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A Nonlinear Model for a Capacitated Location-allocation Problem with Bernoulli Demand using Sub-sources

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Providing permanent services for customers is one of our today's difficulties. In this paper, a capacitated multi-facility location-allocation model with stochastic demands based on a well-known Bernoulli distribution function proposed to solve the problem. In this discrete environment, besides the capacitated facilities, the capacitated sub-sources of each facility could be utilized to satisfy the demands of customers. The proposed model is capable of allowing each facility to be interacted with its sub-source as well as other facility and their sub-sources. The objective function is to find the optimal locations of facilities among a finite number of potential locations and optimal allocation of the demand points (customers) to the operated facilities. Thus, the total sum of establishment costs of the facilities, costs of allocation the costumers to the operated facilities and the expected values of servicing and outsourcing costs are minimized. To display the applicability of the model, a numerical example is provided and computational results are reported.

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1. INTRODUCTION

Facility location problem has been continuously studied in operations research. This is an optimization problem in which the goal is to minimize the cost of operated facilities to serve the demand points. Initially, Weber [1] introduced the location theory in which a single warehouse location problem for minimizing the total distance between a warehouse and a finite number of costumers was considered. Later, many authors focused on the different applications of the location theory. A discrete location theory was addressed by Mirchandani and Francis [2] in which the goal was to minimize the total penalty costs of points that their demands were not satisfied. Francis et al. [3] studied some general models like single/multi facility location problems, quadratic assignment location problems and covering problems. Farahani et al. [4] studied a dynamic single facility location problem beside relocation problem. The goal of this study is not to find locations and/or relocations for the entire horizon, but to find out an optimal or near-optimal solution for the problem over an infinite horizon. Nickel and Puerto [5] studied the network-based location models mainly about median location problem.

The location-allocation (LA) problems are to locate a set of facilities and assign the existing customers to the facilities for satisfying the demands. For the first time, Cooper [6] introduced the location-allocation problem which was applied in several applications such as: emergency service systems, public services, telecommunication networks and etc. Later, a capacitated location-allocation problem was studied by Murtagh and Niwattisyawong [7]. This problem was one of the most prominent subjects in the location-allocation problems field in which the capacity constraints on facilities were considered. More extensive studies on the location-allocation problems could be found in Badri [8] and Hokey et al. [9].

In the deterministic versions of the location-allocation problems, demands of customers are assumed to be well-known, which is not satisfied in most of real world applications. When we are concerned with the determination of demanding customers, we could never exactly know which customer has request for service. Therefore, a reasonable approach is to assume the request for service occurs randomly. This gives rise to the probabilistic versions of the facility location problems. Zhou and Liu [10] represented a capacitated

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location-allocation problem with stochastic demands. By gathering a network simplex algorithm, stochastic simulations and genetic algorithm, the authors proposed a hybrid intelligent algorithm to solve the problem. A queueing location model for locating the emergency vehicles was presented by Marianov and Revelle [11] where the goal was to maximize the availabilities of vehicles. More details about the queueing location problem under uncertainty could be seen in Larson [12], Fernández et al. [13] and Silva and Serra [14]. Kuvelis and Yu [15] investigated the robustness approach of decision making in location analysis. Beraldi and Bruni [16] extended the model was presented by Kuvelis and Yu [15] in which the authors represented a stochastic programming model for emergency medical services. In order to solve the problem, an exact solution method and three heuristic methods was introduced. To survey the relationship between a manufacturer and a retailer in a two-echelon supply chain, a game theory approach was introduced by Hua and Li [17] in which a single period product is produced and sold. Pan et al. [18] represented a problem of looking for the best pricing and ordering of a single product with uncertain demand where the goal was to maximize the expected profit. For defining the uncertainty of each customer's demand, there are two alternatives, probabilistic constraints and multi stage stochastic programs. Chapman and White [19] exerted appropriate level of effort on a model with probabilistic constraints in a classical location set covering problem in order to take into account the randomness of service availability. The proposed model has never been used due to its mathematical complexities. Daskin [20] simplified the model of Chapman and White [19] supposing independence and general busy probabilities among servers and proposed a maximum expected covering location problem (MEXCLP). The objective of MEXCLP model was to maximize the expected value of population coverage so that a fixed number of facilities have to be placed on a network. Furthermore, the author introduced an exchange-based heuristic algorithm that approximates the solution of the problem for all values of the probability of a vehicle being busy. Logendran and Terrell [21] introduced a stochastic uncapacitated transportation plant location-allocation model with the goal of maximizing expected profits. In addition, they proposed several heuristics to solve the model. Laporte et al. [22] considered an uncapacitated facility location problem with uncertainty in demands, selling prices, transportation and production costs. To solve the problem, a stochastic integer linear programming model and a branch and cut solution approach was proposed by the authors. Davoud Pour and Nosraty [23] proposed a meta-heuristic method, namely ant-colony algorithm to solve a multi-facility location problem with rectangular and Euclidean distances. This meta-heuristic algorithm obtains the best solutions for problem instances of up to 20 new facilities. Louveaux and Peeters [3] focused on an uncapacitated facility location problem with stochastic demands in which a set of scenarios were defined and each scenario occurred with some specific probabilities. A capacitated facility location problem with constrained backlogging probabilities was proposed by Silva and de la Figuera [24]. Considering stochastic demands and modeling each facility as an independent queue, the authors introduced a model to obtain a formula for the backlogging probability at a potential facility location. Furthermore, they proposed a heuristic method to solve the formulated model which was based on a reactive greedy adaptive search method. Riis and Anderson [25] studied a capacity expansion problem which is gained by telecommunication networks without location decisions. Considering probabilistic demands distributed based on a well-known probability distribution function, the authors proposed a stochastic integer programming model for the problem. Manzini and Gebennini [26] extended mixed-integer programming optimization models to design and manage dynamic multi-stage and multi-commodity location-allocation problems. A mixed-integer linear programming solver was used to solve the proposed model in the case of complex industrial applications. Many researchers studied the stochastic optimization problems along with Bernoulli demands. Conde [27] studied a robust location–allocation problem with uncertainty in demand coefficients. Using "minimax regret" location problem, they found the optimal location of a new service when fraction of demand points must be served by the utility. To maximize the profits of the two facility supply chain, an outsourcing model and an in-house production model were introduced by Kaya [28]. The author considered the outsourcing model in which the supplier made the effort decision and the in-house model in which the manufacturer decided. To find which contracts were best to use in different cases, he compared several contracts for decentralized supply chain models. Albareda-sambola et al. [29] studied a stochastic generalized assignment problem. Moreover, they proposed an exact algorithm to optimize the expected cost of a recourse function. Subsequently, Albareda-sambola et al. [30] presented a location-routing problem with Bernoulli demands. The authors considered several heuristics and lower bounds to minimize the expected value of recourse function. A discrete capacitated facility location problem with Bernoulli demands was investigated by Albareda-sambola et al. [31]. The authors proposed two different recourse actions associated with two-stage stochastic programming problem. To calculate the expected value of each recourse action, an approximation technique which estimates the impact of stochasticness on the proposed problem was defined by authors. In addition, they introduced a priori solution and a posteriori solution approaches. The priori solution showed the location of the facilities and the allocation of the demand points to the facilities. The posteriori solution indicated an approximation of the expected value of the recourse functions. Berman and Simichi [32] proposed a single-vehicle location-routing problem with Bernoulli demand. They investigated a travelling salesman problem (TSP) and proposed a lower bound on the value of the optimum a priori tour. Jaillet [33] considered a probabilistic traveling salesman problem (PTSP) in which a closed formulation is defined to evaluate the expected value of the length of any given tour. A travelling salesman problem with probabilistic demand was studied by Laporte et al. [34]. The authors presented a linear probabilistic programming model solved by a branch-and-cut approach. Since PTSP problems seemed to be hard due to the combination of the TSP problem and uncertainty in demand, an exact algorithm was proposed by authors. Later, a probabilistic traveling salesman problem was proposed by Bianchi and Campbell [35]. The authors considered a heuristic approach for solving a heterogeneous version of probabilistic travelling salesman problem where all customers have the same probability of being realized.

In this study, a capacitated multi-facility location problem with Bernoulli demand using sub-sources (CMFLPBDS) is considered. The proposed model is to find optimal locations of facilities among a set of candidate locations and optimal allocations of the existing customers with probabilistic demands for the operating facilities. The aim of this stochastic model is to minimize the total sum of establishment costs, costs of allocating the existing demand points to the operated facilities and expected value of outsourcing function. It is assumed that there are a finite number of potential locations for establishment of facilities and each facility could only be opened when a pre-specified minimum number of demand points are allocated to it. The remainder of this paper is organized as follows. Section 2 contains the structure and the definitions of problem. A nonlinear mathematical programming model for capacitated multi-facility location-allocation problem with Bernoulli demands using sub-sources is introduced in section 3. In order to display the efficiency and application of the proposed model, a sample problem is provided in section 4. Finally, section 5 presents conclusions and future researches.

2. MODEL DEFINITION

Let $i \in \{1,2,\ldots,I\}$ be index of potential locations for establishment of the facilities and $j \in \{1,2,\ldots,J\}$ be index of demand points. The request for service of each demand point is denoted by a binary random variable δ_j so that it gets to be 1, if there is a request for service for *j*th demand point otherwise it gets to be zero. Hence, in our paper, it is possible that some demand points request service and others do not. Since, the requests for service of demand points follow a probabilistic distribution function, the random variable δ_j is defined as follows:

$$\delta_j = \begin{cases} 1, & p_j, \\ 0, & 1-p_j, \end{cases} \quad \forall j \in J,$$

According to the Bernoulli distribution, δ_j is considered to be distributed, where parameter p_j shows probability of request for service of the *j*th demand point. Therefore, the probability functions of δ_j could be inscribed as following equality:

$$f\left(\delta_j, p_j\right) = p_j^{\delta_j} (1-p_j)^{1-\delta_j}, \quad \forall j \in J,$$

It means that each demand point has request for service with probability p_j and has no request with probability $1-p_j$. The applied notations in this study are presented as follows:

a_i	Cost of establishment of facility *i*,
d_i	Minimum number of demand points to be allocated to facility *i* if it is operated,
m_i	Maximum number of demand points to be served from facility *i* if it is operated,
n_i	Maximum number of demand points that can be served by sub-source of facility *i* if it is operated,
ϖ_{ij}	Allocation cost of demand point *j* to facility *i*,
ω_{ij}	Service cost of demand point j by facility *i*,
ℓ_i	Cost of satisfying of outsourced demand of facility *i* by its corresponding sub-source,
$\rho_{ii'}$	Cost of satisfying of outsourced demand of facility *i* by facility $i' \in I \setminus i$.

In probability theory and statistics, the Bernoulli distribution, is a discrete probability distribution, which takes value of 1 with success probability p and value 0 with failure probability $1-p$. In our work, the request for service of each demand point is unitary that each demand point has request for service with probability p_j and not any demand with probability $1-p_j$. This behavior is according to the Bernoulli distribution.

Here, it is considered that when a facility is opened, at least l_i demand points have to be allocated to it. We further assume that k_i and u_i are the upper bounds of the number of demand points that can be served from facility *i* and sub-source of facility *i*, respectively. Thoroughly, customer who requests service will be called *demand customer*. In addition, it is considered that each facility is capacitated and has a sub-source which can be used if necessary. These sub-sources are as capacitated as facilities. It is notable that, the capacity constraints in both facilities and sub-sources show the maximum number of demand customers that can be served from these facilities or sub-sources and do not have any effect on the number of demand points that can be assigned to each facility. So, in this paper, exceeding the number of demand points assigned to a facility from its capacity is possible. Hence, to satisfy the additional demand customers, we resort to other facilities or their sub-sources which means *outsourcing process*. Furthermore, the additional demand customers of each facility are called *outsourced demands*. Let J_i and ϕ_i are the set of customers are allocated to *i*th facility and the number of demand customers allocated to *i*th facility (i.e. $\phi_i = \sum_{j\epsilon J_i} \delta_j$), respectively. With respect to the provided notations, it can be stated that if $\phi_i \leq m_i$, then the whole demand customers allocated to *i*th facility, receive services from the corresponding facility. Else, if among the demand points allocated to facility *i*, the number of demand customers exceeds the upper bound m_i. In order to satisfy the additional requests for service, we will resort to the outsourcing strategy. Here, outsourcing can be implemented in different methods. It is considered that each facility $i, i \in I$, has a sub-source with upper bound u_i. The innovation of our model is that we consider a capacitated sub-source for each facility. Furthermore, each facility can resort to other facilities or their sub-sources to serve the outsourced demands. In other words, if $m_i < \phi_i \leq m_i + n_i$, then to satisfy the outsourced demands, first the facility i uses the capacities of its corresponding sub-source. If $\phi_i > m_i + n_i$, then the facility i must first resort to other facilities which are $\phi_{i'} < m_{i'}, i'\epsilon I\backslash i$. If $\phi_{i'} \geq m_{i'}, i'\epsilon I\backslash i$, occurred for all facilities, it can resort to their sub-sources. It is obvious that due to this occurred outsourcing, some additional costs will be included in the problem. It is notable that for each sub-source, its first priority is to service the corresponding facility. That is, each sub-source should be used to service the corresponding facility and then other facilities, if it is necessary. In the followings, there are some decision variables for the model:

$$\underset{\forall i \in I}{y_i} = \begin{cases} 1, & \text{if a facility is operated at ith candidate location,} \\ 0, & \text{otherwise,} \end{cases}$$

$$z_{ij} = \begin{cases} 1, & \text{if costomer } j \text{ is allocated to facility } i, \\ 0, & \text{otherwise,} \end{cases} \quad \forall i \in I, j \in J,$$

$s_{ii'}$ The number of outsourced demands of facility i which satisfied by facility i', $\forall i, i' \in I, i' \neq i$,

$t_{ii'}$ The number of outsourced demands of facility i which satisfied by sub–source of facility i', $\forall i, i' \in I, i' \neq i$,

2.1. Explanation of Outsourcing Function

Primarily, we state the decision variables ζ_i and ϕ_i as follows:

$$\zeta_j = \sum_{j \in J} z_{ij}, \quad \forall i \in I \tag{1}$$

$$\phi_i = \sum_{j \in J} \delta_j z_{ij}, \quad \forall i \in I \tag{2}$$

where, ζ_i is the number of demand points allocated to facility $i, i \in I$. With respect to (2), there is a dependency between probability distribution ϕ_i and the decision variable z_{ij} on the set of $J_i = \{ j \in J : z_{ij} = 1 \}, i \in I$. Considering Bernoulli distribution function for demand of each demand point, we will have:

$$\mathbb{P}_x \ \phi_i = c_i \ = \mathbb{P}_x \ \sum_{j \in J} \delta_j z_{ij} = c_i \ = \sum_{C_i \subset J_i : |C_i| = c_i} \prod_{j \in C_i} p_j \prod_{j \in J_i \backslash C_i} (1 - p_j), \quad \forall i \epsilon I, \tag{3}$$

where, $c_i, i \in I$, is a possible value. Since Equation (3) is a non-homogeneous case, the problem will be complicated. So for simplifying the problem, we will transform this equation to a homogeneous case. It is considered that all the demand points request services with the same probability $p_j = p \ \forall j \epsilon J$.

Therefore, according to Equations (2) and (3), the variable θ_i and p where $\theta_i = \zeta_i$. That is, the distribution of variable ϕ_i depends on the number of demand points assigned to facility i and we have $\mathbb{P}_x[\![\phi_i = c_i]\!] \sim Bin(c_i; \theta_i, p)$. So, the Equation (3) will be transformed to Equation (4):

$$\mathbb{P}_x \ \phi_i = c_i \ = \binom{\theta_i}{c_i} p^{c_i} (1 - \text{p})^{\theta_i - c_i}, \quad c_i = 0, 1, \ldots, \theta_i, \ \forall i \epsilon I. \tag{4}$$

In outsourcing strategy, the sub-source of *i*th facility satisfies the outsourced demands of the given facility with cost per *outsourced demand*. Moreover, the other facilities and their sub-sources satisfy the outsourced demands of *i*th facility with costs $\rho_{ii'}$ and $\rho_{ii'} + \ell_{i'}$, $i, i' \in I, i' \neq i$, per *outsourced demand*, respectively. Considering each facility directly serves its corresponding probabilistic demand customers, then the expected value of serving cost will be evaluated as follows:

$$\mathbb{E}_{\delta}\ \text{service cost} = \sum_{i\in I}\sum_{j\in J} f\left(\delta_j = 1, p_j\right)\omega_{ij} z_{ij} = \sum_{i\in I}\sum_{j\in J} p_j \omega_{ij} z_{ij}, \tag{5}$$

Applying the homogeneous case, this expected value will be simplified as Equation (6):

$$\mathbb{E}_{\delta}\ \text{service cost} = \sum_{i\in I}\sum_{j\in J} p\,\omega_{ij} z_{ij}, \tag{6}$$

According to outsourcing strategy, the expected value of outsourcing cost by enumerating the possible values of ϕ_i is computed as follows:

$$\begin{aligned}
&\mathbb{E}_{\delta}\ \text{outsourcing cost} = \\
&\sum_{i\in I}\sum_{c_i=0}^{\theta_i} \mathbb{P}_x\ \ \phi_i = c_i\ \times \mathbb{E}_{\delta}\ \ \text{outsourcing cost} \mid \phi_i = c_i \\
&= \sum_{i\in I} \ell_i \sum_{c_i = m_i+1}^{m_i+n_i} \binom{\theta_i}{c_i} p^{c_i} \left(1-p\right)^{\theta_i - c_i} \left[c_i - m_i\right] \\
&+ \sum_{i\in I} \sum_{c_i = m_i+n_i+1}^{m_i+n_i+\sum_{i'\in I\setminus i} m_{i'} + \sum_{i'\in I\setminus i} n_{i'}} \binom{\theta_i}{c_i} p^{c_i} \left(1-p\right)^{\theta_i - c_i} [\ell_i n_i \\
&+ \sum_{i'\in I\setminus i} \rho_{ii'} s_{ii'} + \left(\ell_i + \rho_{ii'}\right) t_{ii'}]
\end{aligned} \tag{7}$$

3. PROPOSED MATHEMATICAL MODEL

The proposed problem is to find the optimum locations of facilities among a set of potential locations and the optimum allocations of the existing customers with probabilistic demands to the operated facilities. Note that both facilities and sub-sources are capacitated. The objective function is to minimize a total cost function involving the sum of costs directly proportional to the operated facilities, directly costs proportional to the allocation of demand points and costs directly proportional to the servicing and the outsourcing strategies. Some necessary variables to make the nonlinear mathematical model are presented as follow:

$$\psi_i = \begin{cases} 1, & m_i - c_i \geq 0, \\ 0, & otherwise, \end{cases} \quad \forall i \in I, \tag{8}$$

$$\gamma_{i'} = \begin{cases} 1, & c_{i'} - m_{i'} \geq 0, \\ 0, & otherwise, \end{cases} \quad \forall i' \in I \setminus i, \tag{9}$$

$$\vartheta_{i'} = \begin{cases} 1, & n_{i'} - (c_{i'} - m_{i'})\gamma_{i'} \geq 0, \\ 0, & otherwise, \end{cases} \quad \forall i' \in I \setminus i. \tag{10}$$

Generally, the objective function of the problem can be presented as:

$$\text{Min}\ \sum_{i\in I} a_i y_i + \sum_{i\in I}\sum_{j\in J} \omega_{ij} z_{ij} + \mathbb{E}\ \ \text{service cost} + \text{outsourcing cost} \tag{11}$$

where,

$$\begin{aligned}
&\mathbb{E}_{\delta}\ \ \text{service cost} + \text{outsourcing cost} = \\
&\mathbb{E}_{\delta}\ \ \text{service cost}\ + \mathbb{E}_{\delta}\ \ \text{outsourcing cost}
\end{aligned} \tag{12}$$

Finally, using Equations (6)-(12) the proposed mathematical model is formulated as follow:

$$\begin{aligned}
&\text{Min}\ \sum_{i\in I} a_i y_i + \sum_{i\in I}\sum_{j\in J} \varpi_{ij} z_{ij} + \sum_{i\in I}\sum_{j\in J} p\omega_{ij} z_{ij} \\
&+ \left[\sum_{i\in I} \ell_i \sum_{c_i = m_i+1}^{m_i+n_i} \binom{\theta_i}{c_i} p^{c_i} \left(1-\mathrm{p}\right)^{\theta_i - c_i} \left[c_i - m_i\right] \right. \\
&+ \sum_{i\in I} \sum_{c_i = m_i+n_i+1}^{m_i+n_i+\sum_{i'\in I\setminus i} m_{i'} + \sum_{i'\in I\setminus i} n_{i'}} \binom{\theta_i}{c_i} p^{c_i} \left(1-\mathrm{p}\right)^{\theta_i - c_i} [\ell_i n_i \\
&\left. + \sum_{i'\in I\setminus i} \rho_{ii'} s_{ii'} + \left(\ell_i + \rho_{ii'}\right) t_{ii'}] \right]
\end{aligned} \tag{13}$$

s.t.

$$\sum_{i\in I}\sum_{j\in J} z_{ij} \leq \sum_{i\in I} \left(m_i + n_i\right) y_i, \tag{14}$$

$$\sum_{i\in I} z_{ij} = 1, \qquad \forall j \in J, \tag{15}$$

$$\sum_{j\in J} z_{ij} \geq y_i d_i, \qquad \forall i \in I, \tag{16}$$

$$\sum_{i'\in I\setminus i} s_{ii'} + \sum_{i'\in I\setminus i} t_{ii'} \geq c_i - m_i - n_i, \forall i \in I, \tag{17}$$

$$\sum_{i\in I} s_{ii'} \leq \left(m_{i'} - c_{i'}\right)\psi_{i'}, \quad \forall i' \in I \setminus i, \tag{18}$$

$$\sum_{i\in I} t_{ii'} \leq \left(n_{i'} - \left(c_{i'} - m_{i'}\right)\gamma_{i'}\right)\vartheta_{i'}, \forall i' \in I \setminus i, \tag{19}$$

$$z_{ij} \leq y_i, \qquad \forall i \in I, j \in J, \tag{20}$$

$$2m_{i'}\psi_{i'} - 2c_{i'}\psi_{i'} \geq m_{i'} - c_{i'}, \forall i' \in I \setminus i, \tag{21}$$

$$2c_{i'}\gamma_{i'} - 2m_{i'}\gamma_{i'} \geq c_{i'} - m_{i'}, \forall i' \in I \setminus i, \tag{22}$$

$$\begin{aligned}
&2n_{i'}\vartheta_{i'} - 2c_{i'}\gamma_{i'}\vartheta_{i'} + 2m_{i'}\gamma_{i'}\vartheta_{i'} \geq n_{i'} - c_{i'}\gamma_{i'} + m_{i'}\gamma_{i'}, \\
&\forall i' \in I \setminus i,
\end{aligned} \tag{23}$$

$$y_i, z_{ij}, \psi_{i'}, \gamma_{i'}, \vartheta_{i'} \in \{0,1\}, \forall i \in I, j \in J, i' \in I \setminus i, \tag{24}$$

$$c_i, c_{i'}, s_{ii'}, t_{ii'}\ \text{are integer},\ \forall i \in I, i' \in I \setminus i, \tag{25}$$

The first term of objective function represents the cost of operating the facilities in potential locations. The second expresses the cost of the allocation of demand points to the operated facilities. The third states the costs of serving the demand points. The fourth shows the costs of outsourcing strategy of the operating facilities. Constraints (14) represent the capacity restrictions of facilities. Constraints (15) ensure that each demand point will be assigned to only one facility. Constraints (16) consider the minimum number of demand points to be allocated to any operated facility. The least number of outsourced demands of each facility is stated by constraints (17). Constraints (18) show the maximum number of capacities that can be prepared by each facility in order to satisfy the outsourced demands of other facilities. Constraints (19) indicate the maximum number of capacities that each sub-source can prepares for other facilities. Constraints (20) guarantee that the demand points can only be allocated to the operated facilities. We have swapped equalities (8) with the Equations (21), confirming that when a facility is able to give its capacities to other facilities, then ψ_i gets to be 1, otherwise it gets to be zero. Equations (9) and (10) have been swapped by constraints (22) and (23) and also, guarantee that when a sub-source is able to provide its capacities to other facilities, then ϑ_i is to be 1, otherwise it gets to be zero. Constraints (24) and constraints (25) indicate the binary and integer variables, respectively.

4. NUMERICAL EXAMPLE

To show the efficiency of the proposed stochastic model, a numerical example is provided. This example consists of 8 potential locations for establishment of facilities and 55 demand points. The model finds the optimum locations for establishment of facilities and the optimum allocations of demand customers to the facilities and their sub-sources. Values of the parameters $a_i, \varpi_{ij}, \omega_{ij}, \ell_i$ and $\rho_{ii'}$ are randomly created in the ranges [18, 26], [1, 10], [1, 10], [5, 10] and [14, 20], respectively. Values of the parameters d_i, m_i and n_i are fixed respectively to be 1, 4 and 7 for each facility $i \in I$. Furthermore, the binary parameter δ_j is randomly created based on the Bernoulli distribution. The proposed model has been coded in the LINGO 9.0 software package. The costs of establishing the facilities and satisfying of outsourced demand of each facility by its corresponding sub-source, respectively a_i and ℓ_i are given in Table 1. Also, the request for services of the demand points, δ_j and the allocation costs of demand points, ϖ_{ij} are reported in Tables 2 and 3, respectively. Table 4 shows the value of service costs of demand points, ω_{ij}. The cost of satisfying of outsourced demand of facility i by facility i', $\rho_{ii'}$ is provided in Table 5.

TABLE 1. Values of the parameters a_i and ℓ_i.

I	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
a_i	19	26	18	18	19	24	23	23
ℓ_i	5	7	9	6	8	7	9	10

TABLE 2. Values of the parameter δ_j.

j	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
j	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	
	0	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	

TABLE 3. Values of the parameter ϖ_{ij}.

i \ j	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
1	8	10	4	3	10	7	9	7	7	4	1	6	8	7	2	2	8	5	4	4	3	10	5	2	6	1	6	5
2	2	3	2	8	7	7	7	6	4	2	8	4	6	5	7	3	4	7	10	4	9	9	9	8	3	10	9	9
3	2	5	6	4	1	2	6	3	8	9	8	5	4	1	6	3	2	2	5	8	2	5	6	9	9	4	4	2
4	2	7	8	8	10	7	6	8	9	10	4	9	7	7	5	7	5	8	6	10	10	10	9	7	9	9	8	2
5	3	1	8	6	1	3	2	7	5	3	7	1	4	9	7	7	9	4	7	4	10	8	8	8	8	5	2	2
6	4	1	8	4	7	8	7	5	8	2	1	5	2	8	1	2	5	3	10	4	5	3	6	6	10	7	9	2
7	8	4	4	7	1	5	3	6	3	4	9	2	7	4	8	4	5	10	9	9	9	2	4	7	7	9	3	4
8	7	3	4	3	3	4	4	10	9	7	9	4	7	2	4	6	2	8	9	7	1	9	2	3	4	5	2	9

TABLE 3. Values of the parameter ϖ_{ij} (continued)

i \ j	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
1	8	2	5	6	2	5	8	2	7	8	6	1	4	10	2	8	4	4	6	5	9	4	10	9	10	3	1
2	8	5	9	9	1	8	5	6	10	9	10	4	3	4	4	6	4	9	7	8	3	2	9	3	7	1	4
3	7	3	9	5	4	3	2	8	8	5	9	7	3	1	2	4	8	8	2	4	2	7	2	10	8	2	5
4	2	6	7	5	5	4	10	9	10	3	1	9	7	6	3	6	2	5	3	8	9	9	2	7	4	8	4
5	8	1	6	8	10	5	2	6	3	4	10	2	8	4	5	2	2	9	2	9	5	5	5	3	10	4	5
6	5	1	6	8	7	2	7	4	4	8	2	3	5	6	4	5	4	9	7	3	7	1	10	4	6	8	5
7	3	1	7	4	1	4	2	2	6	1	9	2	5	2	5	9	7	10	5	5	8	9	5	10	9	9	9
8	2	8	2	4	2	10	8	2	5	5	8	8	10	8	7	9	4	9	8	9	5	2	7	8	6	1	4

TABLE 4. Values of the parameter ω_{ij}.

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	8	9	2	9	7	2	4	6	10	10	2	10	10	5	8	2	5	9	8	10	7	1	9	9	7	8	8	5
2	5	10	4	6	3	8	3	6	7	9	10	6	2	2	3	9	3	8	3	9	4	3	3	7	5	4	8	6
3	5	2	3	9	2	8	6	10	2	5	2	10	1	8	8	9	2	5	3	8	5	9	3	3	2	2	9	6
4	2	9	10	6	2	3	4	8	1	1	3	7	8	7	5	6	4	8	3	7	3	4	7	8	2	9	8	5
5	4	9	5	3	9	10	5	2	3	5	6	3	6	7	3	2	4	4	5	6	2	3	8	1	9	8	5	6
6	9	9	4	7	3	1	8	6	5	9	6	7	9	8	6	3	3	9	1	5	3	10	7	6	5	2	7	1
7	4	5	1	10	3	2	4	3	5	4	10	9	1	8	3	5	6	9	5	10	4	7	7	6	7	7	3	2
8	10	8	4	6	2	9	9	8	3	6	1	5	4	2	3	5	2	6	5	7	7	7	1	2	4	6	7	5

i \ j	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
1	7	3	7	1	3	1	2	8	7	4	10	1	5	4	8	8	3	5	5	7	7	8	3	7	7	2	2
2	6	9	4	8	8	4	6	2	1	6	8	9	2	6	5	1	4	2	8	4	6	2	6	3	7	7	8
3	6	2	9	7	4	6	5	2	3	2	3	3	5	1	9	10	5	5	4	9	4	2	8	5	3	5	2
4	5	5	4	6	6	8	8	7	4	8	6	4	9	9	6	7	6	3	4	5	3	9	3	3	3	3	5
5	3	5	10	6	6	3	5	7	7	5	4	10	1	9	9	8	2	3	4	7	2	7	2	7	5	8	7
6	2	6	2	8	8	8	2	7	6	10	7	8	5	5	8	2	2	3	5	8	8	2	5	6	5	7	7
7	10	3	1	6	9	7	3	4	5	10	2	9	7	4	3	5	5	2	6	3	4	6	3	4	7	3	8
8	8	7	10	6	4	2	6	8	5	2	3	2	4	5	6	5	9	6	9	7	10	3	7	4	7	7	2

TABLE 6. The results obtained for the example problem.

Facility index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Number of customers allocated to facilities	12	0	16	4	14	0	0	9
Number of demand customers	12	0	16	2	12	0	0	8
Number of outsourced demands	1	0	5	0	1	0	0	0
Explanation of outsourcing process	$s_{1,4}=1$	0	$t_{3,4}=2$ $t_{3,8}=3$	0	$s_{5,4}=1$	0	0	0

TABLE 5. Values of the parameter $\rho_{ii'}$.

i \ i'	1	2	3	4	5	6	7	8
1	0	15	18	14	20	17	16	19
2	14	0	15	18	19	20	16	17
3	18	17	0	20	14	19	15	15
4	20	20	14	0	16	15	19	18
5	17	16	18	19	0	19	20	15
6	15	15	18	17	16	0	15	14
7	16	18	20	16	15	18	0	20
8	19	14	17	15	18	14	17	0

Moreover, the probability of service request of each demand point is fixed $p = 0.8$. In Table 6, the finding resulting from the example by LINGO software is reported.

In the second row of the table, the non-zero values show the number of customers allocated to each facility. The zero values represent the relevant facilities are not opened and therefore any demand point are not allocated to them. It can be seen that facilities 1, 3, 4, 5 and 8 are opened and the number of allocated demand points are 12, 6, 4, 14 and 9, respectively. Among the allocated demand points of the facilities, a number of demand points have request of demand and other not. The number of demand customers is shown in the third row of the table which is 12, 16, 2, 12 and 8, respectively. Finally, the last row of the table indicates how each facility resorts to other facilities or sub-sources to satisfy its outsourced demands. The obtained results for the example show that facility 1 resorts to facility 4 (i.e., $s_{1,4}$=1), facility 3 resorts to sub-sources of facilities 4 and 8 (i.e., $t_{3,4}$=2 and $t_{3,8}$=3) and facility 5 resorts to facility 4 (i.e., $t_{5,4}$=1) to satisfy theirs outsourced demands. Among the allocated demand points of the facilities, a number of demand points have request of demand and other not. The number of demand customers is shown in the third row of the table which is 12, 16, 2, 12 and 8, respectively, Finally, the last row of the table indicates how each facility resorts to other facilities or sub-sources to satisfy its outsourced demands. The obtained results for the example show that facility 1 resorts to facility 4 (i.e., $s_{1,4}$=1), facility 3 resorts to sub-sources of facilities 4 and 8 (i.e., $t_{3,4}$=2 and $t_{3,8}$=3) and facility 5 resorts to facility 4 (i.e., $t_{5,4}$=1) to satisfy theirs outsourced demands.

In Figure 1, behavior of the proposed model for the example problem is displayed graphically. In this figure, the opened facilities and the number of demand customers are specified. In addition, the opened facilities and their sub-sources which have surplus capacities are marked with green lines and the other opened facilities and their sub-sources with red lines. The surplus capacities and the outsourced demands of the opened facilities and the obtained values of variables ϕ_i, $s_{ii'}$ and $t_{ii'}$ are shown in the figure.

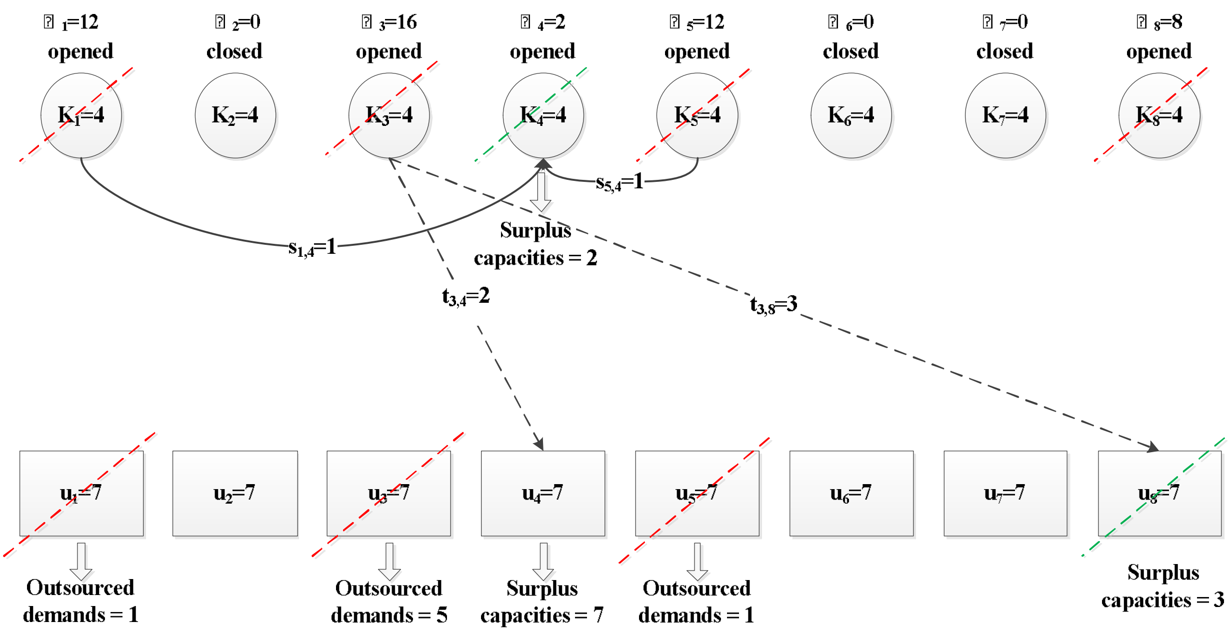

Figure 1. Example specifications

5. CONCLUSION

We proposed a stochastic nonlinear mathematical programming model for the capacitated multi-facility location-allocation problem with Bernoulli demand using the capacitated sub-sources. The objective of the problem is to determine a set of the optimum locations for facilities among a finite number of potential locations and optimum allocations of the existing demand points to the operated facilities, such that the summation of the costs of the operating facilities, allocation of customers and the expected values of servicing and outsourcing costs is minimized. A numerical example was provided to evaluate the efficiency of the proposed model. We solved the example using the LINGO 9.0 software package that led to the global optimum solutions. The experimental results showed that the proposed model was effective in finding the optimum solutions for the presented problem.

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A Nonlinear Model for a Capacitated Location-allocation Problem with Bernoulli Demand using Sub-sources

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چکیده

یکی از مشکلات امروزی ما ارائه سرویسهای دائمی به مشتریان می باشد. در این مقاله یک مدل مکان یابی- تخصیص چندتسهیله ظرفیت دهی شده با تقاضاهای احتمالی براساس تابع توزیع شناخته شده برنولی برای حل این مساله پیشنهاد شده است. به منظور تامین تقاضاهای مشتریان در این محیط گسسته، در کنار تسهیلات ظرفیت دهی شده، منابع فرعی با ظرفیت محدود برای هر تسهیل قابل استفاده می باشند. مدل پیشنهادی به هر تسهیل اجازه می دهد تا با منبع فرعی اش به خوبی رابطه اش با دیگر تسهیلات و منابع فرعیشان در تعامل باشد. تابع هدف به دنبال پیدا کردن مکان های بهینه تسهیلات در میان تعداد معدودی از مکان های بالقوه و تخصیص بهینه نقاط تقاضا (مشتریان) به تسهیلات باز می باشد، بطوریکه جمع کل هزینه های تاسیس تسهیلات، هزینه های تخصیص مشتریان به تسهیلات باز و مقادیر انتظاری هزینه های سرویس دهی و برونسپاری مینیمم شوند. به منظور نمایش قابلیت مدل، یک مثال عددی ارائه شده و نتایج محاسباتی گزارش شده اند.

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