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Free Vibration Analysis of Nonuniform Microbeams Based on Modified Couple Stress Theory: an Analytical Solution

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ABSTRACT

In this study, analytical solution is presented to calculate the free vibration frequencies of nonuniform microbeams. Scale effects are modelled using modified couple stress theory and the microbeam is assumed to be thin while Poisson's ratio effects are also taken into account. Nonuniformity is presented by exponentially varying width among the microbeam while the thickness remains constant. Results are presented for simply-supported, cantilever and clamped boundary conditions. First five natural frequency parameter are calculated for different scale and nonuniformity parameters and effects of each parameter on the results are discussed. Also, to understand the effects of Poisson's ratio, small scale and nonuniformity on the first frequency of the nonuniform microbeam and resonance domain, a comprehensive parametric study is done. This research is important in understanding the dynamic behavior of microbeams and effective designs using variable cross section in this type of microstructures.

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NOMENCLATURE

U	Strain energy (J)	E	Young's modulus (Pa)
T	Kinetic energy (J)	ϑ	Poisson's ratio
μ, λ	Lames constants	θ	Rotation vector
χ	Curvature tensor	ρ	Density (kg/m^3)
u, w	Displacement components (m)	h	Thickness of microbeam (m)
Δ	Variation term	L	Length of microbeam (m)
A_i	Frequency parameters	b_0	Width of microbeam (m)
E	Strain tensor	η	Nonuniformity
σ, γ	Typical and deviatoric stress tensor	ω	Natural frequency term
α	Dimensionless small scale parameter	A	Cross section (m^2)
C_i	Constant coefficients		

1. INTRODUCTION

Free vibration analysis of different structures is a key step in understanding the dynamic behavior of different systems. There has been many studies done in order to comprehend the dynamic behavior of such structures. One of the most important structures used in different mechanical structures are beams. Beams are modeled and studied in different scales such as macrobeams [1-3], microbeams [4-6] and nanobeams [7-10].

In recent years, researchers focused on making a more efficient structure using combined or smart materials such as composites [11-14] and functionally graded materials (FGM) [15-18] to increase the

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efficiency of structures. Microbeams as a part of microstructures have an important role in future designs due to their special behaviors and less space taken. In order to be able to have a more effective design in micro size structures, it is neccesarry to vary the cross section among the length of the microbeam. Small scale structures such as micro/nano beams, plates, shell etc. show a different type of behavior compared to macro sized sctructures and classical continuum theories are unable to predict their behavior with a good precision. For this reason, different non-classical theories such as nonlocal elastic theory [19, 20], strain gradient theory [21], modified couple stress theory [22], etc. are presented to model their static and dynamic behaviors. In all these theories, modified coupled stress theory is one of the well-known theories which is used widely by the researchers to model the microstructures behaviors [23-26].

Utilizing variable cross section beside the significant behavior of mirco scale structures could lead to a more reliable design. There have been some great studies in order to model the vibration in nonuniform microbeams. Akgöz and Civalek [27] studied the free vibration analysis of axially FG nonuniform microbeams. Classical beam theory and modified couple stress theory were used to model the beam. Nonuniformity was presented in three different ways, by linearly varying the width of the beam, linearly varying the thickness of the beam and a combination between them both. Natural frequencies were calculated using Rayleigh–Ritz method.

Shafiei et al. [28] studied the nonlinear vibration of axially FG nonuniform microbeams using modified couple stress theory. Euler–Bernoulli beam theory and Von-Kármán's strain were used to model the beam and its deflections. Nonuniformity was presented by varying the thickness and width of the beam linearly. Frequency parameter was presented for different materials and nonuniformity by presenting numerical solution. They also studied [29] the vibration of the same nonuniform microbeam under a rotation situation and effects of the rotating speed were also presented.

These studies used linear varying in cross section of the beams. Cem Ece [30] presented an exponential varying model for cross section of beams where the free vibration of macro scaled beams was modeled using such kind of nonuniformity. Hosseini Hashemi and Bakhshi Khaniki [31] used the same nonuniformity to model the free vibrations of nanosclaed nonuniformed nanobeams. Eringens nonlocal elastic theory and Euler beam model were employed to achieve the equation of motion and results were calculated by analytically solving the problem. It was shown that using variable cross section for nano scale beams could lead to a great efficiency.

Exponentially variable cross-section beams as a part of nonuniform structures have far less been studied in microscale. In this study, by the frame work of modified couple stress theory, the nonuniformity in microbeams are modeled and analytically solved for different types of boundary conditions and the effects of nonuniformity and microscale effects are investigated. Figure 1 shows a schematic representation of nonuniform microbeams which will be discussed in this paper.

2. PROBLEM FORMULATION

Classical continuum theories are unable to model and predict the behavior of small scale structures which led to use of new nonclassical theories in which the modified couple stress theory is one of the well known theories in modeling behaviors in microstructures. This theory describes that the strain energy of an elastic beam is a function of not only strain tensor, but also the curvature tensor which can be expressed as: [22]

$$U=\frac{1}{2}\int_V(\sigma:\varepsilon+\gamma:\chi)\mathrm{dV} \tag{1}$$

In which σ and γ are the typical and deviatoric part of modified stress tensors and ε and χ are the strain and curvature tensors which could be defined as [32]:

$$\begin{cases}\sigma_{ij}=\lambda_0\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}\\ \gamma_{ij}=2\mu l^2\chi_{ij}\\ \varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad i,\mathrm{j}=x_1,x_2,x_3\\ \chi_{ij}=\frac{1}{2}\left(\frac{\partial\theta_i}{\partial x_j}+\frac{\partial\theta_j}{\partial x_i}\right)\end{cases} \tag{2}$$

where δ is the Kronecker delta, l is the material length scale parameter, λ_0 and μ are the Lames constants and θ is the rotation vector defined as [32]:

$$\theta_{x_1}=0\ ,\ \theta_{x_2}=-\left(\frac{\partial w}{\partial x_1}\right)\ ,\ \theta_{x_3}=0 \tag{3}$$

Displacement relations for thin beams are given as:

$$\begin{aligned}&u(x_1,x_3,t)=-z\ \frac{\partial w\ (x_1,t)}{\partial x_1}\\ &w\ (x_1,x_3,t)=w\ (x_1,t)\end{aligned} \tag{4}$$

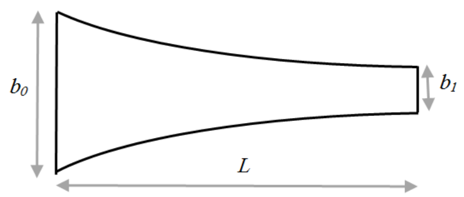


Figure 1. Schematic representation of exponentially non-uniform microbeams

where u and w are the displacement components, x_1 is the longitudinal coordinate measured from the left end of the beam and x_3 is the coordinate measured from the midplane of the beam. The longitudinal strain and symmetric curvature tensors with respect to Euler-Bernoulli beam model are defined as:

$$\begin{aligned} &\sigma_{x_1x_1} = (\lambda_0 + 2\mu)\varepsilon_{x_1x_1} \\ &\gamma_{x_1x_2} = 2\mu l^2 \chi_{x_1x_2} \\ &\varepsilon_{x_1x_1} = \frac{\partial u}{\partial x_1} = -z\,\frac{\partial^2 w}{\partial {x_1}^2} \\ &\chi_{x_1x_2} = \frac{1}{2}\left(\frac{\partial \theta_{x_2}}{\partial x_1} + \frac{\partial \theta_{x_1}}{\partial x_2}\right) \\ &\qquad = -\frac{1}{2}\frac{\partial^2 w\,(x_1, \mathrm{t})}{\partial {x_1}^2} \end{aligned} \tag{5}$$

While other strain and rotation vectors are equal to zero and the Lame's constants are defined as:

$$\begin{cases} \lambda_0 = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \\ \mu = \dfrac{E}{2(1+\nu)} \end{cases} \tag{6}$$

where E is the modulus of elasticity and ν is the Poisson's ratio. Using Equation (1) to Equation (6), the strain energy and kinetic energy in microbeams are written as:

$$\begin{cases} U = \dfrac{1}{2}\displaystyle\int_0^L \left(\left(\frac{E(1-\nu)\mathrm{I}}{(1+\nu)(1-2\nu)} + \mu A l^2\right)\left(\frac{\partial^2 w\,(x_1, \mathrm{t})}{\partial {x_1}^2}\right)^2\right) \mathrm{d}x_1 \\ T = \dfrac{1}{2}\displaystyle\int_0^L\!\!\int_A \left[\rho\left(\frac{\partial w}{\partial t}\right)^2 + \rho z^2\left(\frac{\partial^2 w}{\partial x_1 \partial t}\right)^2\right] dA\,dx_1 \end{cases} \tag{7}$$

where I is the second moment of area of the beam and A is the area of the cross-section which in this study is variable. Also, with respect to Hamilton's principle and taking the first variation we have:

$$\delta\int_0^t L\,dt = \int_0^t \delta(T - U)\,dt = 0 \tag{8}$$

By putting Equation (7) into Equation (8) and doing some calculation, governing equation of motion of general microbeam is achieved as:

$$\begin{aligned} &\rho A\left(\frac{\partial^2 w}{\partial t^2}\right) - \rho I\left(\frac{\partial^4 w}{\partial {x_1}^2 \partial t^2}\right) + \\ &\frac{\partial^2}{\partial {x_1}^2}\left[\left(\frac{E(1-\nu)\mathrm{I}}{(1+\nu)(1-2\nu)} + \mu A l^2\right)\left(\frac{\partial^2 w}{\partial {x_1}^2}\right)\right] = 0 \end{aligned} \tag{9}$$

This equation presents the dynamic behavior of a general microbeam without any external forces. Considering an elastic microbeam with nonuniform variable cross section as shown in Figure 1, the cross section will be a function of the length of the beam defined as:

$$\begin{cases} b(x_1) = b_0 e^{N x_1} \\ b_1 = b_0 e^{Nl} \end{cases} \tag{10}$$

$$\begin{cases} A(x_1) = b_0 h\, e^{N x_1} \\ A_0 = b_0 h \\ I(x_1) = \dfrac{1}{12} b_0 h^3 e^{N x_1} \\ I_0 = \dfrac{1}{12} b_0 h^3 \end{cases} \tag{11}$$

where N is the nonuniformity parameter, I_0 and A_0 are second moment of area and cross section of the microbeam at the left end, b_0 and b_1 are the width of the beam at the left and right end of the microbeam and h is the thickness which is assumed to be constant. By assuming free harmonic motion as:

$$w\,(x_1, t) = W\,(x_1) e^{i\omega t} \tag{12}$$

and substituting Equations (10)-(12) into Equation (9), the equation of motion of nonuniform elastic isotropic microbeam is achieved as:

$$\begin{aligned} &\left(\frac{E(1-\nu)\mathrm{I}}{(1+\nu)(1-2\nu)} + \mu A l^2\right)\left(\frac{\partial^4 W}{\partial {x_1}^4}\right) + \\ &2N\left(\frac{E(1-\nu)\mathrm{I}}{(1+\nu)(1-2\nu)} + \mu A l^2\right)\left(\frac{\partial^3 W}{\partial {x_1}^3}\right) + \\ &\left[\rho I \omega^2 + N^2\left(\frac{E(1-\nu)\mathrm{I}}{(1+\nu)(1-2\nu)} + \mu A l^2\right)\right]\left(\frac{\partial^2 W}{\partial {x_1}^2}\right) - \\ &\rho A \omega^2 W = 0 \end{aligned} \tag{13}$$

It can be seen that by neglecting the nonuniformity parameter N, Equation (13) will become the formal equation of motion of uniform microbeams. Also, by neglecting the micro scale parameter l, equation of motion of nonuniform macro scale beams is achieved and at last by having both $N = 0$ and $l = 0$, the general form of equation of motion of uniform macro beams are achieved.

In order to present Equation (13) in nondimensional form to prevent the scale differences in further calculations, dimensionless variables are defined as:

$$\begin{cases} X = \dfrac{x_1}{L}, W = \dfrac{w}{L}, \alpha = \dfrac{l}{L} \\ \zeta = \dfrac{\sqrt{I}}{L\sqrt{A}}, \eta = NL \\ \lambda^2 = \dfrac{\rho A L^4 \omega^2}{\left(\dfrac{E(1-\nu)\mathrm{I}}{(1+\nu)(1-2\nu)} + \mu A L^2 \alpha^2\right)} \end{cases} \tag{14}$$

where η is the nondimensional nonuniformity parameter, α denotes the dimensionless small scale parameter, X and W are the nondimensional coordinate measured from the left end of the beam along the length and the dimensionless transverse displacement.

Using Equation (14) and rewriting Equation (13) in a nondimensional form we have:

$$\left(\frac{\partial^4 W}{\partial x^4}\right)+2\eta\left(\frac{\partial^3 W}{\partial x^3}\right)+\left(\lambda^2\zeta^2+\eta^2\right)\left(\frac{\partial^2 W}{\partial x^2}\right)-\lambda^2 W=0 \tag{15}$$

3. SOLUTION PROCEDURE

In order to investigate the free vibration of nonuniform microbeams, boundary conditions of the both ends of the microbeam should be chosen. In this study, boundary conditions of the ends of the beam is considered to be simply supported (S), clamped (C) or free (F).

Solution of Equation (15) subjected to either boundary conditions can be written in a general form as:

$$W(X)=C_1e^{(A_1X)}+C_2e^{(A_2X)}+C_3e^{(A_3X)}+e^{(A_4X)} \tag{16}$$

In which A_1 to A_4 are function of natural frequency parameter ω defined as:

$$\begin{cases} A_1=-\dfrac{b}{4a}-S+\sqrt{-4S^2-2p+\dfrac{q}{S}} \\ A_2=-\dfrac{b}{4a}-S-\sqrt{-4S^2-2p+\dfrac{q}{S}} \\ A_3=-\dfrac{b}{4a}+S+\sqrt{-4S^2-2p+\dfrac{q}{S}} \\ A_4=-\dfrac{b}{4a}+S-\sqrt{-4S^2-2p+\dfrac{q}{S}} \end{cases} \tag{17}$$

where undefined terms are presented as:

$$\begin{cases} p=\dfrac{8ac-3b^2}{8a^2} \\ q=\dfrac{b^3-4abc}{8a^3} \\ S=\dfrac{1}{2}\sqrt{-\dfrac{2}{3}p+\dfrac{1}{3a}\left(Q+\dfrac{\Delta_0}{Q}\right)} \\ Q=\sqrt[3]{\dfrac{\Delta_1+\sqrt{\Delta_1^2-4\Delta_0^3}}{2}} \end{cases} \tag{18}$$

$$\begin{cases} a=1 \\ b=\eta \\ c=\lambda^2\zeta^2+\eta^2 \\ e=-\lambda^2 \\ \Delta_0=c^2+12ae \\ \Delta_1=2c^3+27b^2e-72ace \end{cases} \tag{19}$$

Applying boundary conditions in each case leads to an equation for determination of natural frequency. The natural frequency equation are given below for each boundary condition separately.

3. 1. Simply-supported Nonuniform Microbeam

For the microbeams with both ends simply-supported, boundary conditions are defined as:

$$\begin{aligned} &Simply-supported:\ W(0)=0,\ W''(0)=0, \\ &\qquad W(1)=0,\ W''(1)=0 \end{aligned} \tag{20}$$

Applying these conditions into Equation (16) and neglecting the constant parameters C_1 to C_3 leads to:

$$\begin{aligned} &\frac{e^{A_1}L_{11}}{L_{12}}-\frac{e^{A_2}}{A_2^2-A_3^2}\left(\frac{\left(A_1^2-A_3^2\right)L_{11}}{L_{12}}+\left(A_4^2-A_3^2\right)\right)+ \\ &e^{A_3}\left(-1-\frac{L_{11}}{L_{12}}+\frac{(A_1^2+A_3^2)}{A_2^2-A_3^2}\left(\frac{L_{11}}{L_{12}}\right)-\frac{A_3^2-A_4^2}{A_2^2-A_3^2}\right)+e^{A_4}=0 \end{aligned} \tag{21}$$

By solving this equation natural frequencies of simply-supported nonuniform microbeams are achieved and undefined parameters L_{11} and L_{12} are presented as:

$$L_{11}=A_2^2A_3^2\left(e^{A_2}-e^{A_3}\right)+A_2^2A_4^2\left(e^{A_4}-e^{A_2}\right)+A_3^2A_4^2\left(e^{A_3}-e^{A_4}\right) \tag{22}$$

$$L_{12}=A_1^2A_2^2\left(e^{A_1}-e^{A_2}\right)+A_1^2A_3^2\left(e^{A_3}-e^{A_1}\right)+A_2^2A_3^2\left(e^{A_2}-e^{A_3}\right) \tag{23}$$

3. 2. Clamped Nonuniform Microbeam

In the same way, for the microbeams with both clamped ends, boundary conditions are defined as:

$$\begin{aligned} &Clamped:\ W(0)=0,\ W'(0)=0, \\ &\qquad W(1)=0,\ W'(1)=0 \end{aligned} \tag{24}$$

and by substituting Equation (16) into conditions of Equation (24) we have:

$$\begin{aligned} &\frac{e^{A_1}L_{21}}{L_{22}}-\frac{e^{A_2}}{A_2-A_3}\left(\frac{\left(A_1-A_3\right)L_{21}}{L_{22}}-A_3+A_4\right)+ \\ &e^{A_3}\left(-1-\frac{L_{21}}{L_{22}}+\frac{A_1-A_3}{A_2-A_3}\left(\frac{L_{21}}{L_{22}}\right)-\frac{A_3-A_3}{A_2-A_3}\right)+e^{A_4}=0 \end{aligned} \tag{25}$$

By solving this equation, the natural frequencies of both clamped end nonuniform microbeams are achieved and undefined parameters L_{21} and L_{22} are defined as:

$$L_{21}=A_2A_3\left(e^{A_2}-e^{A_3}\right)+A_2A_4\left(e^{A_4}-e^{A_2}\right)+A_3A_4\left(e^{A_3}-e^{A_4}\right) \tag{26}$$

$$L_{22}=A_1A_2\left(e^{A_1}-e^{A_2}\right)+A_1A_3\left(e^{A_3}-e^{A_1}\right)+A_2A_3\left(e^{A_2}-e^{A_3}\right) \tag{27}$$

3. 3. Cantilever Nonuniform Microbeam

At last, by having a clamped condition in one end of the beam and a free end on the other side, the boundary conditions are presented as:

$$\begin{aligned} &Cantilever:\ W(0)=0,\ W'(0)=0, \\ &\qquad W''(1)=0,\ W'''(1)=0 \end{aligned} \tag{28}$$

and in the same way by substituting Equation (16) into conditions of Equation (28) and neglecting the constant parameters C_1 to C_3, the pure equation is achieved as:

$$\frac{2e^{A_1}A_1^2L_{31}}{L_{32}}-\frac{2e^{A_2}A_2^2}{A_2^2-A_3^2}\left(\frac{(A_1-A_3)L_{31}}{L_{32}}-A_3+A_4\right)+ \\ 2e^{A_3}A_3^2\left(-1-\frac{L_{11}}{L_{12}}+\frac{A_1-A_3}{A_2-A_3}\left(\frac{L_{11}}{L_{12}}\right)-\frac{A_3-A_4}{A_2-A_3}\right)+2e^{A_4}A_4^2=0 \quad (29)$$

where L_{31} and L_{32} are defined as:

$$L_{31}=A_2A_3\left(A_2^2e^{A_2}-A_3^2e^{A_3}\right)+A_2A_4\left(A_3^2e^{A_4}-A_2^2e^{A_2}\right)+ \\ A_3A_4\left(A_3^2e^{A_3}-A_4^2e^{A_4}\right) \quad (30)$$

$$L_{32}=A_1A_2\left(A_1^2e^{A_1}-A_2^2e^{A_2}\right)+A_1A_3\left(A_3^2e^{A_3}-A_1^2e^{A_1}\right)+ \\ A_2A_3\left(A_2^2e^{A_2}-A_3^2e^{A_3}\right) \quad (31)$$

4. RESULTS AND DISCUSSION

In this section, in order to illustrate the nonuniformity effects combined with scale effects on free vibration response of micorbeams, frequency parameters are provided for different boundary conditions. To be able to verify the current methodology, results are achieved and compared to those presented in previous literatures. For this reason, by neglecting the scale effect parameter, results are calculated for nonuniform macro beams and compared to the natural frequency parameters calculated by Cem Ece et al. [30]. This verification is presented for first five frequency parameters in Table 1 for simply-supported macrobeams with nonuniformity parameter as $\eta=1$ and 2. In the same way, in Table 2 and Table 3, first five frequency parameters for clamped and cantilever macrobeams are presented in which the results are in a great agreement. Current formulation is also verified by neglecting the nonuniformity in the formulation and calculating the natural frequency term for uniform microbeams.

TABLE 1. Natural frequency parameters for a simply supported nonuniform beam

Classical beam , Simply supported				
	$\eta=1$		$\eta=2$	
	Present	Ref [30]	Present	Ref [30]
Mode 1	9.7729	9.77291	9.4872	9.48725
Mode 2	39.5703	39.57036	39.8523	39.85231
Mode 3	88.9705	88.97052	89.4052	89.40520
Mode 4	158.0841	158.08418	158.5968	158.59689
Mode 5	246.9265	246.92650	247.4862	247.48629

Results are achieved for different scale parameters and compared to those presented in previous literatures [23] in Table 4.

After verifying the problem formulation, by varying different parameters, effects on natural frequency parameter are presented. Poisson's ratio is assumed to be zero in classical beam theory and the geometrical parameters of the microbeams are $L=100\mu m$, $h=1\mu m$ and $b_0=5\mu m$.

TABLE 2. Natural frequency parameters for a clamped nonuniform beam

Classical beam , Clamped				
	$\eta=1$		$\eta=2$	
	Present	Ref [30]	Present	Ref [30]
Mode 1	22.5116	22.51167	22.9377	22.93771
Mode 2	61.8596	61.85968	62.4227	62.42272
Mode 3	121.1079	121.10799	121.7227	121.72272
Mode 4	200.0741	200.07411	200.7186	200.71860
Mode 5	298.7766	298.77661	299.4401	299.44012

TABLE 3. Natural frequency parameters for a cantilever nonuniform beam

Classical beam , Cantilever				
	$\eta=1$		$\eta=2$	
	Present	Ref [30]	Present	Ref [30]
Mode 1	2.8583	2.85833	2.9089	2.90893
Mode 2	20.0391	20.03917	18.1752	18.17520
Mode 3	59.8708	59.87084	58.3886	58.38868
Mode 4	119.0986	119.09862	117.6921	117.69217
Mode 5	198.0696	198.06964	196.7022	196.70224

TABLE 4. Natural frequency parameters for unifrom microbeam with different scale parameters and boundary conditions (MHz)

Simply-supported				
h/l	2	4	6	8
Present	1.39984	1.74893	2.21155	2.73051
Ref [23]	1.3998	1.7489	2.2115	2.7305
Cantilever				
h/l	2	4	6	8
Present	0.49872	0.62297	0.78778	0.97273
Ref [23]	0.4987	0.6230	0.7878	0.9727
Clamped				
h/l	2	4	6	8
Present	3.17321	3.96444	5.01314	6.18963
Ref [23]	3.1732	3.9645	5.0131	6.1896

In Table 5, the first five natural frequency parameters for classical simply supported nonuniform microbeam are calculcated and presented. Nonuniform parameter is assumed to be η=0, 1, 2 while scale effect parameter is presented as α=0, 0.001, 0.003 and 0.005. It can be seen that increasing the scale parameter leads to a lower frequency parameter in all domain for all nonuniformities. Increasing the nonuniform parameter, also leads to lower fequency numbers. For clamped nonuniform microbeams, results are presented in Table 6. Unlike the simply-supported condition, it is shown that increasing the nonuniform parameter leads to higher frequency parameters but same reaction is achieved by increasing the small scale parameter which causes lower frequency parameter.

The reason that increasing the small scale parameter leads to lower frequency parameters is cause of the curvature tensor being mentioned more in the strain energy of the system which causes less rigidity and lower frequency parameters. Also, about the different behaviors seen by varying nonuniformity parameter which is also seen at macro scale problem [30], it can be explained by the situations made by having nonuniform cross section.

TABLE 5. First five natural frequency parameters for classical simply supported nonuniform microbeam

Classical beam , Simply supported

$\eta = 0$

α	0	0.001	0.003	0.005
Mode 1	9.86960	9.5863	7.9529	6.2425
Mode 2	39.47841	38.3454	31.8117	24.9701
Mode 3	88.82643	86.2771	71.5763	56.1827
Mode 4	157.91367	153.3815	127.2468	99.8804
Mode 5	246.74011	239.6587	198.8232	156.0631

$\eta = 1$

α	0	0.001	0.003	0.005
Mode 1	9.77291	9.4924	7.8750	6.1814
Mode 2	39.57036	38.4347	31.8858	25.0283
Mode 3	88.97052	86.4168	71.6924	56.2739
Mode 4	158.08418	153.5472	127.3842	99.9882
Mode 5	246.92650	239.8397	198.9734	156.1810

$\eta = 2$

α	0	0.001	0.003	0.005
Mode 1	9.48725	9.2150	7.6448	6.0007
Mode 2	39.85231	38.7085	32.1130	25.2066
Mode 3	89.40520	86.8393	72.0427	56.5488
Mode 4	158.59689	154.0452	127.7974	100.3125
Mode 5	247.48629	240.3834	199.4245	156.5351

TABLE 6. First five natural frequency parameters for classical clamped nonuniform microbeam

Classical beam , Clamped

$\eta = 0$

α	0	0.001	0.003	0.005
Mode 1	22.37327	21.7311	18.02838	14.15109
Mode 2	61.67281	59.9028	49.69595	39.00805
Mode 3	120.90338	117.4335	97.42394	76.47139
Mode 4	199.85945	194.1235	161.0467	126.4111
Mode 5	298.55552	289.9870	240.576	188.8364

$\eta = 1$

α	0	0.001	0.003	0.005
Mode 1	22.51167	21.86559	18.1399	14.23863
Mode 2	61.85968	60.08431	49.84653	39.12625
Mode 3	121.10799	117.6322	97.58882	76.6008
Mode 4	200.07411	194.332	161.2197	126.5469
Mode 5	298.77661	290.2017	240.7542	188.9762

$\eta = 2$

α	0	0.001	0.003	0.005
Mode 1	22.93771	22.2794	18.48321	14.5081
Mode 2	62.42272	60.63119	50.30023	39.48237
Mode 3	121.72272	118.2293	98.08417	76.98962
Mode 4	200.71860	194.958	161.739	126.9545
Mode 5	299.44012	290.8462	241.2888	189.3959

Exponentially nonuniformed cross-section will put the position of the weakest part of the beam closer to the boundary. For simply supported beams this end could rotate freely, so the effects are well seen by having lower frequency parameter for higher nonuniformity. But in the clamped end condition, this end is fully fixed and both rotations and deflections being prevented so different types of behavior are observed. In the same way, natural frequency parameter for cantilever microbeam is shown in Table 7. It is shown that for all the nonuniform exponentially variable cross sections, adding the scale effect parameter leads to lower frequencies parametes.

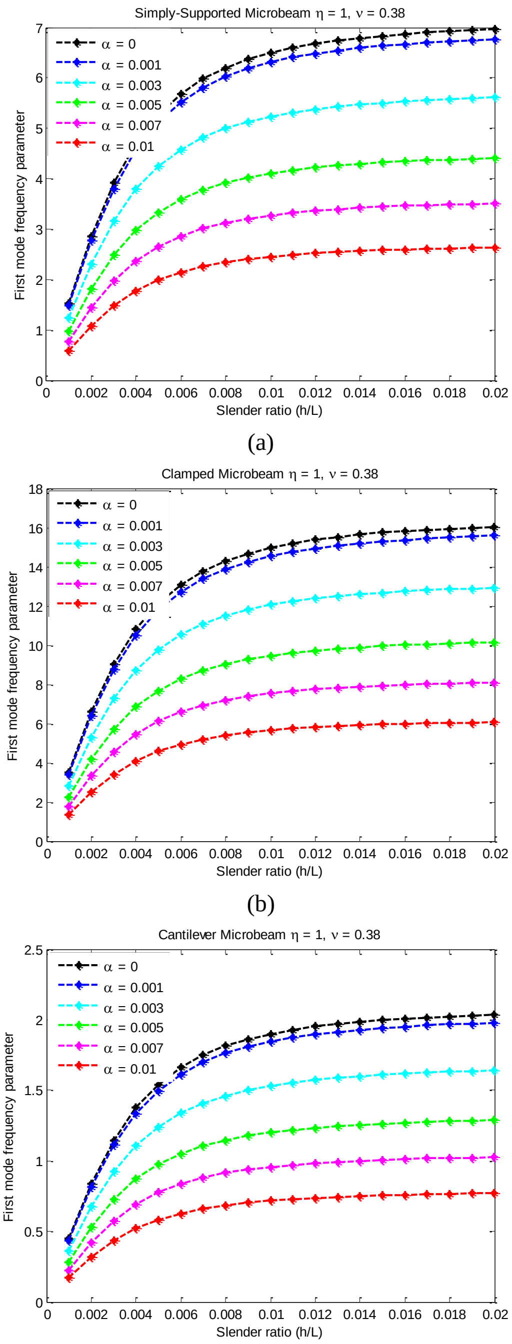
On the other hand, the ratio of the thickness to the length of the microbeam which is called slender ratio is an important parameter affecting the natural frequency.

In Figure 2, these effects are presented for simply-supported, clamped and cantilever nanobeams for different small scale parameters. It is shown that increasing the slender ratio parameter leads to a higher frequency parameter for all boundary conditions. Increasing the slender ratio parameter has the most effect in lower numbers of it and merges to a specific number for higher amounts.

TABLE 7. First five natural frequency parameters for classical cantilever nonuniform microbeam

Classical beam , Cantilever				
$\eta = 0$				
α	0	0.001	0.003	0.005
Mode 1	3.51602	3.41511	2.833209	2.223883
Mode 2	22.03449	21.4021	17.75539	13.93681
Mode 3	61.69721	59.9265	49.71561	39.02349
Mode 4	120.90191	117.432	97.42276	76.47046
Mode 5	199.85953	194.1236	161.0468	126.4112
$\eta = 1$				
α	0	0.001	0.003	0.005
Mode 1	4.72298	4.58743	3.805777	2.987285
Mode 2	24.20168	23.50709	19.50171	15.30756
Mode 3	63.86448	62.03157	51.462	40.39428
Mode 4	123.09790	119.565	99.19229	77.85942
Mode 5	202.06876	196.2694	162.827	127.8085
$\eta = 2$				
α	0	0.001	0.003	0.005
Mode 1	6.25877	6.079143	5.043317	3.958672
Mode 2	26.58350	25.82055	21.42098	16.81406
Mode 3	66.37449	64.46954	53.48456	41.98186
Mode 4	125.68471	122.0776	101.2767	79.49558
Mode 5	204.69531	198.8206	164.9435	129.4698
$\eta = -1$				
α	0	0.001	0.003	0.005
Mode 1	2.85833	2.776296	2.303242	1.807894
Mode 2	20.03917	19.46405	16.14756	12.67478
Mode 3	59.87084	58.15255	48.24392	37.86831
Mode 4	119.09862	115.6805	95.96967	75.32988
Mode 5	198.06964	192.385	159.6045	125.279
$\eta = -2$				
α	0	0.001	0.003	0.005
Mode 1	2.90893	2.825444	2.344016	1.839898
Mode 2	18.17520	17.65357	14.64558	11.49581
Mode 3	58.38868	56.71292	47.0496	36.93084
Mode 4	117.69217	114.3144	94.83635	74.4403
Mode 5	196.70224	191.0569	158.5027	124.4142

Also, Poisson's ratio effects on exponentially varying nonuniform microbeams are also calculated and presented in Figure 3 for all boundary conditions.



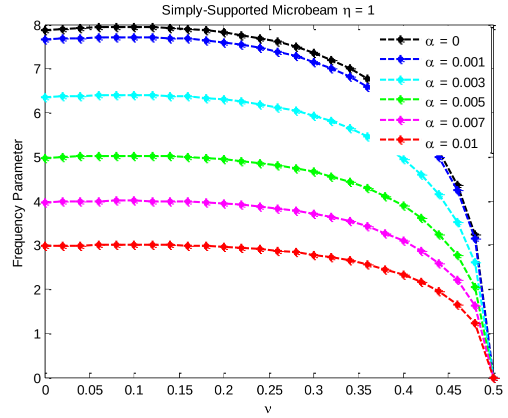
(a)

(b)

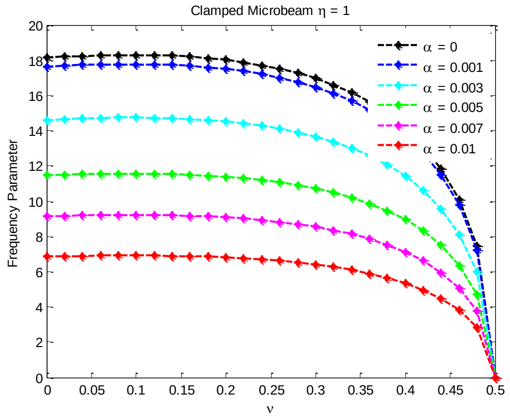
(c)

Figure 2. First mode frequency parameter with respect to scale effect parameter and slender ratio: (a) simply supported, (b) Clamped and (c) Cantilever

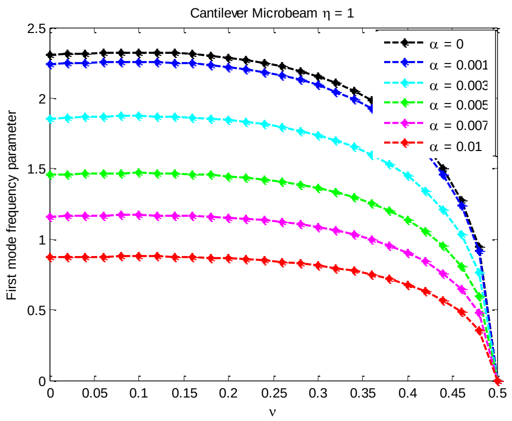
It is seen that increasing the Poisson's ratio from 0 to 0.5 makes a small increase in the first natural frequency parameter at first but then it starts to decrease and reaches to zero. This kind of behavior was independent from the boundary types and happened in the same way for all microbeams.



(a)



(b)



(c)

Figure 3. First mode frequency parameter with respect to scale effect parameter and Poisson's ratio: (a) simply supported, (b) Clamped and (c) Cantilever

5. CONCLUSION

In the present study, free vibration of exponentially variable cross section microbeams is investigated using different types of boundary conditions. Small scale effects are modeled using modified couple stress theory by adding the curvature tensor's effect on strain energy of the microbeam. Governing equation is achieved and a general analytical solution is presented and solved for simply supported, clamped and cantilever boundary conditions. Current methodology is verified by comparing the results with previous literatures in studying uniform microbeams and nonuniform beams. By analytically solving the problem, results revealed that nonuniformity and small scale effects combined with each other have a significant effect on varying the frequency terms. Also, it is shown that these effects are completely different for each boundary condition type. In order to clarify the effects of different parameters such as Poisson's ratio, slender ratio etc. on natural frequencies, parametric study is presented for all types of boundary conditions.

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Free Vibration Analysis of Nonuniform Microbeams Based on Modified Couple Stress Theory: an Analytical Solution

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چکیده

در این تحقیق، حل تحلیلی ارتعاشات آزاد میکروتیرها با سطح مقطع متغیر ارائه شده است. اثرات ابعادی به کمک تئوری تنش کوپل اصلاح شده مدل سازی شده است. میکروتیر باریک فرض شده اما اثرات ضریب پواسون درنظر گرفته شده است. تغییر شکل در سطح مقطع تیر به صورت نمایی در عرض تیر با ضخامت ثابت فرض شده است. نتایج برای شرایط تکیه گاهی دوسر پین شده، دوسرگیردار و یک سر گیردار ارائه شده است. پنج فرکانس طبیعی اول ارتعاشی برای مقادیر مختلف ترم ابعادی و ناهمگنی سطح مقطع بدست آمده است. علاوه بر این، جهت درک مناسب‌تر اثرات ضریب پواسون، اثرات ابعادی و ناهمگنی سطح مقطع روی فرکانس اول ارتعاشی میکروتیرهای سطح مقطع متغیر و شناخت نواحی تشدید، مطالعه پارامتریک جامع ارائه شده است.

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