the coefficients of 
\( \sin (\omega T_0 + \Phi) \) and \( \cos (\omega T_0 + \Phi) \) equal to zero, i.e.,

\[ 2 \omega_0 \left( \frac{dC}{dT_0} + \mu C \right) = 0 \]  \hspace{1cm} (12)

and

\[ 2C \left( \omega_0 \frac{d\Phi}{dT_0} - 3 \alpha \frac{\alpha C^2 - C}{8} \right) = 0 \]  \hspace{1cm} (13)

From (12) we find

\[ C = \tilde{C} e^{\mu T_0} \]  \hspace{1cm} (14)

where \( \tilde{C} \) may be treated as a constant.

Substituting (14) into (13) and solving the differential equation for \( \Phi \), we have

\[ \Phi = -\frac{3 \alpha \tilde{C}^2 - 2 \mu T_0}{16 \mu_0} + \frac{3 \alpha \tilde{F}^2 T_0}{\omega_0} + \tilde{\Phi} \]  \hspace{1cm} (15)

Substituting for \( C \) and \( \Phi \) and using \( T_0 = t \) and \( T_1 = \epsilon t \), the one term approximation \( u_0 \) from (10) can be expressed as

\[ u_0 = \tilde{C} e^{-\mu t} \cos (\omega_0 t - \frac{3 \alpha \tilde{C}^2 e^{-2 \mu t}}{16 \mu_0} + \frac{3 \alpha \tilde{F}^2 T_0}{\omega_0} + \tilde{\Phi}) \]

\[ + 2 \tilde{F} \sin \Omega t \]  \hspace{1cm} (16)

and the first derivative \( \dot{u}_0 \) is

\[ \dot{u}_0 = -\tilde{C} e^{-\mu t} (\alpha + \frac{3 \alpha \tilde{C} e^{-2 \mu t}}{8 \omega_0}) \sin (\omega_0 t - \frac{3 \alpha \tilde{F}^2 T_0}{\omega_0} + \tilde{\Phi}) \]

\[ - \frac{3 \alpha \tilde{C}^2 e^{-2 \mu t} + \frac{3 \alpha \tilde{F}^2}{\omega_0} + \tilde{\Phi} \cdot \epsilon \mu \tilde{C} e^{-\mu t} \cos (\omega_0 t - \frac{3 \alpha \tilde{F}^2 T_0}{\omega_0} + \tilde{\Phi})} {16 \mu_0} \]

\[ + \frac{3 \alpha \tilde{C}^3 e^{-3 \mu t} + \frac{3 \alpha \tilde{F}^3}{\omega_0} + \tilde{\Phi} \cdot \epsilon \mu \tilde{C} e^{-\mu t} \cos (\omega_0 t - \frac{3 \alpha \tilde{F}^2 T_0}{\omega_0} + \tilde{\Phi})} {16 \mu_0} \]  \hspace{1cm} (17)

Applying the initial condition \( u_0(0) = 0 \) to (16) will result in

\[ \tilde{\Phi} = \frac{3 \alpha \tilde{C}^2 \tilde{F}} {16 \mu_0} \]  \hspace{1cm} (18)

This relation expresses \( \Phi \) in terms of \( \tilde{C} \). Similarly, we apply the initial condition \( \dot{u}_0(0) = 0 \) to (17) and obtain

\[ \frac{3 \alpha \tilde{C} \tilde{F}} {8 \omega_0} + \frac{3 \alpha \tilde{C}^2 \tilde{F} \tilde{C} + \omega_0 \tilde{C} + 2 \tilde{F} \Omega = 0} \]  \hspace{1cm} (19)

which is a cubic equation and has three solutions for \( \tilde{C} \).

The one term approximation for \( u \) can now be written as

\[ u = e^{-\mu t} \tilde{C} \left( \alpha e^{-2 \mu t} \sin (\omega_0 t - \frac{3 \alpha \tilde{C}^2 e^{-2 \mu t}}{16 \mu_0} + \frac{3 \alpha \tilde{F}^2 T_0}{\omega_0} + \tilde{\Phi}) \right) \]

\[ + \frac{3 \alpha \tilde{C}^3 e^{-3 \mu t} + \frac{3 \alpha \tilde{F}^3}{\omega_0} + \tilde{\Phi} \cdot \epsilon \mu \tilde{C} e^{-\mu t} \cos (\omega_0 t - \frac{3 \alpha \tilde{F}^2 T_0}{\omega_0} + \tilde{\Phi})} {16 \mu_0} \]  \hspace{1cm} (20)

with \( \tilde{C} \) the solution of (19).

After elimination of secular terms, the differential equation (11) for \( u_t \) becomes

\[ D_0^2 u_t + \alpha^2 u_t = -3 \tilde{C} \tilde{F} (2 \tilde{F}^2 + \tilde{C}^2) \sin \Omega T_0 - 4 \mu \Omega \tilde{F} \cos \Omega T_0 \\
+ 2 \tilde{C} \tilde{F}^2 \sin 3 \Omega T_0 - 3 \alpha C \tilde{F}^2 \sin 2 \Omega T_0 \cos (2(\omega T_0 + \Phi)) \\
+ 6 \alpha \tilde{C} \tilde{F}^2 \cos 2 \Omega T_0 \cos (\omega T_0 + \Phi) - \frac{\alpha C^3}{4} \cos (3(\omega T_0 + \Phi)) \]  \hspace{1cm} (21)

Using the symbolic manipulation system MACSYMA, the particular and homogeneous solutions for \( u_t \) are given.
Differentiating $u_i$ with respect to $t$, an expression for $\dot{u}_i$ is obtained. Then, we apply the initial conditions $u_i(0) = 0$ and $\dot{u}_i(0) = 0$ to the resulting expressions for $u_i$ and $\dot{u}_i$, and solve for the undetermined constants $C_i$ and $C_2$. From (25) we obtain

$$C_2 = 4\mu \Omega \frac{\tilde{F}}{(\omega^2 - \Omega^2)}$$

and from $\dot{u}_i(0) = 0$ we have

$$C_1 = \frac{1}{\omega_0} u_{i\phi}(0) = -\frac{3}{32} \frac{\alpha \tilde{C}^3}{\omega_0^3} (\omega_0 - \alpha) + \frac{3}{8} \frac{\alpha \tilde{C}^2}{\omega_0^2} (\omega_0 - \alpha) + \frac{3}{4} \frac{\alpha \tilde{C}}{\omega_0^4} (\omega_0 - \alpha)$$

Substituting for $C_1$ and $C_2$ into (25) will result in the total solution for $u_i$.

$$u_l = -\frac{1}{2} \frac{\alpha \tilde{C}^2}{\omega_0^2} e^{-2\mu t} \sin \left[2(\omega_0 t + \frac{3}{4} \frac{\alpha \tilde{C}}{\omega_0} (1 - e^{-2\mu t}) + \frac{3}{8} \frac{\alpha \tilde{C}}{\omega_0} (1 - e^{-2\mu t}) \right]$$

Substituting the expressions for $C_1$ and $C_2$ into (25) will result in the total solution for $u_i$.

The two term approximation for $u$ is

$$u = u_0 + \alpha u_1 + O(e^2) = \tilde{C} e^{-\mu t} \sin(\omega_0 t)$$

Substituting $u_{i\phi}$ given in (25).

**THE HYBRID PERTURBATION-GALERKIN METHOD**

In general terms, given the differential equation

$$L(u, \varepsilon) = 0$$

where $u_{i\phi}$ is given in (25).
where ε is a small parameter, the perturbation solution to (29) can be expressed (in the case of a regular expansion) as
\[ u = \sum_{k=0}^{N-1} \epsilon^k u_k + O(\epsilon^N) \]  
(30)

The hybrid Galerkin method is a two-step analysis technique \[6,7\]. The first step involves the computation of the perturbation solution in form of (30) for a particular problem of type (29). The perturbation functions \( u_j \) in (30) are determined from a series of equations obtained by substituting (30) into (29) and setting the coefficients of \( \epsilon^k \) equal to zero, for \( k = 0, 1, \ldots, N - 1 \). In the second step, new amplitudes of the perturbation coordinate functions \( u_k \) are computed by using Galerkin method. Thus, an improved approximate solution \( \tilde{u} \) for \( u \) is sought in the form
\[ \tilde{u} = \sum_{k=0}^{N-1} \lambda_k u_k + O(\epsilon^N) \]  
(31)

where the \( N \) unknown parameters \( \lambda_k = \lambda_k(\epsilon) \) represent the amplitudes of the coordinate functions \( u_k \). To determine these parameters, we substitute \( \tilde{u} \) into the given differential equation (29) and require that the residual is orthogonal to each perturbation coordinate function, i.e.
\[ \int_{\tau}^{\tau} L(\tilde{u}, \epsilon) u_j(t) dt = 0 \quad \text{for} \quad j = 0, 1, \ldots, N - 1. \]  
(32)

Equation 32 represents a set of \( N \) simultaneous equations for the \( N \) unknown amplitudes \( \lambda_0, \lambda_1, \ldots, \lambda_{N-1} \).

The differential equation (29) for the forced Duffing equation (2) can be written as
\[ L(u, \epsilon) = \ddot{u} + 2\epsilon \dddot{u} + \omega^2 u + \epsilon \alpha u^3 - F \sin \omega t = 0. \]  
(33)

For a one term perturbation solution \( u_0 \), a hybrid solution \( \tilde{u} = \lambda_0 u_0 + \dot{\lambda}_0 \dot{u}_0 \) is obtained from (31). Substituting \( \tilde{u} \) into (32) will result in the following equation:
\[ \int_{\tau}^{\tau} \left[ (\lambda_0 \ddot{u}_0 + 2\epsilon \dot{\lambda}_0 \dddot{u}_0 + \omega^2 \dot{\lambda}_0 u_0 + \epsilon \alpha \lambda_0^3 u_0^3 - F \sin \omega t) \right] u_0 dt = 0 \]  
(34)

This equation can be rewritten as the following cubic equation for \( \lambda_0 \)
\[ \lambda_0^3 \int_{0}^{\tau} \epsilon \alpha u_0^4 dt + \lambda_0 \left[ u_0 \dot{u}_0 + 2\epsilon \dot{\lambda}_0 \dot{u}_0 + \omega^2 \dot{\lambda}_0 \dot{u}_0 \right] dt \]
\[- \int_{0}^{\tau} F u_0 \sin \omega t dt = 0 \]  
(35)

Using an integration by parts, we have
\[ \int_{0}^{\tau} \omega \dot{u}_0 dt = \left. \left( \omega \dot{u}_0 \right) \right|_{0}^{\tau} - \int_{0}^{\tau} (\omega^2 \dot{u}_0^2) dt \]  
(36)

Substitution of (36) into (35) eliminates the need to compute \( \ddot{u}_0 \) and hence (35) becomes
\[ \lambda_0^3 \int_{0}^{\tau} \epsilon \alpha u_0^4 dt + \lambda_0 \left[ u_0 \dot{u}_0 + 2\epsilon \dot{\lambda}_0 \dot{u}_0 + \omega^2 \dot{\lambda}_0 \dot{u}_0 \right] dt + \int_{0}^{\tau} (\omega^2 \dot{u}_0^2 + 2\epsilon \dot{\lambda}_0 \dot{u}_0^2) dt \]
\[+ \alpha \omega^2 \dot{u}_0^2 \dot{u}_0 dt - \int_{0}^{\tau} F u_0 \sin \omega t dt = 0 \]  
(37)

For a two term solution \( u = u_0 + \epsilon u_1 \), a hybrid solution \( \tilde{u} = \lambda_0 u_0 + \lambda_1 u_1 \) obtained from (31). Substituting \( \tilde{u} \) into (32) will result in the following equations for \( \lambda_0 \) and \( \lambda_1 \):
\[ \lambda_0 \int_{0}^{\tau} \left[ u_0 \ddot{u}_0 + 2\epsilon \dot{\lambda}_0 \dot{u}_0 \dddot{u}_0 + \omega^2 \dot{\lambda}_0 u_0 \dot{u}_0 \right] dt + \lambda_0^3 \int_{0}^{\tau} \epsilon \alpha u_0^4 dt \]
\[+ \lambda_1 \left[ u_0 \dot{u}_1 + 2\epsilon \dot{\lambda}_1 \dot{u}_1 + \omega^2 \dot{\lambda}_1 \dot{u}_1 \right] dt + \lambda_1^3 \int_{0}^{\tau} \epsilon \alpha u_1^4 dt \]
\[+ 3\lambda_0^2 \lambda_1 \int_{0}^{\tau} \epsilon \alpha u_0^2 u_1^2 dt + 3\lambda_0 \lambda_1^2 \int_{0}^{\tau} \epsilon \alpha u_0^3 u_1^2 dt \]
\[- \int_{0}^{\tau} F u_0 \sin \omega t dt = 0 \]  
(38)
\[ \begin{aligned}
\lambda_0 & \int_{0}^{\tau} \left[ u_0^{\beta_0} + 2 \varepsilon \mu u_0 + \alpha \varepsilon u_0 \right] dt + \lambda_1 \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon u_0^2 dt + \lambda_2 \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon \varepsilon u_0^4 dt \\
& + \lambda_3 \int_{0}^{\tau} \left[ u_0^{\beta_0} + 2 \varepsilon \mu u_0 + \alpha \varepsilon u_0 \right] dt + \lambda_4 \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon u_0^2 dt + \lambda_5 \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon \varepsilon u_0^4 dt \\
& + 3 \lambda_1 \lambda_4 \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon \varepsilon u_0^2 dt + 3 \lambda_2 \lambda_5 \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon u_0^4 dt \\
& - \int_{0}^{\tau} F u_1 \sin \Omega \Omega dt = 0 \quad (39)
\end{aligned} \]

The \( \bar{u}_i \) terms can be eliminated by the identity

\[ \int_{0}^{\tau} \bar{u}_i \bar{u}_j dt = u_i u_j \] \quad (40)

and hence Equations 38 and 39 can be expressed as:

\[ \begin{aligned}
G_i &= a_1 \lambda_0 + a_2 \lambda_1 + a_3 \lambda_1^3 + a_4 \lambda_1^3 + a_5 \lambda_1^2 \\
& + a_6 \lambda_1^2 + a_7 = 0, \quad (41)
\end{aligned} \]

\[ \begin{aligned}
G_2 &= b_1 \lambda_0 + b_2 \lambda_1 + b_3 \lambda_1^3 + b_4 \lambda_1^3 + b_5 \lambda_1^2 \\
& + b_6 \lambda_1^2 + b_7 = 0, \quad (42)
\end{aligned} \]

where \( a_1, a_2, \ldots, a_7 \) are given by

\[ a_1 = u_0 \bar{u}_0^3 + \int_{0}^{\tau} \left[ (\bar{u}_0)^2 + 2 \varepsilon \mu \bar{u}_0 + \alpha \varepsilon \bar{u}_0 \right] dt, \]

\[ a_2 = u_0 \bar{u}_0^5 + \int_{0}^{\tau} \left[ (\bar{u}_0)^2 - \varepsilon \mu \bar{u}_0 + \alpha \varepsilon \bar{u}_0 \right] dt, \]

\[ a_3 = \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon \varepsilon u_0^2 dt, \quad a_4 = \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon u_0^4 dt, \]

\[ a_5 = 3 \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon u_0^2 dt, \quad a_6 = 3 \int_{0}^{\tau} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon u_0^4 dt, \]

\[ a_7 = - \int_{0}^{\tau} F u_1 \sin \Omega \Omega dt. \quad (43) \]

The two nonlinear Equations 41 and 42 must be solved numerically for the unknowns \( \lambda_0 \) and \( \lambda_1 \). To accomplish this, we use Newton's method. This iterative procedure is carried according to the relation

\[ \hat{\lambda}^{(n+1)} = \hat{\lambda}^{(n)} - J(\hat{\lambda})^{-1} G(\hat{\lambda}^{(n)}) \quad (45) \]

where \( \hat{\lambda}^{(0)} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} \) is the approximation at iteration \( n \),

\[ G(\hat{\lambda}) = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \]

is a vector of nonlinear functions, and \( J(\hat{\lambda}) \) is

Jacobian matrix of \( G \) defined as

\[ J(\hat{\lambda}) = \begin{bmatrix} \frac{\partial G_1}{\partial \lambda_0} & \frac{\partial G_1}{\partial \lambda_1} \\ \frac{\partial G_2}{\partial \lambda_0} & \frac{\partial G_2}{\partial \lambda_1} \end{bmatrix} \quad (46) \]

with

\[ \frac{\partial G_1}{\partial \lambda_0} = a_1 + 3 a_4 \lambda_0^2 + 2 a_5 \lambda_0 \lambda_1 + a_6 \lambda_1^2, \]

\[ \frac{\partial G_1}{\partial \lambda_1} = a_2 + 3 a_4 \lambda_1^2 + a_5 \lambda_0^2 + 2 a_5 \lambda_0 \lambda_1, \]

\[ \frac{\partial G_2}{\partial \lambda_0} = b_1 + 3 b_4 \lambda_0^2 + b_5 \lambda_0 \lambda_1 + b_6 \lambda_1^2, \]

\[ \frac{\partial G_2}{\partial \lambda_1} = b_2 + 3 b_4 \lambda_1^2 + b_5 \lambda_0^2 + 2 b_5 \lambda_0 \lambda_1. \]
For the first iteration we use \( \hat{\lambda}^{(0)} = \begin{bmatrix} \lambda_0 \\ \hat{\lambda}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \). Then Equation 45 is repeated until \( |\hat{\lambda}^{(n+1)} - \hat{\lambda}^{(n)}| \leq \delta \) where \( \delta \) is a predetermined small number, e.g., \( 10^{-6} \).

**RESULTS FOR THE HARD NONRESONANT PROBLEM**

The hybrid perturbation-Galerkin method has been applied to the one and two term perturbation solutions as described previously. In this section we discuss the two term solution. In order to study the effects of \( \epsilon \) and \( \Omega \) on the solution to the forced Duffing equation (2), we have set the following parameters to predetermined constants as

\[
F = 1, \alpha = 1, \omega_0 = 1, \mu = 0.2, \quad \tau = 10\pi \quad (47)
\]

while varying the parameters \( \epsilon \) and \( \Omega \). We have also applied the fourth-order Runge-Kutta method with a step size of \( \pi/50 \) to (2) and compared the perturbation and hybrid solutions to this numerical solution.

To obtain the nonresonant excitation only, the value of \( \Omega \) must be away from \( \omega_0 = 1.0 \) (primary resonance). We have computed the perturbation, hybrid perturbation-Galerkin, and numerical solutions to the differential equation (2) for many different values of \( \Omega \) and \( \epsilon \). In Figure 1 through Figure 4 the two-term perturbation and the two-

![Graph 1](image1)

**Figure 1.** \( \Omega = 0.5, \epsilon = 0.02 \)

![Graph 2](image2)

**Figure 2.** \( \Omega = 0.5, \epsilon = 0.1 \)

![Graph 3](image3)

**Figure 3.** \( \Omega = 0.7, \epsilon = 1 \)

![Graph 4](image4)

**Figure 4.** \( \Omega = 1.2, \epsilon = 0.5 \)

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term hybrid solutions are compared with the numerical solution for some selected cases using the values in [47].
The first two figures illustrate the solutions for $\Omega = 0.5$ and $\epsilon = 0.02, 0.10$. For small values of $\epsilon$ both the perturbation and hybrid solutions are very close to the numerical solution. As $\epsilon$ increases, we observe that the perturbation solution diverges from the numerical solution and the hybrid method results in a better approximation to the solution of (2).

As $\Omega$ approaches $\omega_0 = 1.0$ ($0.5 < \Omega < 1.0$), we observe that the amplitude and shape of the perturbation solution deviates very rapidly from the numerical solution for increasing $\epsilon$. For these cases, the hybrid solution improves the perturbation solution significantly. This is illustrated in Figure 3. When $\Omega$ increases above the primary resonance, the hybrid solution again provides a much better approximation to the solution of the differential equation (2) than the perturbation solution. Figure 4 is a representative of these cases. For values of $\Omega > 2.0$ while $\epsilon$ remains small (up to $\epsilon = 0.5$) the perturbation and hybrid solutions are almost the same as the numerical solution. As $\epsilon$ increases, the hybrid method does not appear to improve the perturbation solution.

Table 1 contains the hybrid coefficients $\lambda_1$ and $\lambda_2$ for the hard nonresonant case for some selected values of $\Omega$ and $\epsilon$.

### TABLE 1

<table>
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<th>$\epsilon$</th>
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<th>$\lambda_2$</th>
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### CONCLUSIONS

We have computed the perturbation and hybrid perturbation-Galerkin solutions for the forced Duffing equation and have demonstrated that in general the hybrid method improves the perturbation solution. For values of $\Omega$ larger than $\omega_0 = 1.5$ (above the primary resonance), it appears that the perturbation solution by itself is very close to the numerical solution and hence it is difficult to observe the improvements contributed by the hybrid method. The usefulness of the hybrid method is best demonstrated for values of $\Omega$ near the resonance ($0.5 < \Omega < 1.0$ and $1.0 < \Omega < 1.5$).

We also note that this paper represents one of the first applications of the hybrid method to time dependent problems (see [12]), since previous applications have been largely restricted to boundary value problems ([2], [7], [8]). We feel that the method (or some suitably modified version of it) will be useful for time dependent problems in even broader application areas (e.g. partial differential equations). We are currently investigating some particular applications involving partial differential equations with encouraging initial results.
ACKNOWLEDGEMENT

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REFERENCES