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Bayes Interval Estimation on the Parameters of the Weibull Distribution for Complete and Censored Tests

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ABSTRACT

A method for constructing confidence intervals on parameters of a continuous probability distribution is developed in this paper. The objective is to present a model for an uncertainty represented by parameters of a probability density function. As an application, confidence intervals for the two parameters of the Weibull distribution along with their joint confidence interval are derived. The model admits complete data, as well as censored data. The estimation accuracy of the proposed model is compared to those of the existing procedures by a numerical method. The validation analysis shows that the estimation accuracy of the proposed model lead to an encouraging conclusion. It is shown that improper use of available information in the data that affects the width of the confidence intervals obtained by the existing procedures. It does not affect the coverage of the proposed confidence interval method.

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1. INTRODUCTION

Interval estimation for unknown parameters of a continuous pdf from a sample data requires sufficient statistics. Sufficient statistics arise in nearly every aspect of statistical inference. Intuitively, a sufficient statistic captures all information from sample data that are relevant to estimating the values of the unobservable parameters. In other words, a sample data is used for estimating the underlying probability distribution of a random variable that the data has been drawn from it. For example, for the family of normal distributions, the pair $(X_1+X_2+...+X_n$ and $X_1^2+X_2^2+...+X_n^2)$ are sufficient statistics. This means that the conditional probability distribution of the data $X_1, X_2,..., X_n$ given the values of $X_1+X_2+...+X_n$ and $X_1^2+X_2^2+...+X_n^2$ does not depend on the mean and the variance of the normal distribution. Similarly, for the family of Poisson distributions, the sum $X_1+X_2+...+X_n$ is sufficient for the parameter λ and for the family of uniform distributions on $(0,\theta)$, the $Max\{X_1,..., X_n\}$ is sufficient for θ. A sufficient statistic often does not exist, for example no sufficient statistic is known for the Weibull distribution [1]. This could be a problem that may prevent construction of confidence intervals for widely used distributions. There are two alternative approaches to deal with this type of problem. The first alternative is to use the Monte Carlo simulation procedure, where the results are usually summarized in some tables (see for example [1].) Since the Monte Carlo procedures are very time-consuming or require extensive simulation tables, this approach could be cumbersome [2]. The other approach for constructing an exact confidence interval uses a nonsufficient statistic; examples are given by Yang et al. [2], Chen [3], and Lawless [4]. This approach is discussed in section 3. Aljuaid [5] proposed Bayes estimators for two parameters exponentiated inverted Weibull distribution when sample is available from complete and Type-II censoring scheme. The estimators were compared with the corresponding maximum likelihood estimators using Monte Carlo simulation. Ng [6]

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derived maximum likelihood estimators (MLE) of the parameters of a modified Weibull distribution based on a progressively Type-II censored sample. He also constructed an approximate confidence intervals for the parameters based on the s-normal approximation and log-transformed MLE. He numerically evaluated the coverage probabilities of the proposed confidence intervals. Soliman [7] developed MLE, Bayes estimators for some life parameters as well as the parameters of the Burr XII model based on progressively Type-II censored samples. He obtained the Bayes estimators using both the symmetric squared-error and asymmetric general-entropy loss functions. This was performed based on the conjugate prior for the shape parameter and discrete prior for the other parameter. Zaindin and Sarhan [8] proposed MLE and least squares estimators for the parameters of the modified Weibull distribution based on Type-II censored data. Moreover, assuming a log-concave prior density function for the shape parameter of a Weibull distribution, and letting the scale parameter to have a conjugate prior distribution for a given shape parameter, Kundu [9] developed Bayesian inference of unknown parameters of the progressively censored Weibull distribution. He employed the Lindley's approximation to compute the Bayes estimates and the Gibbs sampling procedure to calculate the credible intervals. Perdon´a et al. [10] investigated the properties of the modified Weibull model, a three-parameter model that allows U-shaped hazards to be accommodated. They presented inferences for this model's parameters based on both complete and censored samples. While none of the above-mentioned studies and to the best of authors' knowledge no research provides exact confidence intervals on the parameters of a two-parameter Weibull distribution, this paper first presents a new method for constructing confidence intervals for the parameters of continuous distributions. The novelty comes from an uncertainty, where the uncertainty is due to a lack of information about unknown parameters, and the modeling. Although the proposed method, as demonstrated in section 2, does not use sufficient statistics, still provides identical results to those obtained from sufficient statistics. Then, section 3 presents an application of this method in construction of confidence intervals on the shape and scale parameters of a two-parameter Weibull distribution. The model considers complete and censored data. Two existing methods that only allow the Type-II censoring are selected to evaluate and to compare the performances of the proposed method.

2. THE METHOD

The theory behind the new confidence interval method for the parameters of continuous distributions is proposed in this section. It is first shown that constructing a confidence interval procedure is equivalent to obtaining a probability distribution. Then, the steps involved in the proposed procedure are presented and followed by selection process of the prior distribution.

2. 1. Confidence Interval Determination is Equivalent to Finding a Probability Distribution

Let $f_X(x;\theta)$ be the probability density function (pdf) of a random variable X with an unknown parameter θ. A confidence interval $I_{1-\alpha}$, which has been defined using a series of observations, is a set in which the probability of presence of θ is $1-\alpha$ [11]. Suppose there are two confidence intervals $I_{1-\alpha}$ and $I_{1-\beta}$ based on the same vector $\mathbf{x}$ of observations, in which the latter is a subset of the former. Then, the probability of presence of θ in $A=I_{1-\alpha}-I_{1-\beta}$ is $\beta-\alpha$ and by assigning different values to α and β, one can find the probability of a presence of θ in different regions. In other words, constructing confidence intervals for a parameter is equivalent to finding a probability distribution that determines the probability of presence of θ in different regions.

2. 2. The Similarity Between the Bayes Equation and the Confidence Interval Methods

A two-step process for constructing a CI is proposed as follows:

1. Prior to construction of the confidence interval, no information is available on the parameter $\boldsymbol{\theta}$. In fact, $\boldsymbol{\theta}$ can take any quantity among all possible values with an equal chance. This is called uncertainty about $\boldsymbol{\theta}$.
2. By constructing the confidence interval, we have condition the uncertainty based on the observations.

As a result, the stages involved in constructing a confidence interval are the same as those of the Bayesian estimation. This will lead to find a Bayesian based approach to construct a CI. First, a prior distribution should be identified to model the uncertainty, and then information from a given data is used to design the CI. The unknown parameter is treated as a random variable and the observed data are utilized to obtain the posterior distribution. To conduct the above two-step process, let $f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\theta})$ be the joint pdf of the random vector $\mathbf{X}$, which represents the observations taken from the distribution with unknown parameter vector $\boldsymbol{\theta}$. The second step is discussed first, and the first step is presented later (in section 4) , based on the outcome of step two. The posterior distribution in the second step is obtained as follows:

$$f_{\Theta|X}(\boldsymbol{\theta}\mid\mathbf{x})=\frac{f_{X|\Theta}(\mathbf{x}\mid\boldsymbol{\theta})f_{\Theta}(\boldsymbol{\theta})}{\int_{\forall\boldsymbol{\theta}} f_{X|\Theta}(\mathbf{x}\mid\boldsymbol{\theta})f_{\Theta}(\boldsymbol{\theta})d\,\boldsymbol{\theta}} \quad (1)$$

where, $f_{\Theta}(\boldsymbol{\theta})$ is the prior distribution of the parameter vector $\boldsymbol{\theta}$ that represents uncertainty about $\boldsymbol{\theta}$, and $f_{X|\Theta}$ is the likelihood function.

2. 3. The Prior Distribution Code Validation The key for modeling uncertainty (i.e. finding a proper $f_{\Theta}(\boldsymbol{\theta})$) is to note the integration in Equation (1) and the integration used later on the posterior distribution to construct CI. Therefore, $d\boldsymbol{\theta}$ must be defined. This is explained in the following example:

2. 3. 1. Example Let $X \approx N(\mu,\sigma^2)$ and define the distribution of X given (μ,σ^2) be a function in $(R\times R^{+},S)$ {for all random variables with normal distribution} with $X(a,b)=X_{a,b}$. Since μ and σ are considered two random variables, the X is actually the random variable of our interest. Since for a normal random variable with mean μ and variance σ^2 we know what dx is, we aim to define $d\mu$ or $d\boldsymbol{\theta}$ in terms of dx of a defined normal random variable. For example we can say $X \approx N(\mu,\sigma^2)=\sigma\times X(0,1)+\mu$. If we are interested in $d\sigma^2$ in Equation (1) (and assuming μ is known), then $dX(\mu,\sigma^2)=1/(2\sigma)\times X(0,1)d\sigma^2$. In fact if in Equation (1), we are using $d\sigma^2$ instead of $X(\mu,\sigma^2)$, we should be careful to use $d\sigma^2$ with coefficient $1/(2\sigma)\times X(0,1)$ or just coefficient $1/\sigma$, for the integration to make sense. Since we are using $f_{\Theta}(\boldsymbol{\theta})$ as an adjustment for $d\boldsymbol{\theta}$, in this example we use $f_{\sigma^2}(\sigma^2)=1/\sigma$. In section 2. 4, a more formal procedure to calculate $f_{\Theta}(\boldsymbol{\theta})$ is developed when $\boldsymbol{\theta}$ is the only unknown parameter and in section 2. 5 we will develop the procedure for a case of more than one unknown parameter.

2. 4. Interval Estimation for one Unknown Parameter Suppose there is only one unknown parameter $\boldsymbol{\theta}$ and $X(\boldsymbol{\theta})$ is a function which takes value with parameter $\boldsymbol{\theta}$. We are looking for a transformation for converting every $X(\boldsymbol{\theta})$ to reference random variable (for continuous random variables such transformation always exists since for every continuous random variable an inverse of its distribution function has uniform distribution between 0 and 1). To make calculation easier, let $O(X(\boldsymbol{\theta}_1),\boldsymbol{\theta}_1)=O(X(\boldsymbol{\theta}_2),\boldsymbol{\theta}_2)$ for every possible $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, where $O(X(\boldsymbol{\theta}),\boldsymbol{\theta})$ is the transformation. Next, we calculate $dO(X(\boldsymbol{\theta}),\boldsymbol{\theta})/d\boldsymbol{\theta}$ for both sides and eliminate repeated values, then the remaining simplified terms (which should be a function of $\boldsymbol{\theta}$) is the prior distribution $f_{\Theta}(\boldsymbol{\theta})$. For example for a normal distribution with mean 0 and unknown variance σ^2 for $X_1 \approx N(0,\sigma_1^2)$ and $X_2 \approx N(0,\sigma_2^2)$, these transformations can be obtained by dividing the random variables by their standard deviations, to transform the random variables to standard normal variables, $Z_1=(x_1-0)/\sigma_1, Z_2=(x_2-0)/\sigma_2$ that do not depend on their variances, i.e. $g_{Z_1}(z_1)$ and $g_{Z_2}(z_2)$ for all σ_1 and σ_2.

Next, the differentials of the transformations $g_{Z_1}(z_1)$ and $g_{Z_2}(z_2)$ with respect to the unknown parameters are obtained, and after the simplification, the coefficient for $d\theta$ is chosen to be $f_{\Theta}(\theta)$. In this example, the differentials are equal, that is $-X_1 d\sigma_1^2/(2\sigma_1^3)=-X_2 d\sigma_2^2/(2\sigma_2^3)$. Since $Z_1=Z_2$, the simplification implies $d\sigma_1^2/\sigma_1^2=d\sigma_2^2/\sigma_2^2$. This means identical differentials are obtained by selecting $f_{\theta}(\boldsymbol{\theta})=f_{\sigma}(\boldsymbol{\sigma})=1/\boldsymbol{\sigma}$. Appendix A shows that a prior pdf is independent of a transformation of random variables. To further ascertain the accuracy of this statement, the proposed method was employed to build confidence intervals on:

a. a parameter of a normal distribution where the other parameter is known
b. the parameter of an exponential distribution, and
c. an unknown parameter of the uniform distribution.

This leads to identical results obtained from the other existing confidence interval methods listed below (see Motaei [12] for more detailes).

2. 5. Interval Estimation for more than one Unknown Parameter This section presents interval estimation for unknown parameters of a pdf having more than one parameter. This method is similar to the one presented in section 2. 4. Suppose there are two unknown parameters, namely θ_1 and θ_2, where the vector of observations is denoted by $\mathbf{x}=(x_1,x_2,...,x_n)$. Then, to obtain $f_{\Theta_1|X}(\theta_1\mid\mathbf{x})$, from Bayes' equation:

$$f_{\Theta_1|X}(\theta_1\mid\mathbf{x})=\frac{f_{X|\Theta_1}(\mathbf{x}\mid\theta_1)f_{\Theta_1}(\theta_1)}{\int_{\forall\theta_1} f_{X|\Theta_1}(\mathbf{x}\mid\theta_1)f_{\Theta_1}(\theta_1)d\theta_1} \tag{2}$$

where, $f_{X|\Theta_1}(\mathbf{x}\mid\theta_1)$ can be derived as:

$$f_{X|\Theta_1}(\mathbf{x}\mid\theta_1)=\int_{\forall\theta_2} f_{X|\Theta_1,\Theta_2}(\mathbf{x}\mid\theta_1,\theta_2)f_{\Theta_2|\Theta_1}(\theta_2\mid\theta_1)d\theta_2 \tag{3}$$

Note that $f_{X|\Theta_1}(\mathbf{x}\mid\theta_1)$ can be improper (if the integration is not equal to 1), but $f_{\Theta_1|X}(\theta_1\mid\mathbf{x})$ is a proper density function.

The method described in section 2. 4 can be used to obtain $f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta_1)$, in which θ_2 is unknown and θ_1 is known. The $f_{\Theta_1}(\theta)$ is determined from a vector of transformed random variables $\mathbf{X'}=(X'_1,X'_2,...,X'_n)$, where $X_i^{'}=g_{X_i}(x_i)$ does not depend on θ_2, and the only unknown parameter is θ_1. Note that, independent X_i s do not necessarily lead to independent $X_i^{'}$ s. The distribution of X_i s is obtained first, then using the weight $f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta_1) \Big/ \int_{\forall\theta_2} f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta_1)d\theta_2$, $X_i^{'}$ is obtained. The denominator of the weight is added to convert $f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta_1)$ to a proper distribution. This is clarified by the following example.

2. 5. 1. Example Consider a vector of dependent random variables $\mathbf{Y}=(Y_1,Y_2,...,Y_k)$ and a random variable Z where the conditional values of $\mathbf{Y}$ on Z are independent. Then, $f_{Y_i}(y_i)=\int_{\forall z} f_{Y_i|Z}(y_i|z)f_Z(z)dz$. Since θ_2 is unknown, it is treated as a random variable. Hence, the distribution of X_i is obtained as follows.

$$f_{X_i|\Theta_1}(x_i \mid \theta_1)=\int_{\forall\theta_2} f_{X_i|\Theta_1,\Theta_2}(x_i \mid \theta_1,\theta_2)\frac{f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta_1)}{\int_{\forall\theta_2} f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta_1)d\theta_2}d\theta_2 \tag{4}$$

To construct a confidence interval on θ_1 based on the sample $\mathbf{x}^{'}$, any transformation can be used (see section 2. 4). Then, the cumulative distribution of any continuous distribution follows a uniform distribution between 0 and 1, and the transformation to obtain $f_{\Theta_1}(\theta)$ is derived as:

$$\begin{aligned}&\int_{x'}\int_{\forall\theta_2} f_{X'|\Theta_1,\Theta_2}(x'|\theta'_1,\theta_2)\frac{f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta'_1)}{\int_{\forall\theta_2} f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta'_1)d\theta_2}d\theta_2 dx^{'} = \\ &\int_{x''}\int_{\forall\theta_2} f_{X'|\Theta_1,\Theta_2}(x''|\theta''_1,\theta_2)\frac{f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta''_1)}{\int_{\forall\theta_2} f_{\Theta_2|\Theta_1}(\theta_2 \mid \theta''_1)d\theta_2}d\theta_2 dx^{''}\end{aligned} \tag{5}$$

where, $X^{'}$ is a random variable with $\Theta_1=\theta'_1$ and $X^{''}$ is a random variable with $\Theta_1=\theta''_1$.

The proposed method was employed to construct confidence intervals on the two unknown parameters of both a uniform distribution and a normal distribution. The results were identical to those obtained from using other existing confidence interval estimation methods [12]. In a case of more than two unknown parameters, the proposed method can be used recursively. It should be mentioned that in some special cases, the proposed method could even be employed more easily. The following theorem can be helpful in this regard.

2. 5. 2. Theorem Suppose X is a continuous random variable with unknown parameters $\theta_1,\theta_2,...,\theta_m$ that has the independent parameters property. Then we have:

$$f_{\Theta_i|\mathbf{X}}(\theta_i \mid \mathbf{x})= \int_{\forall\theta_j;j\neq i}\left[\frac{f_{\mathbf{X}|\Theta_1,\Theta_2,...,\Theta_m}(\mathbf{x} \mid \theta_1,\theta_2,...,\theta_m)\Delta_{\Theta_1}(\theta_1)..\Delta_{\Theta_m}(\theta_m)}{\int_{\forall\theta_j} f_{\mathbf{X}|\Theta_1,\Theta_2,...,\Theta_m}(\mathbf{x} \mid \theta_1,\theta_2,...,\theta_m)\Delta_{\Theta_1}(\theta_1)..\Delta_{\Theta_m}(\theta_m)d\theta_j}\right]d\theta_j \tag{6}$$

The proof of this theorem for the case of two unknown parameters is given in the Appendix B, which can be extended to more than two parameters cases.

Section 3 presents an application of the proposed methodology, to construct confidence interval for the two unknown parameters of Weibull distribution.

3. INTERVAL ESTIMATION OF PARAMETERS OF WEIBULL DISTRIBUTION

Due to its flexibility, the Weibull distribution has been extensively applied in reliability and life data analysis. The statistical methods for tests of hypotheses and constructions of confidence intervals on the parameters of this distribution usually depend on using extensive statistical tables, or estimation from Weibull graphical papers. In practice, utilization of these methods could be quite cumbersome and inconvenient.

In this section, the proposed method is employed to construct confidence intervals on the two parameters of the Weibull distribution for complete data, and for the Type-I and the Type-II censored data. Notation and definitions are given in section 3. 1, then the performances of the proposed procedure are compared against two of the existing confidence interval estimation methods. Section 3. 2 presents a comparison of the proposed method with Yang et al. [2] method on confidence interval of the scale parameter. The second comparison involves building a joint confidence interval for both the shape and the scale parameters is described in section 3. 3 with Chen [3] method. These two methods can only be used for either Type-II censored data or uncensored data. Further, neither of the two methods uses sufficient statistics. Section 3. 4 discusses the effects of using nonsufficient statistics on the accuracy of the confidence interval methods. Section 3. 5 presents an application of the proposed method to build confidence intervals on the two parameters of the Weibull distribution. Finally, numerical experiments are presented in section 3. 6 for analysis and validation of the proposed model, followed by conclusions in section 4.

3. 1. Notation and Definitions The pdf of the Weibull distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$, is $f_X(x) = \beta\alpha^{-\beta}x^{\beta-1}e^{-(x/\alpha)^\beta}$, for $x > 0$. Suppose we have a sample of n observations from a Weibull distribution. The i^{th} observation and the i^{th} smallest observation is denoted by x_i and $x_{(i)}$, respectively, and any observations less than a fixed value r, is defined as a Type-I censoring. If the data consists of k smallest observations (k is a fixed number), then it is a Type-II censoring, where k/n is called the degree of censorship.

3. 2. The Exact Confidence Interval on the Scale Parameter This section briefly describes the Yang et al. [2] method of constructing a confidence interval for only the scale parameter of the Weibull distribution for a complete, and for a Type-II censored data.

Let $S(\beta) = \sum_{i=1}^{k} x_{(i)}^{\beta} + (n-k)x_{(k)}^{\beta}$. Then it can be shown that $2S(\beta)/\alpha^{\beta}$ follows a chi-square distribution with $2k$ degrees of freedom. Hence, a $100(1-\delta)\%$ confidence interval for the scale parameter α can be easily obtained as:

$$\left\{\left(\frac{2S(\hat{\beta})}{\chi^2_{2k,\delta/2}}\right)^{1/\hat{\beta}}, \left(\frac{2S(\hat{\beta})}{\chi^2_{2k,1-\delta/2}}\right)^{1/\hat{\beta}}\right\} \quad (7)$$

where, $\hat{\beta}$ is the MLE of the shape parameter and $\chi^2_{v,\delta}$ denotes the upper δ-percentile of a chi-squared χ^2 random variable. Since the shape parameter is estimated, as Yang et al. [2] indicated, there will be a convergence problem in the confidence. As a result, they replaced the chi-squared distribution with a distribution that has an exact mean and a variance of $2S(\hat{\beta})/\alpha^{\hat{\beta}}$. Moreover, they used a modified unbiased version of MLE, to estimate the shape parameter with the following confidence interval for α:

$$\left\{\left(\frac{2S(\hat{\beta})}{c\chi^2_{2k,\delta/2} - 2k(c-1)}\right)^{1/\hat{\beta}}, \left(\frac{2S(\hat{\beta})}{c\chi^2_{2k,1-\delta/2} - 2k(c-1)}\right)^{1/\hat{\beta}}\right\} \quad (8)$$

where, c is a constant that depends only in the degree of censorship.

3. 3. Joint Confidence Interval for the Scale and Shape Parameters Chen [3] presented a joint confidence interval on the scale and shape parameters of the Weibull distribution. This method is similar to Yang et al. [2] procedure that can only be applied for a Type-II censored data or for an uncensored data.

They defined $S(\beta) = \sum_{i=1}^{k} x_{(i)}{}^{\beta} + (n-k)x_{(k)}{}^{\beta}$ and $\xi(\beta) = \sum_{i=1}^{k} x_{(i)}{}^{\beta} + (n-k)x_{(k)}{}^{\beta} - nx_{(1)}^{\beta}/n(k-1)x_{(1)}^{\beta}$.

$2S(\beta)/\alpha^{\beta}$ follows a chi-square distribution with $2k$ degrees of freedom and $\xi(\beta)$ has an F distribution with $2k-2$ and 2 degrees of freedom. Furthermore, $S(\beta)$ and $\xi(\beta)$ are independent. Hence, a $100(1-\delta)\%$ joint confidence interval for the scale parameter α and the shape parameter β were given as:

$$\begin{cases} \phi\left(F_{\frac{1+\sqrt{1-\delta}}{2}}(2k-2,2)\right) \le \beta \le \phi\left(F_{\frac{1-\sqrt{1-\delta}}{2}}(2k-2,2)\right) \\ \left(\dfrac{2S(\hat{\beta})}{\chi^2_{2k,\frac{1-\sqrt{1-\delta}}{2}}}\right)^{1/\hat{\beta}} \le \alpha \le \left(\dfrac{2S(\hat{\beta})}{\chi^2_{2k,\frac{1+\sqrt{1-\delta}}{2}}}\right)^{1/\hat{\beta}} \end{cases} \quad (9)$$

where, $F_{\delta}(\upsilon_1,\upsilon_2)$ is the upper δ-percentile of a $F(\upsilon_1,\upsilon_2)$ random variable.

3. 4. Effects of a Non-sufficient Statistics on the Interval Estimation of an Unknown Parameter If a confidence interval is constructed based on a sufficient statistic, the corresponding probability distribution is referred by $f_{\Theta|\boldsymbol{X}}(\theta \mid \boldsymbol{x})$, and if it is constructed based on a statistic $n(\boldsymbol{X})$, we use $f_{\Theta|n(\boldsymbol{X})}(\theta \mid n(\mathbf{x}))$ to refer to this probability distribution.

Suppose $n(\boldsymbol{X})$ is a sufficient statistic. Then $f_{\Theta|n(\boldsymbol{X})}(\theta \mid n(\boldsymbol{x})) = f_{\Theta|\boldsymbol{X}}(\theta \mid \boldsymbol{x})$. Since there is a vector of observations on the right-hand side, we have $f_{\Theta|\boldsymbol{X}}(\theta \mid \boldsymbol{x}) = f_{\Theta|\mathbf{X},n(\boldsymbol{X})}(\theta \mid \mathbf{x}, n(\boldsymbol{x}))$. As a result, $f_{\Theta|n(\boldsymbol{X})}(\theta \mid n(\mathbf{x})) = f_{\Theta|\mathbf{X},n(\boldsymbol{X})}(\theta \mid \mathbf{x}, n(\mathbf{x}))$ which implies $\frac{f_{\Theta,n(\boldsymbol{X})}(\theta, n(\mathbf{x}))}{f_{n(\boldsymbol{X})}(n(\mathbf{x}))} = \frac{f_{\Theta,\mathbf{X},n(\boldsymbol{X})}(\theta, \mathbf{x}, n(\mathbf{x}))}{f_{\mathbf{X},n(\boldsymbol{X})}(\mathbf{x}, n(\mathbf{x}))}$ That is equivalent to $f_{\boldsymbol{X}/n(\boldsymbol{X})}(\boldsymbol{x}/n(\mathbf{x})) = f_{\boldsymbol{X}/\Theta,n(\boldsymbol{X})}(\boldsymbol{x}/\theta, n(\mathbf{x}))$.

In other words, a statistic is sufficient if and only if the probability distribution of the vector of observations conditioned on a statistic does not depend on an unknown parameter. This fact leads to the conclusion that none of the statistics used in sections 3. 2 and 3. 3 are sufficient.

Since nonsufficient statistics do not properly utilize the available data and discard some information, confidence intervals constructed from nonsufficient statistics are expected to be wider than those obtained from sufficient statistics are. This is further clarified by utilizing partial data for an interval estimation of an unknown parameter. Suppose a vector of observations from a normal distribution is available for constructing a confidence interval on the mean. An interval estimation from only one half of the observations will be wider than the one from the full set of observed data. However, as discussed in section 2. 1, every confidence interval is equivalent to a probability distribution. Moreover, for any two different probability distributions, there are two regions, in which the probability of occurrence is greater in one. Hence, when the width of a confidence interval is referenced, it is an indication of the width of the critical confidence interval. A critical confidence interval is the tightest possible confidence interval made using a given probability coverage. In summary, we expect critical confidence intervals based on nonsufficient statistics to be wider than the critical confidence interval constructed from a sufficient statistic.

In a nonsufficient statistics, only partial information is utilized, and all invalid information is excluded. As a result, the coverage of the confidence intervals is not affected by a statistical non-sufficiency. This fact has been further clarified by the experiments on the coverage of exact confidence intervals presented by Yang et al. [2].

3. 5. Interval Estimation for Parameters of Weibull Distribution An application of the proposed method on two parameters of Weibull distribution is presented in this section. Consider the cumulative distribution function (cdf) of the Weibull distribution given below:

$$F_X(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}} \quad ; \quad x > 0 \tag{10}$$

where, $\alpha > 0$ is the scale and $\beta > 0$ is the shape parameter. First it is shown that, the Weibull distribution has independent parameter property. The following approach from the cumulative distribution function is proposed to obtain the transformation froms of the pdf of two Weibull random variables, $X_1 \approx Weibull(\alpha_1, \beta)$ and $X_2 \approx Weibull(\alpha_2, \beta)$

Let $X_1/\alpha_1 \approx Weibull(1,\beta)$ and $X_2/\alpha_2 \quad Weibull(1,\beta)$. Thus

$$\frac{X_1}{\alpha_1} = \frac{X_2}{\alpha_2} \Rightarrow \frac{X_1}{\alpha_1^2} d\alpha_1 = \frac{X_2}{\alpha_2^2} d\alpha_2 \Rightarrow \frac{1}{\alpha_1} d\alpha_1 = \frac{1}{\alpha_2} d\alpha_2 \tag{11}$$

Now, let $Y_1 \approx Weibull(\alpha, \beta_1)$ and $Y_2 \approx Weibull(\alpha, \beta_2)$. Then

$$\left(\frac{Y_1}{\alpha}\right)^{\beta_1} = \left(\frac{Y_2}{\alpha}\right)^{\beta_2} \Rightarrow \left[\ln\left(\frac{Y_1}{\alpha}\right)\right]\left(\frac{Y_1}{\alpha}\right)^{\beta_1} d\beta_1 = \left[\ln\left(\frac{Y_2}{\alpha}\right)\right]\left(\frac{Y_2}{\alpha}\right)^{\beta_2} d\beta_2 \Rightarrow \frac{1}{\beta_1}\ln\left(\frac{Y_1}{\alpha}\right)^{\beta_1} d\beta_1 = \frac{1}{\beta_2}\ln\left(\frac{Y_2}{\alpha}\right)^{\beta_2} d\beta_2 \Rightarrow \frac{1}{\beta_1} d\beta_1 = \frac{1}{\beta_2} d\beta_2 \tag{12}$$

Hence, the Weibull distribution has independent parameter property and the confidence interval can be derived using:

$$f_{\alpha,\beta|X}(\alpha_1,\beta_1 \mid \boldsymbol{x}) = \frac{\frac{1}{\alpha\beta} f_{X|\alpha,\beta}(\boldsymbol{x} \mid \alpha_1,\beta_1)}{\int_{\beta=0}^{+\infty}\int_{\alpha=0}^{+\infty} \frac{1}{\alpha\beta} f_{X|\alpha,\beta}(\boldsymbol{x} \mid \alpha_1,\beta_1)\, d\alpha\, d\beta} \tag{13}$$

Then, $f_{X|\alpha,\beta}(\boldsymbol{x} \mid \alpha_1,\beta_1)$ can be obtained from Type-I or from Type-II censored data. For a computational ease, we replace the ranked data with the unranked ones.

In a sample of Type-I censored data, k_1 observations are less than or equal to a fixed value r, denoted by $x_{(i)}$; $i = 1,2,...,k_1$, and $(n-k_1)$ observations re greater than r. Thus

$$f_{X|\alpha,\beta}(\boldsymbol{x} \mid \alpha_1,\beta_1) = \frac{\beta_1^{k_1}}{\alpha_1^{k_1\beta_1}}\left(\prod_{i=1}^{k_1} x_{(i)}\right)^{\beta_1-1} e^{-\frac{\sum_{i=1}^{k_1} x_{(i)}^{\beta_1} + (n-k_1) r^{\beta_1}}{\alpha_1^{\beta_1}}} \tag{14}$$

In a Type-II censored data, values of the k smallest observations, denoted by $x_{(i)}$; $i = 1,2,...,k$, are known and $(n-k)$ observations are greater than $x_{(k)}$. Hence,

$$f_{X|\alpha,\beta}(\boldsymbol{x} \mid \alpha_1,\beta_1) = \frac{\beta_1^{k}}{\alpha_1^{k\beta_1}}\left(\prod_{i=1}^{k} x_{(i)}\right)^{\beta_1-1} e^{-\frac{\sum_{i=1}^{k} x_{(i)}^{\beta_1} + (n-k) x_{(r)}^{\beta_1}}{\alpha_1^{\beta_1}}} \tag{15}$$

Equations (14) and (15) can be shown in a unique form as follows:

$$f_{X|\alpha,\beta}(\boldsymbol{x} \mid \alpha_1,\beta_1) = \frac{\beta_1^{K}}{\alpha_1^{K\beta_1}}\left(\prod_{i=1}^{K} x_{(i)}\right)^{\beta_1-1} e^{-\frac{\sum_{i=1}^{K} x_{(i)}^{\beta_1} + (n-K) R^{\beta_1}}{\alpha_1^{\beta_1}}} \tag{16}$$

where, $K = k_1$ and $R = r$ are used for the Type-I censored data, and $K = k$ and $R = x_{(r)}$ are used for the Type-II censored data. Then the joint confidence interval is derived from:

$$f_{\alpha,\beta|X}(\alpha_1,\beta_1 \mid \boldsymbol{x}) = \frac{\frac{\beta_1^{K-1}}{\alpha_1^{K\beta_1+1}}\left(\prod_{i=1}^{K} x_{(i)}\right)^{\beta_1-1} e^{-\frac{\sum_{i=1}^{K} x_{(i)}^{\beta_1} + (n-K) R^{\beta_1}}{\alpha_1^{\beta_1}}}}{\int_{\beta=0}^{+\infty}\int_{\alpha=0}^{+\infty} \frac{\beta_1^{K-1}}{\alpha_1^{K\beta_1+1}}\left(\prod_{i=1}^{K} x_{(i)}\right)^{\beta_1-1} e^{-\frac{\sum_{i=1}^{K} x_{(i)}^{\beta_1} + (n-K) R^{\beta_1}}{\alpha_1^{\beta_1}}} d\alpha\, d\beta} \tag{17}$$

As a result, for a confidence interval (Bayesian) on the scale parameter we have

$$f_{\alpha|X}(\alpha_1 \mid x) = \int_{\beta_1=0}^{+\infty} f_{\alpha,\beta|X}(\alpha_1, \beta_1 \mid x) d\beta_1 \quad (18)$$

The Bayesian confidence interval on the shape parameter is obtained from:

$$f_{\beta|X}(\beta_1 \mid x) = \int_{\alpha_1=0}^{+\infty} f_{\alpha,\beta|X}(\alpha_1, \beta_1 \mid x) d\alpha_1 \quad (19)$$

Then, the $100(1-\delta)\%$ confidence intervals for the above two parameters are the regions on which the integration of the posterior distribution (Equations (17), (18) or (19)) is equal to $(1-\delta)$.

3. 6. Experiments This section presents some experiments that are designed to evaluate the performances of the proposed methodology for the intervals estimation of the parameters of Weibull distribution. It is assumed that the scale parameter $\alpha = 1$ and the shape parameter $\beta = 2$. The confidence intervals based on samples of size n = 20 and 50 with m replications are constructed. Three confidence levels are assumed at: $1-\delta = 0.90$, 0.95, and 0.99. Furthermore, all experiments are coded and run in MATLAB (R2008a). Table 1 shows the average confidence interval coverage of the proposed method (column P), the average coverage for the scale parameter of the Yang et al. [2] method (column Y), and the average coverage for the joint confidence interval presented by Chen [3] (column C). Note that the results shown on columns Y and C are based on a Type-II censored data. The number of confidence intervals used for experiments appears in the column under "Runs." The results shown on Table 1 indicate that all methods provide good performances in terms of coverage. Moreover, these results show that nonsufficient statistics do not affect the coverage of the confidence intervals. For Type-I censored data, the average coverage and width of the estimated confidence intervals on the scale parameter, the shape parameter, and both parameters using the proposed method are given in Table 2. For the r value, the 7th decil (d_7), 8th decil (d_8), and 9th decil (d_9) of the Weibull distribution are chosen. Based on the results given in Table 2, in all scenarios of Type-I censored data, the estimated confidence intervals from the proposed method have exact coverage as well as tight widths. Table 3 shows the results obtained on the average width of the critical confidence intervals for the scale parameter from the proposed method, and the method developed by Yang et al. [2]. It also contains the results obtained for the average joint confidence intervals of the proposed method and the Chen [3] procedure. These results show that in all cases the widths of the critical confidence intervals from the proposed method are less than those obtained from the other two methods.

TABLE 2. The average coverage and length of the methods for Type-I censoring data

			Coverage			**Length**			
		r	α	β	**Joint**	α	β	**Joint**	**Runs**
n=20	$1-\delta=0.9$	$d_7=1.097$	0.892	0.888	0.910	0.571	1.623	1.348	1000
		$d_8=1.269$	0.910	0.900	0.907	0.492	1.471	1.030	1000
		$d_9=1.517$	0.908	0.895	0.912	0.442	1.339	0.832	1000
n=20	$1-\delta=0.95$	$d_7=1.097$	0.956	0.942	0.952	0.724	1.919	1.936	1000
		$d_8=1.269$	0.944	0.954	0.948	0.608	1.739	1.404	1000
		$d_9=1.517$	0.946	0.948	0.946	0.539	1.584	1.115	1000
n=20	$1-\delta=0.99$	$d_7=1.097$	1.000	1.000	1.000	1.198	2.613	4.436	100
		$d_8=1.269$	1.000	0.990	1.000	0.924	2.346	2.718	100
		$d_9=1.517$	0.990	1.000	0.990	0.744	2.146	1.933	100
n=50	$1-\delta=0.9$	$d_7=1.097$	0.914	0.916	0.918	0.312	1.004	0.441	1000
		$d_8=1.269$	0.907	0.913	0.912	0.283	0.912	0.363	1000
		$d_9=1.517$	0.891	0.889	0.888	0.267	0.827	0.311	1000
n=50	$1-\delta=0.95$	$d_7=1.097$	0.957	0.957	0.957	0.377	1.203	0.596	1000
		$d_8=1.269$	0.953	0.941	0.950	0.341	1.091	0.488	1000
		$d_9=1.517$	0.945	0.953	0.950	0.320	0.990	0.414	1000
n=50	$1-\delta=0.99$	$d_7=1.097$	1.000	1.000	1.000	0.529	1.557	1.141	100
		$d_8=1.269$	0.980	1.000	0.990	0.482	1.414	1.065	100
		$d_9=1.517$	0.980	1.000	0.990	0.454	1.287	0.94	100

TABLE 1. Comparing the coverage of the methods for Type-II censoring data

n	$1-\delta$	Degree of censorship	α P	β P	Joint P	α Y	Joint C	Runs
20	0.9	0.4	0.914	0.9	0.916	0.888	0.886	1000
20	0.9	0.5	0.885	0.911	0.913	0.902	0.897	1000
20	0.9	0.6	0.885	0.901	0.894	0.91	0.893	1000
20	0.9	0.7	0.886	0.895	0.894	0.886	0.902	1000
20	0.9	0.8	0.908	0.898	0.913	0.885	0.909	1000
20	0.9	0.9	0.908	0.902	0.911	0.911	0.914	1000
20	0.9	1.0	0.901	0.891	0.896	0.918	0.918	1000
		average	0.898	0.8997	0.905	0.900	0.903	
20	0.95	0.4	0.963	0.96	0.961	0.951	0.963	1000
20	0.95	0.5	0.949	0.95	0.949	0.963	0.951	1000
20	0.95	0.6	0.941	0.954	0.95	0.944	0.953	1000
20	0.95	0.7	0.937	0.944	0.96	0.948	0.948	1000
20	0.95	0.8	0.937	0.944	0.956	0.947	0.942	1000
20	0.95	0.9	0.961	0.942	0.948	0.941	0.951	1000
20	0.95	1.0	0.953	0.954	0.952	0.949	0.952	1000
		average	0.949	0.9497	0.954	0.949	0.951	
20	0.99	0.4	0.994	0.993	0.992	0.992	0.995	1000
20	0.99	0.5	0.988	0.995	0.987	0.991	0.991	1000
20	0.99	0.6	0.984	0.991	0.987	0.984	0.989	1000
20	0.99	0.7	0.985	0.991	0.987	0.987	0.989	1000
20	0.99	0.8	0.993	0.992	0.991	0.993	0.987	1000
20	0.99	0.9	0.993	0.990	0.992	0.988	0.993	1000
20	0.99	1.0	0.996	0.986	0.989	0.987	0.996	1000
		average	0.990	0.991	0.989	0.989	0.991	
50	0.9	0.4	0.908	0.908	0.908	0.888	0.912	1000
50	0.9	0.5	0.910	0.913	0.915	0.907	0.915	1000
50	0.9	0.6	0.912	0.910	0.912	0.903	0.913	1000
50	0.9	0.7	0.912	0.907	0.913	0.897	0.905	1000
50	0.9	0.8	0.891	0.912	0.896	0.898	0.885	1000
50	0.9	0.9	0.891	0.889	0.897	0.885	0.882	1000
50	0.9	1.0	0.893	0.912	0.91	0.915	0.892	1000
		average	0.902	0.907	0.907	0.899	0.901	
50	0.95	0.4	0.963	0.953	0.953	0.958	0.958	1000
50	0.95	0.5	0.950	0.957	0.950	0.953	0.963	1000
50	0.95	0.6	0.957	0.953	0.960	0.957	0.967	1000
50	0.95	0.7	0.947	0.950	0.953	0.946	0.936	1000
50	0.95	0.8	0.947	0.948	0.947	0.943	0.933	1000
50	0.95	0.9	0.938	0.948	0.943	0.943	0.943	1000
50	0.95	1.0	0.947	0.949	0.953	0.957	0.943	1000
		average	0.95	0.951	0.951	0.951	0.949	
50	0.99	0.4	1.000	0.990	1.000	0.990	1.000	100
50	0.99	0.5	1.000	0.990	0.990	0.990	1.000	100
50	0.99	0.6	1.000	1.000	1.000	0.980	0.990	100
50	0.99	0.7	0.990	1.000	0.990	0.930	0.990	100
50	0.99	0.8	0.980	1.000	0.990	0.960	0.970	100
50	0.99	0.9	0.980	1.000	0.990	0.960	0.960	100
50	0.99	1.0	0.980	1.000	0.990	0.990	0.980	100
		average	0.990	0.997	0.990	0.970	0.980	

TABLE 3. Comparing average lengths of the methods for Type-II censoring data

n	$1-\delta$	Degree of censorship	α P	α Y	Joint P	Joint C	Runs
20	0.9	0.4	1.027	1.765	3.576	4.251	1000
20	0.9	0.5	0.650	1.262	2.186	2.579	1000
20	0.9	0.6	0.504	1.061	1.570	1.790	1000
20	0.9	0.7	0.448	1.026	1.230	1.383	1000
20	0.9	0.8	0.401	1.016	1.007	1.149	1000
20	0.9	0.9	0.374	0.984	0.848	1.039	1000
20	0.9	1.0	0.364	0.914	0.718	1.071	1000
20	0.95	0.4	1.377	2.137	5.580	6.065	1000
20	0.95	0.5	0.883	1.558	3.232	3.682	1000
20	0.95	0.6	0.666	1.292	2.231	2.551	1000
20	0.95	0.7	0.554	1.220	1.695	1.943	1000
20	0.95	0.8	0.496	1.181	1.367	1.609	1000
20	0.95	0.9	0.462	1.171	1.142	1.459	1000
20	0.95	1.0	0.443	1.133	0.961	1.498	1000
20	0.99	0.4	2.593	3.092	10.009	10.923	1000
20	0.99	0.5	1.260	2.205	4.467	6.442	1000
20	0.99	0.6	0.856	1.817	2.661	4.393	1000
20	0.99	0.7	0.683	1.756	1.919	3.315	1000
20	0.99	0.8	0.597	1.687	1.526	2.729	1000
20	0.99	0.9	0.549	1.695	1.275	2.465	1000
20	0.99	1.0	0.521	1.617	1.083	2.525	1000
50	0.9	0.4	0.508	0.957	1.047	2.061	1000
50	0.9	0.5	0.400	0.725	0.713	1.290	1000
50	0.9	0.6	0.337	0.612	0.533	0.913	1000
50	0.9	0.7	0.299	0.602	0.426	0.707	1000
50	0.9	0.8	0.276	0.584	0.356	0.591	1000
50	0.9	0.9	0.263	0.576	0.307	0.539	1000
50	0.9	1.0	0.255	0.547	0.266	0.564	1000
50	0.95	0.4	0.624	1.155	1.448	2.848	1000
50	0.95	0.5	0.488	0.872	0.973	1.776	1000
50	0.95	0.6	0.407	0.735	0.721	1.254	1000
50	0.95	0.7	0.360	0.699	0.573	0.969	1000
50	0.95	0.8	0.330	0.687	0.477	0.808	1000
50	0.95	0.9	0.314	0.676	0.409	0.737	1000
50	0.95	1.0	0.304	0.661	0.353	0.772	1000
50	0.99	0.4	1.011	1.571	2.915	4.938	100
50	0.99	0.5	0.702	1.165	1.925	3.025	100
50	0.99	0.6	0.572	0.974	1.274	2.117	100
50	0.99	0.7	0.504	0.917	1.114	1.628	100
50	0.99	0.8	0.482	0.892	1.071	1.352	100
50	0.99	0.9	0.542	0.908	0.859	1.224	100
50	0.99	1.0	0.467	0.874	0.679	1.269	100

4. CONCLUSION

In this paper, a novel method for constructing exact confidence intervals on the parameters of continuous distribution was proposed. It was shown that the proposed methodology could produce the same exact confidence intervals from sufficient statistics; it could also construct reliable confidence intervals in the absence of sufficient statistics, through a numerical integration method. As an application, this method was employed to construct confidence intervals for the shape and scale parameters of a Weibull distribution, as well as their simultaneous confidence region for complete data, and for a Type-I and Type-II censored data. The numerical experiments indicated that this method lead to the tightest confidence intervals for Type-II censored data, and to the exact confidence intervals for a Type-I censored data.

Although the proposed method is powerful and easy to implement, for some distribution a closed form formula for a prior distribution cannot be easily derived, therefore numerical methods are preferred.

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APPENDIX A

Independence of Prior PDF from Transformation of Random Nariables For continuous random variables with one unknown parameter, it is shown that $f_\Theta(\theta)$ is independent of the transformation. Let X be an arbitrary continuous random variable with only one unknown parameter. If $\theta=\theta_1$ we denote X by X_1 and if $\theta=\theta_2$, then X is denoted by X_2. Suppose there exist two transformations $O(X_1;\theta_1)=O(X_2;\theta_2)$ and $Q(X_1;\theta_1)=Q(X_2;\theta_2)$. Let h_O and h_Q be the transformations that convert the random variables generated by O and Q , and $H=h_Q^{-1}(h_O)$ is a mapping that converts O to Q. This transformation exists for every continuous random variable. Furthermore we show that:

$$\frac{dQ(X_1;\theta_1)}{d\theta}\div\frac{dQ(X_2;\theta_2)}{d\theta}=\frac{dO(X_1;\theta_1)}{d\theta}\div\frac{dO(X_2;\theta_2)}{d\theta} \quad \text{(A.1)}$$

Let $\frac{dQ(X_1;\theta_1)}{d\theta}\div\frac{dQ(X_2;\theta_2)}{d\theta}=k$. Then

$$k=\frac{d}{d\theta}\left(h\left(O\left(X_1;\theta_1\right)\right)\right)\div\frac{d}{d\theta}\left(h\left(O\left(X_2;\theta_2\right)\right)\right)$$
$$\left(\frac{d}{d\theta}O\left(X_1;\theta_1\right)\frac{d}{dO\left(X_1;\theta_1\right)}h\left[O\left(X_1;\theta_1\right)\right]\right)\div \quad \text{(A.2)}$$
$$\left(\frac{d}{d\theta}O\left(X_2;\theta_2\right)\frac{d}{dO\left(X_2;\theta_2\right)}h\left[O\left(X_2;\theta_2\right)\right]\right)$$

Since $O(X_1;\theta_1)=O(X_2;\theta_2)$, we have $\frac{d}{dO(X_1;\theta_1)}h\left[O(X_1;\theta_1)\right]=\frac{d}{dO(X_2;\theta_2)}h\left[O(X_2;\theta_2)\right]$ and it follows that $\frac{d}{d\theta}O(X_1;\theta_1)\Big/\frac{d}{d\theta}O(X_2;\theta_2)=k$. Therefore, $f_\Theta(\theta)$ is independent of the transformation. Moreover, using the change of variable technique in Equation (1), it can be shown that by selecting either θ or any function of θ leads to equivalent results.

APPENDIX B.

Prof of the Theorem Suppose X is a continuous random variable with unknown parameters $\theta_1,\theta_2,...,\theta_m$ with an independent parameter property. Then we have:

$$f_{\Theta_i|\boldsymbol{x}}(\theta_i\,|\,\boldsymbol{x}) = \int_{\forall\theta_j;j\neq i}\left[\frac{f_{\boldsymbol{X}|\Theta_1,\Theta_2,...,\Theta_m}(\boldsymbol{x}|\theta_1,\theta_2,...,\theta_m)\Delta_{\Theta_1}(\theta_1)..\Delta_{\Theta_m}(\theta_m)}{\int_{\forall\theta_j} f_{\boldsymbol{X}|\Theta_1,\Theta_2,...,\Theta_m}(\boldsymbol{x}|\theta_1,\theta_2,...,\theta_m)\Delta_{\Theta_1}(\theta_1)..\Delta_{\Theta_m}(\theta_m)}\right]d\theta_j \quad \text{(B.1)}$$

Proof: The proof is given for the case of two unknown parameters. It can be easily extended for the cases of more than two unknown parameters.

For the first parameter of a two dimensional parameter vector $\boldsymbol{\theta}=[\theta_1,\theta_2]$, we are to prove the following expression

$$f_{\Theta_1|\boldsymbol{X}}(\theta_1\,|\,\boldsymbol{x}) = \int_{\theta_2}\frac{f_{\boldsymbol{X}|\Theta_1,\Theta_2}(\boldsymbol{x}\,|\,\theta_1,\theta_2)\Delta_{\Theta_1}(\theta_1)\Delta_{\Theta_2}(\theta_2)}{\int_{\theta_1}\int_{\theta_2} f_{\boldsymbol{X}|\Theta_1,\Theta_2}(\boldsymbol{x}\,|\,\theta_1,\theta_2)\Delta_{\Theta_1}(\theta_1)\Delta_{\Theta_2}(\theta_2)d\theta_2 d\theta_1}d\theta_2 \quad \text{(B.2)}$$

By the independent parameters property we have $f_{\Theta_2|\Theta_1}(\theta_2\,|\,\theta_1)=\Delta_{\Theta_2}(\theta_2)$. Then, expression (B.3) can be written as:

$$f_{\boldsymbol{X}|\Theta_1}(\boldsymbol{x}\,|\,\theta_1) = \int_{\theta_2} f_{\boldsymbol{X}|\Theta_1,\Theta_2}(\boldsymbol{x}\,|\,\theta_1,\theta_2)f_{\Theta_2|\Theta_1}(\theta_2\,|\,\theta_1)d\theta_2 = \int_{\theta_2} f_{\boldsymbol{X}|\Theta_1,\Theta_2}(\boldsymbol{x}\,|\,\theta_1,\theta_2)\Delta_{\Theta_2}(\theta_2)d\theta_2 \quad \text{(B.3)}$$

We first prove $f_{\Theta_1}(\theta_1)=\Delta_{\Theta_1}(\theta_1)$. To do this, according to lemma 1, $\Delta_{\Theta_2}(\theta_2)$ is independent of transformation and $f_{\boldsymbol{X}|\Theta_1}(\boldsymbol{x}\,|\,\theta_1)$ is obtained uniquely. Hence, the random variable that is obtained in this step is unique and according to lemma 1, $f_{\Theta_1}(\theta_1)$ is unique and independent of transformation.

Let us introduce two transformations to obtain $\Delta_{\Theta_1}(\theta_1)$ and $f_{\Theta_1}(\theta_1)$. Let $F_X(x;\theta_1,\theta_2)$ be the CDF of a random variable X with known parameters θ_1,θ_2 and $F_Y^{-1}(y;\theta_1,\theta_2); 0\le y\le 1$ is the inverse of this function. $F_X(x;\theta_1)$ and $F_Y^{-1}(y;\theta_1); 0\le y\le 1$ are defined similarly. Furthermore, we have:

$$F_X(x;\theta_1) = \int_{\theta_2} F_X(x;\theta_1,\theta_2)\frac{\Delta_{\Theta_2}(\theta_2)}{\int_{\theta_2}\Delta_{\Theta_2}(\theta_2)d\theta_2}d\theta_2 \quad \text{(B.4)}$$

Then, the transformations are defined as:

$$\frac{\Delta_{\Theta_1}(\theta_1^{'})}{\Delta_{\Theta_1}(\theta_1^{''})} = \frac{\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{'},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{'},\theta_2)dx}{\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{''},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{''},\theta_2)dx} \quad \text{(B.5)}$$

$$\frac{f_{\Theta_1}(\theta_1^{'})}{f_{\Theta_1}(\theta_1^{''})} = \frac{\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{'})} f_{X|\Theta_1}(x;\theta_1^{'})dx}{\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{''})} f_{X|\Theta_1}(x;\theta_1^{''})dx} \quad \text{(B.6)}$$

Since θ_2 s in the numerator and denominator of expression (12) are equal, we have:

$$\frac{\Delta_{\Theta_1}(\theta_1^{'})}{\Delta_{\Theta_1}(\theta_1^{''})} = \frac{\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{'},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{'},\theta_2)dx}{\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{''},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{''},\theta_2)dx} =$$

$$\frac{\frac{\Delta_{\Theta_2}(\theta_2)}{\int_{\theta_2}\Delta_{\Theta_2}(\theta_2)}\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{'},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{'},\theta_2)dx}{\frac{\Delta_{\Theta_2}(\theta_2)}{\int_{\theta_2}\Delta_{\Theta_2}(\theta_2)}\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{''},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{''},\theta_2)dx} \quad \text{(B.7)}$$

Also, since $\Delta_{\Theta_2}(\theta_2)$ does not depend on θ_1, we have:

$$\Delta_{\Theta_1}(\theta_1^{'})\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{''},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{''},\theta_2)\frac{\Delta_{\Theta_2}(\theta_2)}{\int_{\theta_2}\Delta_{\Theta_2}(\theta_2)d\theta_2}dx =$$

$$\Delta_{\Theta_1}(\theta_1^{''})\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{'},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{'},\theta_2)\frac{\Delta_{\Theta_2}(\theta_2)}{\int_{\theta_2}\Delta_{\Theta_2}(\theta_2)d\theta_2}dx \quad \text{(B.8)}$$

Note that $\Delta_{\Theta_1}(\theta_1)$ does not depend on θ_2 and by integration on θ_2 we have:

$$\Delta_{\Theta_1}(\theta_1^{'})\int_{\theta_2}\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{''},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{''},\theta_2)\frac{\Delta_{\Theta_2}(\theta_2)}{\int_{\theta_2}\Delta_{\Theta_2}(\theta_2)d\theta_2}dx$$
$$=\Delta_{\Theta_1}(\theta_1^{''})\int_{\theta_2}\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{'},\theta_2)} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{'},\theta_2)\frac{\Delta_{\Theta_2}(\theta_2)}{\int_{\theta_2}\Delta_{\Theta_2}(\theta_2)d\theta_2}dxd\theta_2 \quad \text{(B.9)}$$

Therefore,

$$\Delta_{\Theta_1}(\theta_1^{'})\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{''})}\left[\int_{\theta_2} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{''},\theta_2)\Delta_{\Theta_2}(\theta_2)d\theta_2\right]dx$$
$$=\Delta_{\Theta_1}(\theta_1^{''})\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}(y;\theta_1^{'})}\left[\int_{\theta_2} f_{X|\Theta_1,\Theta_2}(x;\theta_1^{'},\theta_2)\Delta_{\Theta_2}(\theta_2)d\theta_2\right]dx \quad \text{(B.10)}$$

Hence,

$$\Delta_{\Theta_1}\left(\theta_1^{'}\right)\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}\left(y;\theta_1^{''}\right)} f_{X|\Theta_1}\left(x;\theta_1^{''}\right)dx = \Delta_{\Theta_1}\left(\theta_1^{''}\right)\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}\left(y;\theta_1^{'}\right)} f_{X|\Theta_1}\left(x;\theta_1^{'}\right)dx \quad \text{(B.11)}$$

Thus,

$$\frac{\Delta_{\Theta_1}\left(\theta_1^{'}\right)}{\Delta_{\Theta_1}\left(\theta_1^{''}\right)} = \frac{\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}\left(y;\theta_1^{'}\right)} f_{X|\Theta_1}\left(x;\theta_1^{'}\right)dx}{\frac{d}{d\theta_1}\int_{x=-\infty}^{F_Y^{-1}\left(y;\theta_1^{''}\right)} f_{X|\Theta_1}\left(x;\theta_1^{''}\right)dx} = \frac{f_{\Theta_1}\left(\Theta_1=\theta'\right)}{f_{\Theta_1}\left(\Theta_1=\theta''\right)} \quad \text{(B.12)}$$

In other words, $f_{\Theta_1}\left(\Theta_1=\theta\right)=\Delta_{\Theta_1}\left(\theta\right)$. Now using expression (2), we have:

$$f_{\Theta_1|\boldsymbol{x}}\left(\theta_1 \mid \boldsymbol{x}\right) = \frac{\Delta_{\Theta_1}\left(\theta_1\right)\int_{\theta_2} f_{\boldsymbol{X}|\Theta_1,\Theta_2}\left(\boldsymbol{x}\mid\theta_1,\theta_2\right)\Delta_{\Theta_2}\left(\theta_2\right)d\theta_2}{\int_{\theta_1}\Delta_{\Theta_1}\left(\theta_1\right)\int_{\theta_2} f_{\boldsymbol{X}|\Theta_1,\Theta_2}\left(\boldsymbol{x}\mid\theta_1,\theta_2\right)\Delta_{\Theta_2}\left(\theta_2\right)d\theta_2 d\theta_1} \quad \text{(B.13)}$$

Again since $\Delta_{\Theta_1}\left(\theta\right)$ does not depend on θ_2, we have:

$$f_{\Theta_1|\boldsymbol{x}}\left(\theta_1 \mid \boldsymbol{x}\right) = \frac{\int_{\theta_2} f_{\boldsymbol{X}|\Theta_1,\Theta_2}\left(\boldsymbol{x}\mid\theta_1,\theta_2\right)\Delta_{\Theta_1}\left(\theta_1\right)\Delta_{\Theta_2}\left(\theta_2\right)d\theta_2}{\int_{\theta_1}\int_{\theta_2} f_{\boldsymbol{X}|\Theta_1,\Theta_2}\left(\boldsymbol{x}\mid\theta_1,\theta_2\right)\Delta_{\Theta_1}\left(\theta_1\right)\Delta_{\Theta_2}\left(\theta_2\right)d\theta_2 d\theta_1} \quad \text{(B.14)}$$

This concludes the proof.

Bayes Interval Estimation on the Parameters of the Weibull Distribution for Complete and Censored Tests

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چکیده

در این مقاله ابتدا یک روش عمومی جدید برای ساخت فواصل اطمینان برای پارامترهای یک توزیع احتمال پیوسته توسعه داده می شود. هدف ارائۀ روشی برای مدلسازی عدم قطعیت موجود در پارامترهای این توزیعهاست.به عنوان یک کاربرد، فواصل اطمینان برای دو پارامتر توزیع وایبول به همراه ناحیۀ اطمینان توام آنها به دست می آید. این مدل هم داده های کامل و هم داده های سانسور شده می پذیرد. دقت تخمین به دست آمده توسط مدل با دقت تخمین روشهای موجود با استفاده از روشهای عددی مقایسه می شود. تحلیل اعتبار نشان می دهد که دقت تخمین روش پیشنهادی امیدوار کننده است. همچنین نشان داده می شود که استفادۀ نامناسب اطلاعات موجود که بر عرض فواصل اطمینان حاصل از روشهای موجود اثر می گذارد بر میزان همپوشانی روش پیشنهادی اثرگذار نیست.

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