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TECHNICAL NOTE

# Free Vibration Analysis of a Sloping-frame: Closed-form Solution versus Finite Element Solution and Modification of the Characteristic Matrices

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*ABSTRACT*

This article deals with the free vibration analysis and determination of dynamic characteristics of a sloping-frame. First of all, a closed-form solution is proposed and then, a numerical analysis is performed for some verification purposes. The closed-form solution is developed by solving the frame equations of motion, directly. For this reason, some mathematical techniques are utilized, such as Fourier transform and the well-known complementary solutions. In this way, some differential equations must be solved, and several boundary conditions should be satisfied. Herein, the more accurate derivation of one of twelve boundary conditions is the most important challenge of this paper. This boundary condition is expressed as three distinctive versions, and the free vibration parameters of the frame for the three versions are obtained. Moreover, these results are obtained by the use of the finite element method. In this comparison process, some differences are observed between the closed-form and the numerical results. This fact motivated us to propose some modifications in the characteristic matrices of the finite element model of the frame. This modification makes the results of the finite element method similar to the results of the first version of the closed-form solution. Finally, the natural frequencies and mode shapes are presented for a wide range of angles of the sloping member.

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## 1. INTRODUCTION

The free vibration analysis of beams and frames is an important problem in the structural engineering. In view of that, many researchers have devoted themselves to the study of this field, with more concentration on the beams [1-10] in contrast with the frames [11]. Moreover, the study of the closed-form solution of vibrating frame structures along with the numerical solution and the comparison of these two approaches, in order to examine the accuracy and the precision of the numerical ones, such as Finite Element Method (FEM), Boundary Element Methods (BEM), etc., is hardly considered in the literature. Herein, some of the relevant recent papers in this field will be cited. For example, Kim et al. [1] derived the frequency equations for Euler –Bernoulli beams with general restraints in a matrix form using Fourier series. The problem of free vibrations of a uniform beam with intermediate constraints and ends elastically restrained against rotation and translation had been studied by Albarracin et al. [2]. Also, Li [3] used an alternative discretization scheme based on the Galerkin method, instead of the Fourier method for solving the governing differential equation of beams. As an interesting new work on the frame structures, the free vibration of a frame with intermediate constraints and elastic restraints has been investigated by Albarracin et al. [11]. In contrast to what described above, the sufficient information was not found for sloping-frames with variable slopes.

This article deals with the free vibration analysis and determination of dynamic characteristics of a sloping-frame which consists of three members; a horizontal, a vertical, and an inclined member. The both ends of the frame are clamped, and the members are rigidly connected at joint points. The individual members of the frame are assumed to be governed by the transverse

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vibration theory of an Euler-Bernoulli beam. To solve this classical problem, a closed-form solution is firstly proposed and then, a numerical analysis is performed for some verification purposes. The closed-form solution is developed by solving the frame equations of motion, directly. For this reason, some mathematical techniques are utilized, such as Fourier transform and the well-known complementary solutions. In this way, some differential equations must be solved, and several boundary conditions should be satisfied. Herein, the more accurate derivation of one of twelve boundary conditions is the most important challenge of this paper. This boundary condition is expressed as three distinctive versions, and the free vibration parameters of the frame are attained for these three versions. Moreover, the results are obtained by the use of the FEM. In this comparison process, some differences are observed between the closed-form and the numerical results. This fact motivated us to propose some modifications in the characteristic matrices of the finite element model of the frame, with focus on the mass matrix in this article. Finally, the natural frequencies and mode shapes are presented for a wide range of angles of the sloping member.

## 2. DEFINITION OF THE PROBLEM

The free vibration analysis of the frames could be considered as a Boundary Value Problem (BVP) which involves one partial differential equation (PDE) and several boundary conditions (BCs). The differential equation is derived by means of Newton's second law of motion. Moreover, conditions specified at restraints/connections, give known information about deflections, slopes, moments and shears or their relations, are called boundary conditions. The PDE and BCs are widely explained in the next sections. It should be noted that the mentioned differential equation contains partial derivatives with respect to two independent variables, herein, space and time. Also, the dependent variable (the unknown function) is the deflection of the bending member at each space and time.

A well-known technique to eliminate one of the independent variables is to implement the integral transforms, such as Fourier transform which will be used in this study. Obviously, when the PDE consists of only two independent variables, this technique converts the partial differential equations into the ordinary one (ODE). Herein, the time variable is removed by transferring the problem from the time domain to the frequency domain. Consider a frame with two clamped restraints and one vertical, one horizontal and one inclined member as shown in Figure 1. The members at joint points are rigidly connected.

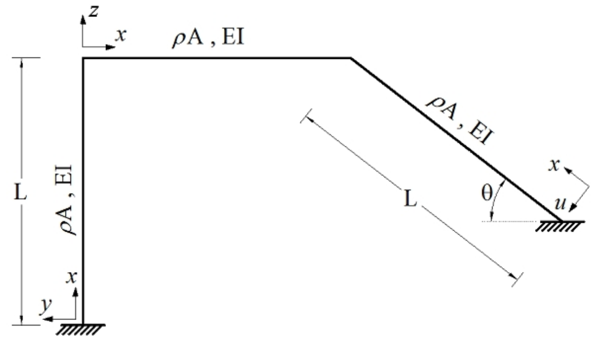


**Figure 1.** The sloping- frame

The individual members of the frame are assumed to be governed by the Euler–Bernoulli beam theory. The geometrical and mechanical properties and the length of three uniform members are the same. The flexural rigidity of the member is denoted by EI. Also, $\rho$ is the mass density and $A$ is the cross-sectional area of the bending member. The angle among the inclined member and the horizontal direction is shown by $\theta$ which is assumed between 0° and 90°. Furthermore, the displacement functions for the vertical, horizontal and inclined members are $y$ , $z$ and $u$ , respectively.

### 2. 1. Differential Equation

Herein, a uniform Euler-Bernoulli beam is considered as an individual member of the frame shown in Figure 1. The equation of motion for free flexural vibrations of this uniform elastic beam ignoring shear deformation and rotary inertia effects is:

$$\mathrm{EI}\,\frac{\partial^4 y(x,t)}{\partial x^4} + \rho\mathrm{A}\,\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{1}$$

where $y(x,t)$ is the lateral displacement at distance $x$ along the length of the beam and time $t$ , EI is the flexural rigidity of the beam, $\rho$ denotes the mass density and $A$ is the cross-sectional area of the beam. As it is clear, Equation (1) is expressed in the time domain which the Fourier transform can easily convert it into the frequency domain as:

$$Y^{IV} - \alpha^4 Y = 0 \qquad ; \qquad \left(\alpha^4 = \frac{\rho\mathrm{A}}{\mathrm{EI}}\omega^2\right) \tag{2}$$

where $Y(x,\omega)$ is the Fourier transform of $y(x,t)$ and $\omega$ denotes the circular frequency. Obviously, Equation (2) is a homogenous ordinary differential equation with the following complementary solution:

$$Y(x,\omega) = c_1 \sin\alpha\, x + c_2 \cos\alpha\, x + c_3 \sinh\alpha\, x + c_4 \cosh\alpha\, x \tag{3}$$

Due to the fact that three members of the frame shown in Figure 1 have the same $\rho\mathrm{A}$ , EI and $\omega$ , the general form of the above-mentioned solution is similar for all members. However, the coefficients are different and consequently, each element has an individual displacement function: $Y(x,\omega)$ for vertical member (repeat Equation (3)), $Z(x,\omega)$ for the horizontal and $U(x,\omega)$ for inclined one:

$$Z(x,\omega) = c_5 \sin\alpha\, x + c_6 \cos\alpha\, x + c_7 \sinh\alpha\, x + c_8 \cosh\alpha\, x \tag{4}$$

$$U(x,\omega) = c_9 \sin\alpha\, x + c_{10} \cos\alpha\, x + c_{11} \sinh\alpha\, x + c_{12} \cosh\alpha\, x \tag{5}$$

From now on, capital letters $Y, Z$ and $U$ are replaced by $y$, $z$ and $u$ for simplicity in notation.

### 2. 2. Boundary Conditions (B.C.s)

As it has been mentioned before, the main target of this paper is to investigate the boundary conditions which are usually used in the exact solution of the free vibration of frames. Therefore, in this section the boundary conditions of the system would be focused more.

As it is clear from Equations (3)-(5), the entire system has 12 unknown constants, which can be solved through the satisfaction of 12 boundary conditions. These boundary conditions are thoroughly illustrated in this section in the following order: the natural boundary conditions of the fixed ends, the continuity conditions for slopes, moments and displacements between each pair of the three members meeting at the joints, and the equation of equilibrium of forces and moments which is governed and expressed in three versions. At clamped ends the natural boundary conditions are:

$$y(0) = 0\ ;\ y'(0) = 0\ ;\ u(0) = 0\ ;\ u'(0) = 0 \tag{6-9}$$

All members are rigidly connected. Therefore, the slope functions are continuous and the rotations of the members are identical at connections:

$$y'(\mathrm{L}) = z'(0)\ \ ;\ \ u'(\mathrm{L}) = z'(\mathrm{L}) \tag{10-11}$$

Also, moments at both joint points must be in equilibrium as it is clarified in Figure 2:

$$\mathrm{EI}\ y''(\mathrm{L}) = \mathrm{EI}\ z''(0)\ \ ;\ \ \mathrm{EI}\ u''(\mathrm{L}) = -\mathrm{EI}\ z''(\mathrm{L}) \tag{12-13}$$

Other boundary conditions will be obtained by considering the “bending deformation diagram” shown in Figure 3. It should be mentioned, this figure is sketched based on the fundamental hypothesis of the Euler-Bernoulli theorem, i.e., neglecting the axial deformation. By this, the length of all elements remains invariable.

$$y(\mathrm{L}) = u(\mathrm{L})\sin\theta\ ;\ z(0) = 0\ ;\ y(\mathrm{L}) = -z(\mathrm{L})\tan\theta \tag{14-16}$$

Up to now, eleven boundary conditions are provided. In the next three sections, the 12th boundary condition which is the last one would be illustrated in three distinctive versions.

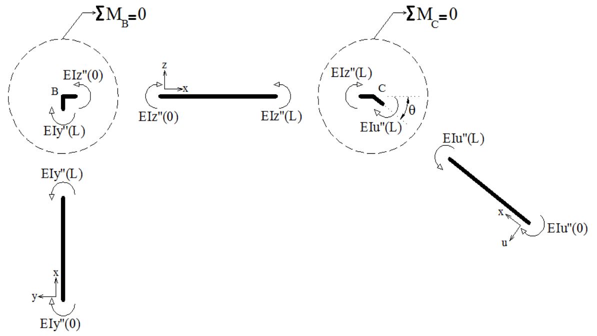


**Figure 2.** Equilibrium of moments at connections

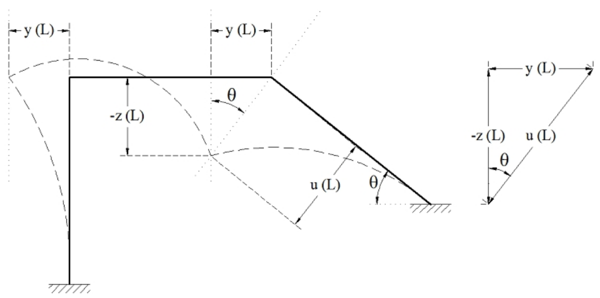


**Figure 3.** Bending deformation diagram based on neglecting of the axial deformation

### 2. 2. 1. The First Version of 12th B.C.

The first version of 12th boundary condition is based on the equilibrium of axial forces for the horizontal member. Let us consider the free body diagram shown in Figure 4. The equilibrium of the forces in the vertical and horizontal directions at joint points B and C gives the axial forces at both sides of the beam. Therefore, $\mathrm{N}_2$ and $\mathrm{N}_3$ are obtained as:

$$\mathrm{N}_2 = \mathrm{EI}\ y'''(\mathrm{L}) \tag{17}$$

$$\mathrm{N}_3 = \frac{\mathrm{EI}}{\sin\theta}[z'''(\mathrm{L})\cos\theta - u'''(\mathrm{L})] \tag{18}$$

The horizontal displacement for all points of the beam is constant and equal to $y(\mathrm{L})$. Consequently, the beam horizontal acceleration is $\ddot{y}(\mathrm{L})$, and by Newton’s second law of motion in the horizontal direction, the resultant of $\mathrm{N}_2$ and $\mathrm{N}_3$ is equal to the mass of the beam times the horizontal acceleration $\ddot{y}(\mathrm{L})$

$$\mathrm{N}_2 - \mathrm{N}_3 = m\,\ddot{y}(\mathrm{L}) \tag{19}$$

Substituting $m = \rho\mathrm{L}$, $\ddot{y}(\mathrm{L}) = -\omega^2 y(\mathrm{L})$, $\omega^2 = \mathrm{EI}\alpha^4/\rho$, Equation (17) and (18) into Equation (19), 12th boundary condition becomes:

$$y'''(\mathrm{L}) + \frac{1}{\sin\theta}u'''(\mathrm{L}) - \frac{1}{\tan\theta}z'''(\mathrm{L}) = -\alpha^4\,\mathrm{L}\ y(\mathrm{L}) \tag{20}$$

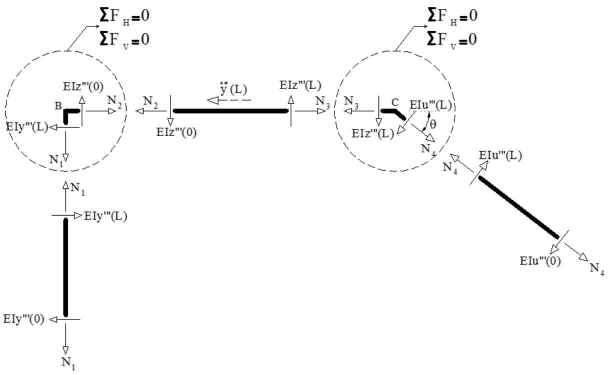


**Figure 4.** Free body diagram of the beam for the first version

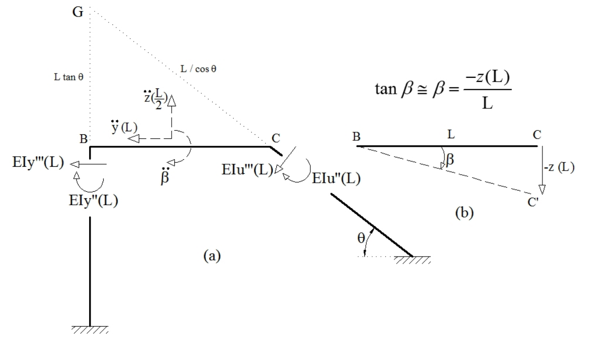


**Figure 5.** (a) Free body diagram of the beam for the second version and (b) Beam as a rigid bar

**2. 2. 2. The Second Version of 12th B.C.** Utilizing the equilibrium of moments instead of the equilibrium of forces (Figure 5-a) and assuming the beam as a rigid bar with a central concentrated inertia (Figure 5-b) leads to the second version of 12$^{th}$ boundary condition. Equilibrium of moments will be considered about point G , located at the intersection point of lines extended from the vertical and inclined members. Similar to the previous case, the beam horizontal acceleration is $\ddot{y}(\mathrm{L})$ for all points. However, the beam vertical acceleration is variable for each point. Herein, the beam vertical acceleration is assumed equal to the vertical acceleration at the middle point of the beam, i.e., $\ddot{z}(\mathrm{L}/2)$. Moreover, the rotational acceleration is $\ddot{\beta}$ (while $\beta$ is the rotation of the rigid bar as shown in Figure 5b), and the rotary inertia is $\mathrm{I}=m\mathrm{L}^2/12$. Using Newton's second law of motion as the equation of moment equilibrium about point G gives:

$$\sum \mathrm{M_G} = \mathrm{I}\ddot{\beta} + (ma)\,d \tag{21}$$

where $m=\rho\mathrm{L}$ is the beam mass, $a$ is the translational accelerations of the beam middle point, and $d$ dentes the distance between the beam middle point and point G, then:

$$\begin{aligned}&\mathrm{EI}\left( y'''(\mathrm{L})\cdot \mathrm{L}\tan\theta + u'''(\mathrm{L})\cdot\frac{\mathrm{L}}{\cos\theta} + y''(\mathrm{L}) + u''(\mathrm{L}) \right) = \\ &\frac{\rho \mathrm{L}^3}{12}\ddot{\beta} + \rho\mathrm{L}\cdot\ddot{y}(\mathrm{L})\cdot\mathrm{L}\tan\theta - \rho\mathrm{L}\cdot\ddot{z}(\mathrm{L}/2)\cdot\frac{L}{2}\end{aligned} \tag{22}$$

Substituting $\ddot{\beta}=-\omega^2\beta$, $\ddot{y}(L)=-\omega^2 y(L)$, $\ddot{z}(L/2)=-\omega^2 z(L/2)$ and $\omega^2=\mathrm{EI}\alpha^4/\rho$ into Equation (23), 12$^{th}$ boundary condition would be obtained as:

$$\begin{aligned}&y''(\mathrm{L}) + u''(\mathrm{L}) + \mathrm{L}\left( y'''(\mathrm{L})\tan\theta + \frac{u'''(\mathrm{L})}{\cos\theta} \right) = \\ &\mathrm{L}^2\alpha^4\left( \frac{z(\mathrm{L}/2)}{2} + \frac{z(\mathrm{L})}{12} - y(\mathrm{L})\tan\theta \right)\end{aligned} \tag{23}$$

**2. 2. 3. The Third Version of 12th B.C.** In fact, the third version of 12$^{th}$ boundary condition could be considered as a modified form of the second version. Similar to that, the new version is based on the moment equilibrium of the beam about point $G$ in Figure 5. However, in contrast to the second version, it is assumed that beam is flexible. Neglecting the rigidity assumption of the beam causes major changes in the formulation which are presented in the following section.

Consider a small portion of the beam as shown in Figure 6. In deformed frame, this portion transfers to the new position with new horizontal and vertical distances from the previous position $y(L)$ and $z(x)$ respectively. Consequently, translational accelerations in horizontal and vertical directions become $\ddot{y}(L)$ and $\ddot{z}(x)$.

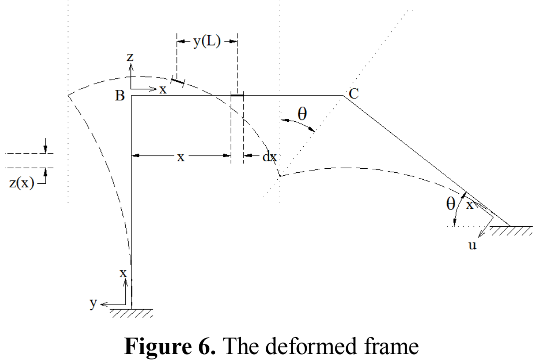


**Figure 6.** The deformed frame

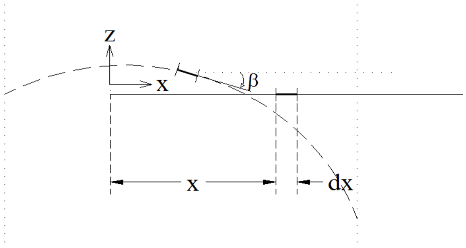


**Figure 7**. A small portion of the beam

Also as shown in Figure 7, the rotation of the portion is $\beta(\cong \tan\beta)$ which equals to $-z'(x)$. The minus sign is due to the clockwise rotation of $\beta$. The clockwise moment caused by the (H)orizontal acceleration of the portion about point $G$, as shown in Figure 8, can be expressed as:

$$(d\mathrm{M_G})_\mathrm{H} = \ddot{y}(\mathrm{L})\, dm \cdot \mathrm{L} \tan\theta \tag{24}$$

where $dm$ is the mass of the portion which equals to $\rho\, dx$. The term $\ddot{y}(\mathrm{L})\, dm$ is the force caused by the acceleration $\ddot{y}(L)$. The total moment about point $G$ will be obtained by integrating Equation (24) as follows:

$$(\mathrm{M_G})_\mathrm{H} = \int_0^\mathrm{L} \mathrm{L}\tan\theta\, \ddot{y}(\mathrm{L})\, dm \tag{25}$$

Substituting $\ddot{y}(L) = -\omega^2 y(L)$ and integrating gives:

$$(\mathrm{M_G})_\mathrm{H} = -\rho \mathrm{L}^2 \omega^2 y(\mathrm{L}) \tan\theta \tag{26}$$

Similarly, the clockwise moment caused by the (V)ertical acceleration about point $G$ becomes:

$$(d\mathrm{M_G})_\mathrm{V} = -x \ddot{z}(x)\, dm \tag{27}$$

The term $\ddot{z}(x)\, dm$ is the force caused by the acceleration $\ddot{z}(x)$ in the portion. Substituting $\ddot{z}(x) = -\omega^2 z(x)$ and integrating Equation (27) leads to the total moment about point $G$:

$$(\mathrm{M_G})_\mathrm{V} = \omega^2 \rho \int_0^\mathrm{L} x z(x)\, dx \tag{28}$$

In addition to the moments caused by the horizontal and vertical accelerations, i.e., $(\mathrm{M_G})_\mathrm{H}$ and $(\mathrm{M_G})_\mathrm{V}$, the (R)otational acceleration $\ddot{\beta}$ also creates a moment about point $G$. This moment for the mentioned portion is:

$$(d\mathrm{M_G})_\mathrm{R} = \ddot{\beta} \cdot d\mathrm{I} \tag{29}$$

where $d\mathrm{I}$ is the rotary inertia of the portion. To obtain $d\mathrm{I}$, let us consider a bar of length $x$ as shown in Figure 9. The rotary inertia of this bar is equal to $\rho x^3/12$. Differentiation of this term yields the rotary inertia of the portion of length $dx$ as $\rho x^2 dx/4$. Substituting $\ddot{\beta} = \omega^2 z'(x)$ and $dI = \rho x^2 dx/4$ leads to the total moment caused by the rotational acceleration of all beam points.

$$(\mathrm{M_G})_\mathrm{R} = \frac{1}{4}\rho\omega^2 \int_0^\mathrm{L} x^2 z'(x)\, dx \tag{30}$$

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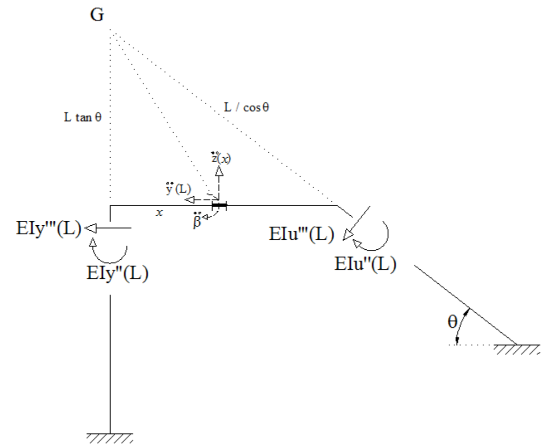


**Figure 8.** Free body diagram of the beam for the third version

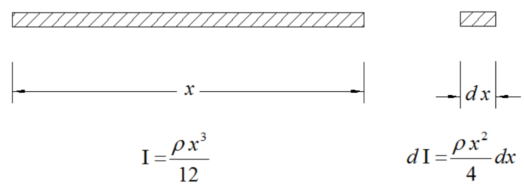


**Figure 9.** Rotary inertia of the bar

Consider Figure 8 once more. By Newton's second law of motion as the equilibrium of clockwise moments about point G yields:

$$\mathrm{EI}\left( y'''(\mathrm{L}) \cdot \mathrm{L}\tan\theta + u'''(\mathrm{L}) \cdot \frac{\mathrm{L}}{\cos\theta} + y''(\mathrm{L}) + u''(\mathrm{L}) \right) = (\mathrm{M_G})_\mathrm{H} + (\mathrm{M_G})_\mathrm{V} + (\mathrm{M_G})_\mathrm{R} \tag{31}$$

Substituting Equations (27), (29) and (31), and $\omega^2 = \mathrm{EI}\alpha^4/\rho$ into Equations (32) gives the third version of the 12th boundary condition as:

$$y''(\mathrm{L}) + u''(\mathrm{L}) + \mathrm{L}\left( y'''(\mathrm{L})\tan\theta + \frac{u'''(L)}{\cos\theta} \right) = \alpha^4 \left( -\mathrm{L}^2 y(\mathrm{L})\tan\theta + \int_0^\mathrm{L} x z(x)\, dx + \frac{1}{4}\int_0^\mathrm{L} x^2 z'(x)\, dx \right) \tag{32}$$

As it is clear, the above equation is in integral form in contrast to the previous versions.

## 3. DETERMINATION OF THE CLOSED-FORM SOLUTION

Substituting Equations (3)–(5) in boundary conditions (6)–(16) in the vicinity of 12th boundary condition in each version, one obtains a set of 12 homogeneous equations in the constants $c_i, i = 1,2,\ldots,12$. Since the system is homogeneous for existence of a non-trivial solution, the determinant of coefficients must be equal to zero. This procedure yields the frequency equation as:

$$S(\mathrm{L},\alpha,\theta)=0 \tag{33}$$

## 4. MODIFICATION OF MASS MATRIX

Up to now, the free vibration analysis of under study system has been performed utilizing two different approches. The closed-form solution was selected as an accurate method along with the famous numerical technique, i.e., finite element method, in order to assess the accuracy of the numerical one, exclusively to obtain the natural frequencies and the mode shapes of the mentioned sloping frame. Subsequent to derivation of the required formulas and producing the necessary programs, we solved the problem quantitatively and obtained the numerical results. In this process, the first thing that seemed interesting was some differences between the results of the two mentioned approaches. By this presupposition that the closed-form solution is more accurate and could be assumed as the base of comparison, we tried to match the finite element results with the closed-form ones. For this purpose, firstly, the usual attempts, such as increasing the element numbers and the degree of interpolation functions, were examined thoroughly and no magnificent change was not observed. Therefore, the modification of the characteristic matrices, i.e., mass and stiffness matrices, was preferred as the next stage of modification. Herein, adjustment of the mass matrix was considered and investigated comprehensively. To do this, several added mass, the translational and rotational ones were examined, the results of the modified model were obtained and the required comparisons were carried out. In the next section, some important results are shown and the others are omitted for brevity. It should be mentioned that the best modification which coincided appropriately with the closed-form results was to add the single translational mass which is excited only by the horizontal acceleration of the beam. Moreover, the amount of the mentioned mass should be equal to the beam mass.

## 5. RESUTS

As explained in the previous section, the finite element method was utilized as a numerical analysis approach in the present study such as many other studies which are carried out recently in this field. Therefore, a special purpose computer program was developed based on the theory explained in the previous section. The program utilizes both the original and the modified mass matrices. There is an option available in this analysis tool to add one horizontal added mass. Also, a simple code was produced based on the formulation proposed in this article as a closed-form solution to obtain the natural frequencies and mode shapes of the frame.

Utilizing the above-mentioned programs, the free vibration responses of sloping-frame are obtained for several values of $\theta$, as a discrete spectrum ($\theta$ = 0, 5, 10,..., 90$^\circ$). The first value represents L-shape frame, while $\theta=90^\circ$ represents the well-known portal frame. It should be mentioned that response quantities which will be presented in this section are the values of dimensionless eigen frequency $\lambda$. These values are obtained for various amounts of angle $\theta$. For the first five modes, the amount of eigenvalues $\lambda$ is plotted versus the values of angle $\theta$ as shown in Figures 10-14. As it is clear, these values are similar for the modified finite element method with added mass and the first version of closed-form solution. Furthermore, some comparisons are illustrated in each figure for the analytical and numerical approaches. This would help to capture a feeling of the accuracy obtained in the closed-form solution versus the finite element technique. However, prior to this presentation, it is worthwhile to have a glance at the comparison for natural frequencies. It is observed that the response of the first version of the closed-form solution in each mode matches very well with the modified finite element response, such that it is hardly distinguishable from the corresponding exact curve.

The approximate linear trend of the eigenvalues versus the mode numbers is displayed in Figure 15 for the first version of closed-form solution (similar to FEM with added mass). Also, Figure 16 illustrates the effect of the number of bending elements (which is denoted by abbreviation NE) on the accuracy of the modified finite element response. This investigation is carried out for $\theta=30$ at the first 10 modes using several NE. Moreover, some mode shapes are illustrated for some values of $\theta$ = 0,30, 60, 90$^\circ$ as shown in Figures 17-20.

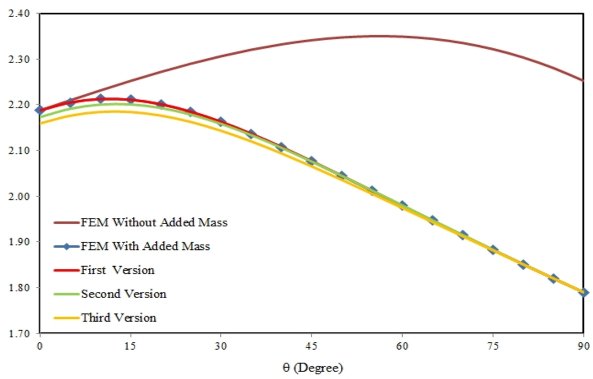


**Figure 10.** The first mode eigenvalues of sloping-frame

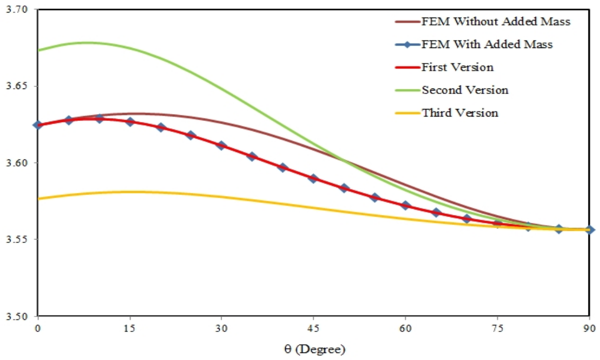


**Figure 11.** The second mode eigenvalues of sloping-frame

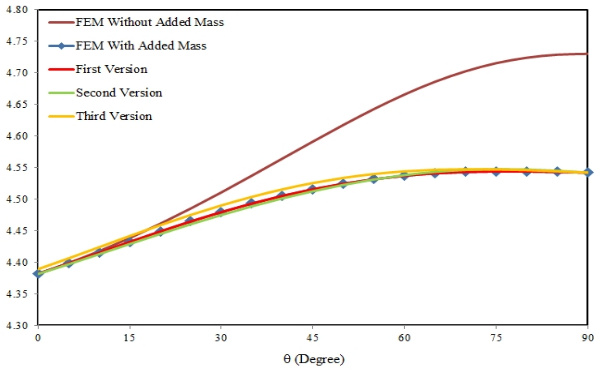


**Figure 12.** The third mode eigenvalues of sloping-frame

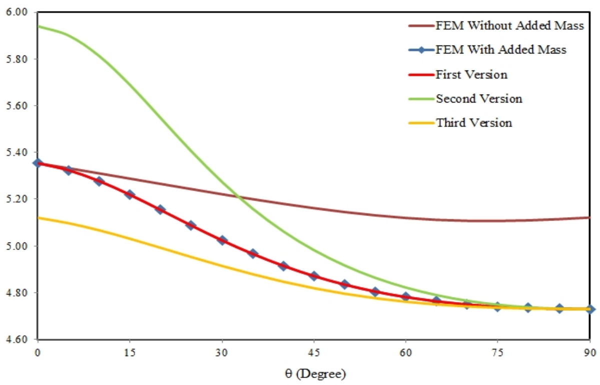


**Figure 13.** The fourth mode eigenvalues of sloping-frame

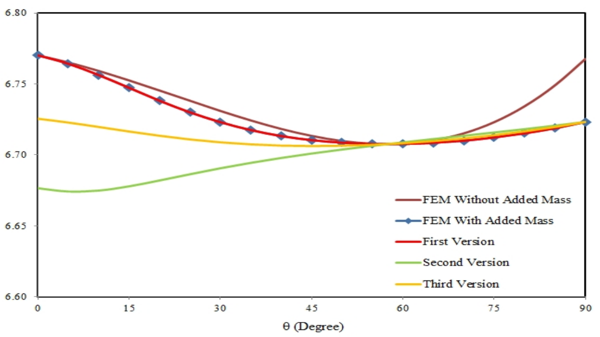


**Figure 14.** The fifth mode eigenvalues of sloping-frame

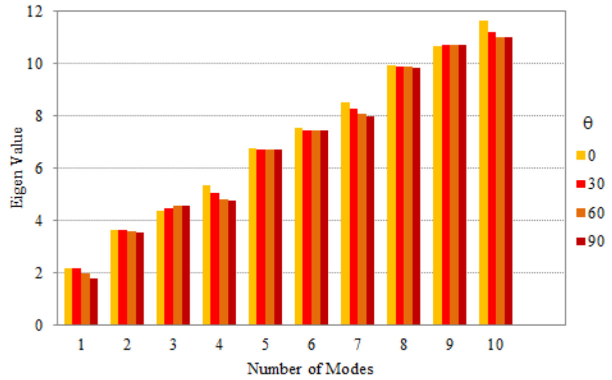


**Figure 15.** Eigenvalues of the first version of closed-form solution (similar to FEM with added mass)

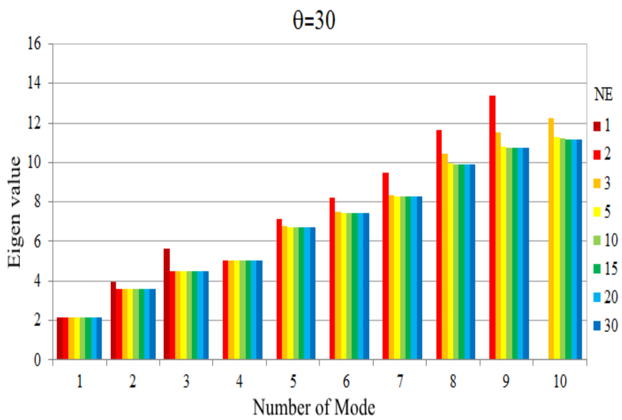


**Figure 16.** Effect of number of element (NE) on eigenvalues for $\theta = 30^{\circ}$


**Figure 17.** The first five mode shapes of the sloping-frame with $\theta = 90^{\circ}$

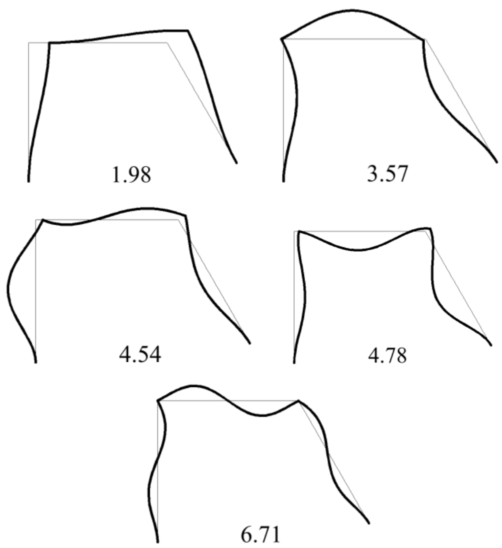


**Figure 18.** The first five mode shapes of the sloping-frame with $\theta = 60^o$

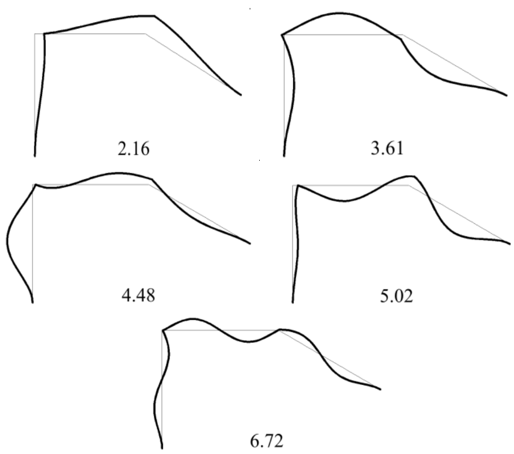


**Figure 19.** The first five mode shapes of the sloping-frame with $\theta = 30^o$

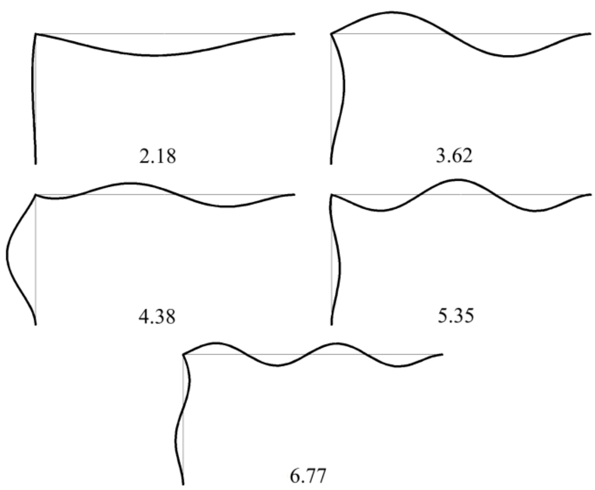


**Figure 20.** The first five mode shapes of the sloping-frame with $\theta = 0^o$

## 6. CONCLUSION

The free vibration analysis of a sloping-frame was studied and the dynamic characteristics of the system were determined. For this reason, a closed-form solution was developed which was based on the satisfaction of the both differential equations and boundary conditions, simultaneously. In this way, the $12^{th}$ boundary condition which could be considered as a more challenging equation of the problem was expressed as three versions. Also, the finite element method was utilized and the results were obtained to be compared with those of the closed-form solutions. The natural frequencies and mode shapes were presented for different values of the angle $\theta$ in the inclined member. Moreover, two special cases were presented and discussed. Overall, the main conclusions obtained by the present study can be listed as follows:

* It is observed that natural frequencies of the original finite element method are generally smaller than the natural frequencies obtained in all three versions of the analytical approach. However, it is noted that the results corresponding to the modified finite element model with a change in the mass matrix are getting closer to the exact results and coincides with the first version of the closed-form solution. Moreover, the comparison between the analytical and numerical techniques in relation with the variation of the angle $\theta$ in inclined member for several modes are thoroughly discussed, as follow:

* In the first mode, the difference between the natural frequency of the original finite element method and the results of the analytical approach increases gradually as the angle $\theta$ grows. The similar trend is observed for the third and fourth mode. On the contrary, the mentioned difference increases for the median values of the angle $\theta$ in the second mode.

* In general, the parameter $\theta$ has a significant effect on the frequency parameters of the sloping-frame. This fact could be clearly proved by the comprehensive sensitivity analysis that was carried out in the result section. Also, this is true for all three versions of the analytical solution considered.

* Almost in all modes, a single special value of $\theta$ could be introduced as an optimum value which lead to the maximum/minimum amount of the natural frequency. For example, in the first mode, this optimum value is about ten degrees (lead to max.) and in the fifth mode, it is about sixty degrees (lead to min.).

* In the special case of $\theta = 90^o$ which sloping-frame converts into the portal frame, all three versions of the analytical approach converge into the same natural frequency, approximately for all modes. In fact, there is no considerable difference between several models presented to express the $12^{th}$ boundary condition of the frame in the case $\theta = 90^o$.

* As it is obvious from the presented results for all investigated modes, the finite element solutions converges into the first version of closed-form solution, of course by imposing the proposed modification on the mass matrix.

* Although, the analytical approach is not in general as simple and programmable as the numerical techniques, it is a reliable approach and more accurate. Furthermore, it is much easier than the experimental efforts. Moreover, it has been shown that the analytical technique could solve the special case of the sloping-frame.

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# Free Vibration Analysis of a Sloping-frame: Closed-form Solution versus Finite Element Solution and Modification of the Characteristic Matrices

**TECHNICAL NOTE**

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**چکیده**

این مقاله به بررسی تحلیل ارتعاش آزاد و مشخص کردن پارامترهای دینامیکی یک قاب شیبدار می‌پردازد. ابتدا، یک روش دقیق ارائه شده است و سپس به منظور صحت‌سنجی، تحلیل عددی انجام گرفته است. روش دقیق بر اساس حل مستقیم معادلات حرکت قاب است. به این منظور، از برخی روش‌های ریاضی مانند تبدیل فوریه و جواب کامل استفاده شده است. در این حالت، یکسری از معادلات دیفرانسیل بایستی حل شوند و شرایط مرزی نیز باید ارضا شوند. مهمترین چالش این مقاله دقیق-ترین بدست آوردن دوازدهمین شرط مرزی است. این شرط مرزی به سه صورت مجزا بدست آمده است و پارامترهای ارتعاش آزاد برای هر سه حالت بدست آمده است. علاوه‌براین، این پارامترها با استفاده از روش اجزای محدود نیز بدست آمده‌اند. در این جا، برخی تفاوت‌ها بین نتایج حاصله مشاهده می‌شود. این واقعیت ما را به انجام برخی اصلاحات در ماتریس-های ویژگی روش اجزای محدود قاب ترغیب نمود. این اصلاحات باعث یکسان شدن نتایج حاصل از روش اجزای محدود با نتایج حالت اول روش دقیق می‌شود. در نهایت، فرکانس‌ها طبیعی و شکل‌مودهای قاب برای زوایای مختلف ارائه شده‌اند.

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