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Large Deflection Analysis of Compliant Beams of Variable Thickness and Non-homogenous Material under Combined Load and Multiple Boundary Conditions

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ABSTRACT

This paper studies a new approach to analyze the large deflection behavior of prismatic and non-prismatic beams of non-homogenous material under combined load and multiple boundary conditions. The mathematical formulation has been derived which led to a set of six first-order ordinary differential equations. The geometrically nonlinear problem has been solved numerically using the multiple shooting method combined with a quadratic programming technique. The results obtained by presented method have been compared with the existing with less complex cases in the literature. The method may be applied to the design of compliant mechanisms, nonlinear springs or any related subject.

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1. INTRODUCTION

Conventional mechanisms have been studied and extensively implemented in past centuries. The use of flexibility in structures and mechanisms was mostly avoided because of the difficulty in design and manufacturing. In recent decades, compliant mechanisms (CM) have become an important area in mechanical science which is rapidly developing due to dramatic improvement of technology, need for high precision, smal scale and low final cost. Compliant mechanisms are usually monolithic and transmit motion and forces by means of their flexible members. Therefore CMs' members usually undergo large deflection without exceeding the yield stress of the material. The advantages of these mechanisms compared to traditional rigid-body mechanisms are due to the absence of rigid body kinematic joints. Some of the many advantages of these mechanisms includes lower degree of noise, backlash and lubrication as well as wear; resulting in structures with very low maintenance cost and high precision. Compliant mechanisms are widely used in product design [1–3], design for systems without assembly proccess[4], Micro Electro Mechanical Systems (MEMS) [5], nonlinear springs [6, 7], stages for precision engineering [8], statically balanced mechanisms [9, 10].

In many engineering applications each segment of a compliant mechanism has been modeled as a straight beam. Although, the complaint mechanisms may undergo large deflection, strains in beam segments remain small, hence, the governing equations can be obtained from the Euler–Bernoulli beam theory. The analysis of compliant mechanisms may require development of the governing equations as a general system of nonlinear algebraic equations (NAE) that rely on a complex model and kinematics of a deflected beam. Therefore, the principle of superposition cannot be applied and large deflections may not be easy to determine by any known methods. Usually elliptic integrals [11], finite difference method [12], finite element method [13] and some numerical methods such as shooting method [14] were used for analyzing large static deflection of flexible structures. Large deflection analysis based on solving the two-point nonlinear boundary value problem (BVP) has been investigated by C. C. Lan [15]. He has shown that the problem of initially straight or curved beam can be formulated by a set of four first-order ordinary differential equations which have been numerically solved using the scheme of multiple shooting method with two numerical solvers

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including unbounded Gauss-Newton Method and bounded Gauss-Newton Method [15]. An approximate power series solution using an incremental linearization approach in global coordinate model was implemented to investigate deformed shape of compliant mechanisms with large displacement and rotation [16].

In that work, the concentrated loads were applied to a compliant mechanism of initially curved members. The analytical homotopy analysis method (HAM) has been used to investigate the large deformation of a cantilever beam under point follower load at the free tip [17, 18]. An analytical analysis investigation on large deflection of cantilever beams subjected to end point load has also been studied using the Adomian decomposition method (ADM) [19] and direct nonlinear solution (DNS) [20]. A comparative review of different methods regarding computation time, implementation convenience and accuracy was reported in the literature [20]. Nonlinear elastic behavior of materials, makes the problems more complex. Large deflection of a linear viscoelastic slender cantilever beam subjected to a time dependent concentrated load at the free end has been investigated by Vaz and Caire [21]. Solano investigated exact solutions for the large deflection of a cantilever beam made of Ludwick type material under a concentrated loading at the free end as well as a uniformly distributed vertical load [22]. An initially curved partially compliant bi-stable mechanism was also studied by Tolou and Herder [10].

Most of the beams which have been investigated in the above mentioned studies are the classical type of compliant beams which were subjected to tip loads, while a few of them were subjected to distributed loads; with variable thickness or nonlinear materials [22, 23, 16]. In addition, Lee has carried out the Post-buckling analysis of uniform columns under combination of a uniformly distributed axial load and concentrated load at the free end [24] using Butcher's numerical integration procedure. Then Merli et al. have analyzed the nonlinear bending problem of a constant cross-section extensible beam with simply supported ends subjected to a uniformly distributed load [25].

In the above studies, the numerical integration was derived using a nonlinear Timoshenko beam and two different linearization schemes, namely, multistep transversal linearization (MTrL) and multi-step tangential linearization (MTnL).

In this article, the previous works have been extended by analyzing the large deflection of a variable thickness beam. The non-homogenous material under general loading cases consisting of concentrated end tip force, end tip moment, and distributed loading were also considered. Large deflections analysis of beam was carried out considering multiple boundary conditions including, clamped, simply supported, free and sliding. The mathematical formulation of this problem yields to a set of six first-order ordinary differential equations consisting components of slope, moment, forces and positions of each point of the beam in global Cartesian coordinates. The shooting method has been used to solve the governing equations numerically. The two point boundary value problem (BVP) is transformed into an initial value problem (IVP). The solution starts by assuming an initial conditions and continues with integration of the equations using the Runge-Kutta algorithm [26].

By adjusting the assumed initial values using iterative methods, the terminal boundary conditions will be convergence to an acceptable degree of accuracy. This adjustment is implemented by minimization of errors in terminal boundary conditions using an optimization algorithm such as Newton's or quasi-Newton's method. This procedure has a simple formulation and high accuracy with extreme convergence speed [26]. Unlike the finite element method which requires fine meshing to increase the accuracy, this method does not depend on element size and correspondingly may reduce the computation time. Without loss of generality, the shear and axial deformations are assumed insignificant because of their small values compared to flexural deformation of slender beams. Several numerical examples are presented covering prismatic and non-prismatic cantilever beams subjected to a general loading consisting of moments, distributed loads and concentrated loads in vertical and horizontal directions. The beams are inextensible, linear elastic and initially straight with various boundary conditions at both ends.

This paper is organized as follows: the moment-curvature relationship and governing equation are first derived in section 2. 1. In section 2. 2, constraints and loading cases are introduced; section 2. 3 presents the shooting method as the numerical solver. The results are portrayed graphically and presented numerically in section 3 and compared with those of the numerical solution obtained from FEM. Discussion on obtained results are presented in section 4 and finally, section 5 represents the conclusions.

2. MATHEMATICAL MODEL

Figure 1 shows the deflected configuration of a non-prismatic compliant beam of length L and variable stiffness $EI(s)$ subjected to concentrated forces F_x, F_y, bending moment M_e at the end point and non-uniform horizontal and vertical distributed loads $W_x(s)$, $W_y(s)$, where s denotes the location along the deflected axis of the beam. The boundary conditions of the beam at points O and B are assumed to be clamped, simply supported, sliding or free. As mentioned before, the material is assumed to be isotropic with linear elastic stress-strain behavior and for the slender beam the axial and shear deformations are assumed to be negligible.

2. 1. Governing Equation Figure 2 shows the free body diagram of a compliant beam segment from point (x_s, y_s) to the tip of the beam.

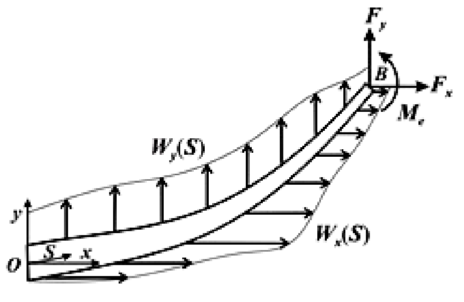


Figure 1. Non-prismatic compliant beam under generalized loading

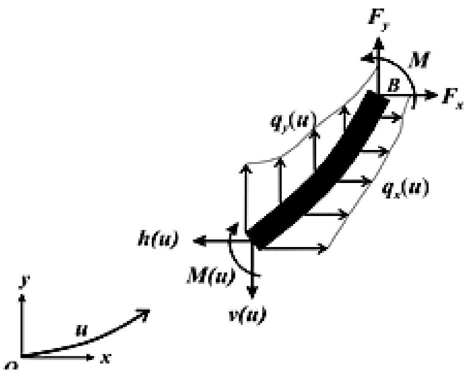


Figure 2. Free body diagram of the compliant beam segment

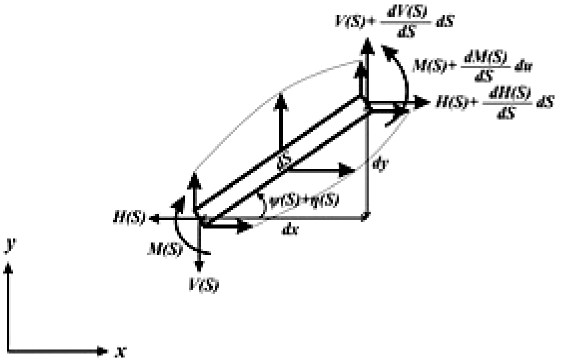


Figure 3. Infinitesimal element in global coordinate system

Horizontal and vertical equilibrium force equations of this segment are:

$$H(s) = \int_S^L W_x(s)ds + F_x \tag{1}$$

$$V(s) = \int_S^L W_y(s)ds + F_y \tag{2}$$

Consider an infinitesimal element of length ds of the beam as shown in Figure 3.

Applying the moment equilibrium of the element yields:

$$\frac{dM(s)}{dS}ds + V(s)dx - H(s)dy = 0 \tag{3}$$

The link is characterized by a nondimensional arc length $u \in [0,1]$ along its neutral axis that is defined as $u = s/L$. From Euler–Bernoulli law and the geometrical relationship of the infinitesimal element:

$$M(u) = \frac{EI(u)}{L}\frac{d}{du}\psi(u) \tag{4}$$

$$\frac{d}{du}x = L\cos(\psi(u) + \eta(u)) \tag{5}$$

$$\frac{d}{du}y = L\sin(\psi(u) + \eta(u)) \tag{6}$$

where Ψ and η are the angle of rotation and initial slope of the beam, respectively. As shown in Equation 4, both non-uniform moment of inertia, $I(u)$, and non-homogenous Young's modulus, $E(u)$, can be incorporated. Substituting Equations (4), (5) and (6) into Equation (3), results in:

$$\begin{aligned}&\frac{1}{L^2}\frac{d}{du}(EI(u)\psi'(u)) + V(u)\cos(\psi(u) + \eta(u)) - \\ &H(u)\sin(\psi(u) + \eta(u)) = 0\end{aligned} \tag{7}$$

The prime over a variable denotes the first derivative with respect to u. The detail of above derived equations are found in the literature [23].

Equations (1), (2), (5), (6) and (7) are the governing equations of the large deflection behavior of cantilever beams of non-uniform thickness and non-homogeneous material subjected to various types of loadings. In order to proceed solving the equations use the direct multiple shooting method, the beam is divided into N arbitrary subsegments by N-1 control points. In each subsegment, η, EI are constant and W_x, W_y are uniform. As the number of control points increases, it can be shown the accuracy of the results increases.

On the other hand the computational procedure and consequently, the CPU time increased. Dividing the beam in the subsegments less than 10 (i.e $N \le 10$) is usually sufficient to determine the shape of a compliant beam.

Equations (1), (2), (5), (6), and (7) form a system of coupled first-order ordinary differential equations as follows:

$$q_i' = \begin{Bmatrix}\psi_i \\ \psi_i' \\ x_i \\ y_i \\ H_i \\ V_i\end{Bmatrix}' = \begin{Bmatrix}\psi_i' \\ \frac{L_i^2}{EI_i}(H_i\sin(\psi_i + \eta_i) - V_i\cos(\psi_i + \eta_i)) \\ L_i\cos(\psi_i + \eta_i) \\ L_i\sin(\psi_i + \eta_i) \\ -L_i(W_x)_i \\ -L_i(W_y)_i\end{Bmatrix} \tag{8}$$

$, \ i = 1 \, to \, N$

Each set of governing Equation (8) can be applied to any sub segment. In each sub segment the non-dimensional parameter u_i along with neutral axis of the beam follows the following relationship:

$$0 = u_0 < u_1 < ... < u_N = 1 \tag{9}$$

The $q = [q_1^T, q_2^T, ... q_N^T]^T$ is the vector of unknowns of the flexible beam. Equation (8) is a set of ordinary differential equations which can be solved numerically by algorithms such as "ode45" in MATLAB® [29], using an initial guess for vector q. However, the problem of the beam is a BVP which is then treated as an IVP.

2. 2 Boundary Conditions Both ends of the beam are constrained, which can be expressed by constraint equations. The equations may be divided into four classes including force, position, moment and slope constraints depending on the joint type. The joints are also categorized in four types including clamped, simply supported, free and sliding. The joints along with the boundary conditions are shown in Table 1.

TABLE 1 . Boundary conditions and joint types

	Force	Position		Moment	Slope
Clamp	NA*	$\begin{cases} x_i = const \\ y_i = const \end{cases}$		$\psi_i = 0$	NA
Simply Support	NA	$\begin{cases} x_i = const \\ y_i = const \end{cases}$		$\frac{EI_i}{L_i}\psi_i' = M_e$	NA
Free	$\begin{cases} H_i = F_x \\ V_i = F_y \end{cases}$	NA		$\frac{EI_i}{L_i}\psi_i' = M_e$	NA
Sliding	Along x axis	$H_i = F_x$	$y_i = const$	$\frac{EI_i}{L_i}\psi_i' = M_e$	NA
	Along y axis	$V_i = F_y$	$x_i = const$	$\frac{EI_i}{L_i}\psi_i' = M_e$	NA

*NA: not available and are to be determined during computation

The bending moment and forces are normalized as

$$\bar{F}_x = \frac{L^2}{EI(0)}F_x, \quad \bar{F}_y = \frac{L^2}{EI(0)}F_y, \quad \bar{W}_x = \frac{L^3}{EI(0)}W_x, \quad \bar{W}_y = \frac{L^3}{EI(0)}W_y, \quad \bar{M} = \frac{L}{2\pi EI(0)}M \tag{10}$$

When a beam is divided into N subsegments, it has $6 \times (N-1)$ continuity conditions at each control point as shown in Equation (11) for the forces, positions, moment and slope in each subsegment.

$$j = 1 \; to \; N-1: \begin{cases} \psi_j(u_j) = \psi_{j+1}(u_j) \\ \frac{EI_j}{L_j}\psi_j'(u_j) = \frac{EI_{j+1}}{L_{j+1}}\psi_{j+1}'(u_j) \\ x_j(u_j) = x_{j+1}(u_j) \\ y_j(u_j) = y_{j+1}(u_j) \\ h_j(u_j) = h_{j+1}(u_j) \\ v_j(u_j) = v_{j+1}(u_j) \end{cases} \tag{11}$$

From Equation (8), a beam has $6 \times N$ differential equations, a number that must be equal to the number of constraint equations in order to be solved to obtain $6 \times N$ unknowns. There are 6 constraints at both end points which are presented in Table 1. As a result, a beam with N sub segments has $6 \times N$ constraint equations. So, the number of equations is equal to the number of boundary conditions. Constraint equations including boundary conditions and continuity equations lead to a vector of nonlinear algebraic equations:

$$F(q) = 0 \tag{12}$$

2. 3. Multiplie Shooting Method The shooting method is an approach for solving a BVP by converting it to an IVP. The unknown initial values need to be assumed, and then the IVP to be solved to obtain the boundary values subsequently. If the obtained results are not equal to the predefined boundary conditions, an iterative approach such as Newton's method will be applied. The initial guess will iteratively be changed until the boundary conditions on the other side are satisfied. Similar to solve nonlinear equations, the simple Shooting method has the following two major concerns in implementation:

- The solution is sensitive to the initial guess. For highly nonlinear or unstable problems, the initial guess has to be close enough to a solution. Wrong initial guesses may lead to singularities or cause the solutions not exist on bounds.
- Slow convergence of solutions. Solving nonlinear systems utilizes root finding techniques that typically make these methods converging at slower rate. Also the round off errors that emerge in each iteration may slow down the convergence rate of the solutions.

The multiple shooting method was originally developed by Keller [26]. This procedure divides the interval into small subdivisions and solves a BVP by implementing a simple shooting method in each of the smal intervals until the results converge to an acceptable degree of accuracy. Considering uniqueness and existence of solutions of BVP, the Gauss-Newton method has been used to solve BVP which begins with $6 \times N$ initial guesses q_0. It solves q iteratively, until Equation (12) has been satisfied. Therefore, it is possible to approximate Equation (13) in each iteration (k^{th}) by the first-order function:

$$F(q) \cong F(q_k) + J(q_k)(q - q_k) \tag{13}$$

In this equation, q_k is the initial guess and $J(q_k)$ is the Jacobian matrix evaluated numerically at q_k using the finite difference method. Calculation of Jacobian matrix is very time consuming, hence, it the Quasi- Newton approach such as Broyden's method [27] can be used to approximate the Jacobian matrix in each iteration. To find vector q by solving Equation (13), when the $\min \| F(q) \|$ become less than the tolerance ε the iteration will be terminated:

$$\min \| F(q) \| \le \varepsilon \tag{14}$$

Next, the approximation Equation (15) is squared and expanded:

$$\| J_k q - b_k \|^2 = (q^t J_k^t J_k q - 2b_k^t J_k q + b_k^t b_k) \tag{15}$$

To simplify the notation, the vectors $J(q_k)$ and $F(q_k)$ are defined as J_k and F_k, respectively. Where b_k is a constant vector in each iteration and can be defined as follows:

$$b_k = J_k q_k - F_k \tag{16}$$

Since $b_k^t b_k$ is a constant vector in each iteration, it can be eliminated from Equation (15). The remaining problem is to develop in the standard form of the quadratic programming problem (QP) which can be solved using various methods such as "fmincon" or "quadprog" in MATLAB®:

$$\min(\frac{1}{2} q^t H q + F^t q) \; ; q_{lb} \le q \le q_{ub} \tag{17}$$

Solving the quadratic programming problem leads to find the vector q, that is the initial guess for the next step q_{k+1}. This iteration procedure will be repeated until Equation (14) is satisfied for a given ε.

3. RESULTS

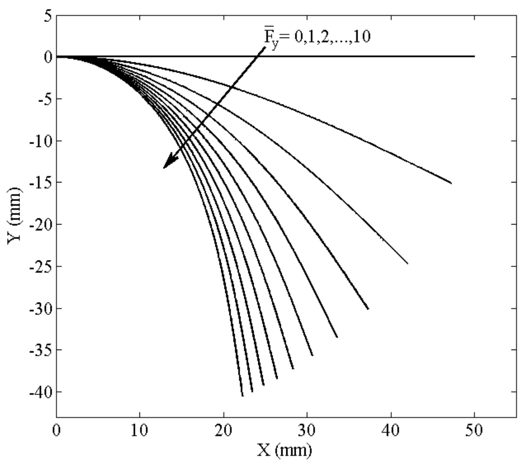
Numerical examples will now be given to illustrate the validity of the present study and to investigate the force-deflection behavior of the beam under different boundary conditions. The nonlinear FEM result is obtained by the co-rotational procedure implemented in ANSYS® [30], in which the beam 3 element has been used to perform large-displacement static analysis. The tolerance ε in all examples was set at 1e-8. The adopted dimensions in the numerical example are presented in Table 2.

TABLE 2. Geometrical properties adopted for the numerical examples.

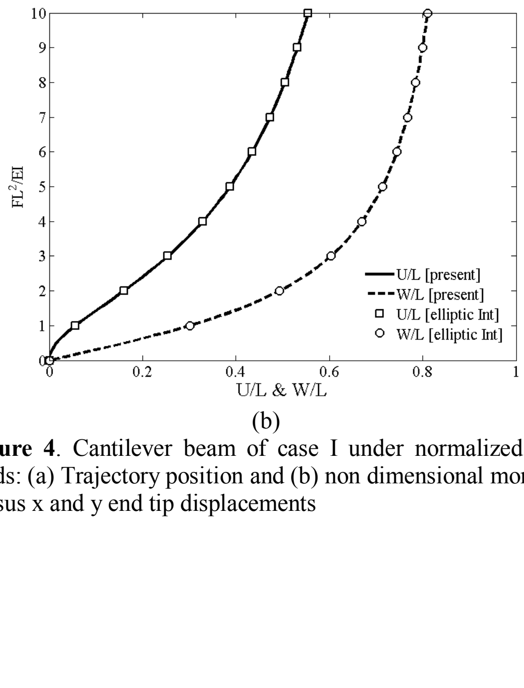
Length, *L* [mm]	50
In-plane thickness, *h* [mm]	0.4
out-plane thickness, *b* [mm]	6
Young's modulus, *E* [GPa]	115 (titanium Ti-6 Al-2 V)[31]

3. 1. Case I: Beam with Clamped-free Boundary Condition under End Point Force and Moment

Figure 4(a) shows the trajectory position of an initially straight cantilever beam made up of linear elastic material under normalized end point concentrated load $\bar{F}_y$ (0 to 10 in –y direction). Figure 4(b) shows the non dimensional loads versus non dimensional displacements of end tip in x and y directions (*U*/*L*, *W*/*L* respectively). In Table 3 the results are compared with the elliptic integral solution [28].



(a)

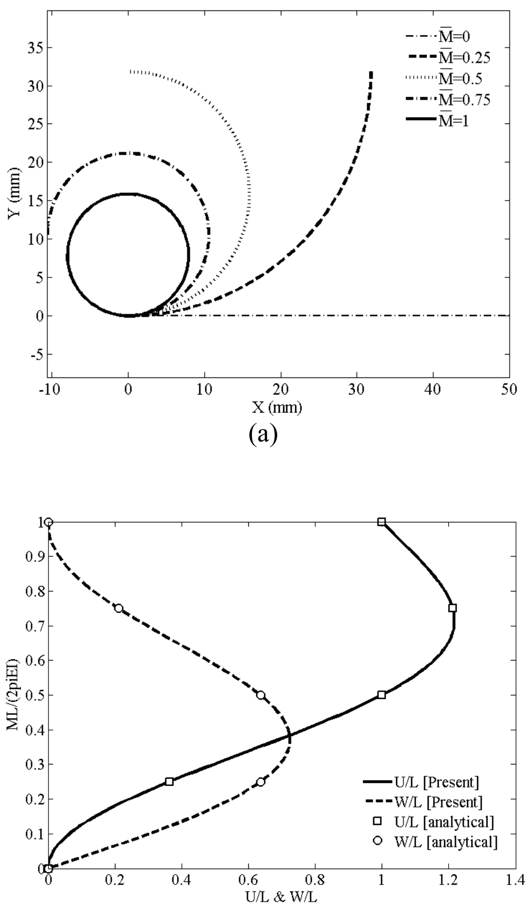


(b)

Figure 4. Cantilever beam of case I under normalized end loads: (a) Trajectory position and (b) non dimensional moment versus x and y end tip displacements

TABLE 3. Comparison of present study results with those of the elliptic integral solution for case I under end point force

$\bar{F}_y$	$\frac{U}{L}$		$\frac{W}{L}$	
	Ref [27]	Present Study	Ref [27]	Present Study
1	0.05643	0.0564	0.30172	0.3017
2	0.16064	0.1606	0.49346	0.4935
3	0.25442	0.2544	0.60325	0.6033
4	0.32894	0.3289	0.66996	0.6700
5	0.38763	0.3876	0.71379	0.7138
6	0.43459	0.4346	0.74457	0.7446
7	0.47293	0.4729	0.76737	0.7674
8	0.50483	0.5048	0.78498	0.7850
9	0.53182	0.5318	0.79906	0.7991
10	0.55500	0.5550	0.81061	0.8106



(a)

(b)

Figure 5. Cantilever beam of case I under normalized end moments: (a) Trajectory position and (b) non dimensional moment versus x and y end tip displacements

TABLE 4. Comparison of presented study with analytical method for case I under end moment

$\bar{M}$	$\frac{U}{L}$		$\frac{W}{L}$	
	analytical	present study	analytical	present study
0.25	0.6366	0.6366	0.3634	0.3634
0.5	0.6366	0.6366	1.0000	1.0000
0.75	0.2122	0.2122	1.2122	1.2122
1	0.0000	0.3718e-8	1.0000	1.0000

The trajectory position of a cantilever beam subjected to normalized end moment $\bar{M}$ (from 0 to 1) and four load steps are shown in Figure 5(a). Since all terms including forces, i.e. H(s) and V(s), in Equation (7) are zero, the problem can have an analytical solution ($\psi(u) = 2\pi\bar{M} \times u$). Non dimensional loads versus non dimensional displacements and slopes are plotted in Fig. 5(b). In Table 4 a comparison of analytical solutions and the present numerical study is provided.

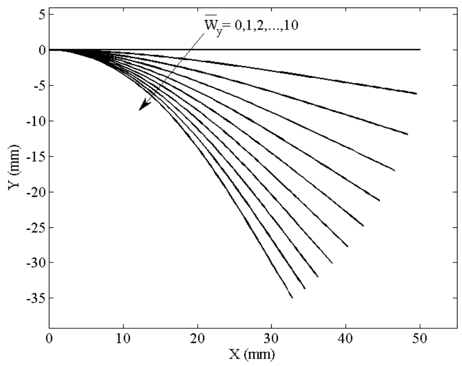
3. 2. Case II: Beam with Various Boundary Conditions under Uniform Distributed Loading

An inextensible beam with various boundary conditions such as clamped-free, clamped-sliding and simply supported-sliding under normalized uniformly distributed load $\bar{W}_y$ (0 to 10 in –y direction) has been investigated using the present method. The results are compared to those from a finite element solution. Since the load is uniform, the beam is considered without sub-segments (N=1).

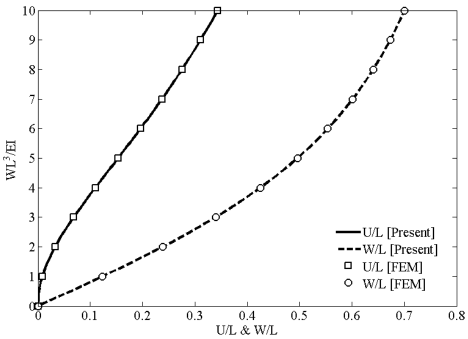
Non dimensional force displacements of the beam for (a) clamped-free, (b) clamped-sliding and (c) simply support-sliding boundary conditions are shown in Figures 6, 7 and 8, respectively.

The results are compared to existing results in the previous studies [22]. As may be seen in Figures 6 there are not any difference between results of presented method and those of FEM, while there some differences as shown in Figures 7 and 8. That was due to the more complexity of supports of the beam.

Table 5 demonstrates the normalized loads versus normalized endpoint displacements and slopes under (a) clamped-free, (b) clamped-sliding and (c) simply supported-sliding boundary conditions.

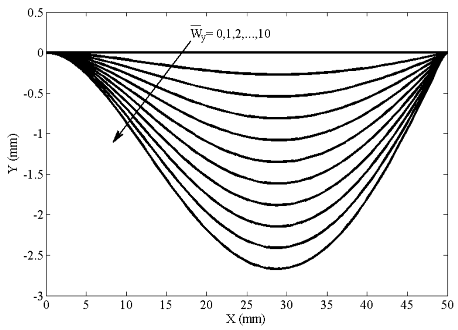


(a)

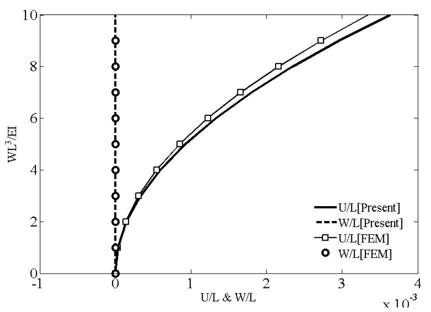


(b)

Figure 6. Beam of case II with clamped-free boundary condition: (a) Trajectory position and (b) non dimensional load- displacement



(a)



(b)

Figure 7. Beam of case II with clamped-sliding boundary condition: (a) Trajectory position and (b) non dimensional load- displacement.

TABLE 5. Normalized displacements and slopes for (a) clamped-free, (b) clamped-sliding and (c) simply supported-sliding boundary conditions

$\overline{W}_y$	$\frac{U}{L}$	$\frac{W}{L}$	$\frac{\psi}{2\pi}$
1	0.0088	0.1235	0.0263
2	0.0331	0.2385	0.0512
3	0.0685	0.3396	0.0738
4	0.1099	0.4252	0.0937
5	0.1533	0.4959	0.1109
6	0.1963	0.5539	0.1258
7	0.2372	0.6015	0.1386
8	0.2756	0.6406	0.1496
9	0.3110	0.6731	0.1592
10	0.3436	0.7002	0.1675

(a) Clamp-Free

$\overline{W}_y$	$\frac{U}{L}$	$\frac{\psi}{2\pi}$
1	0.2e-8	0.0022
2	0.0001	0.0045
3	0.0003	0.0068
4	0.0006	0.0090
5	0.0009	0.0112
6	0.0013	0.0135
7	0.0018	0.0157
8	0.0023	0.0179
9	0.0030	0.0201
10	0.0036	0.0222

(b) Clamp-Sliding

$\overline{W}_y$	$\frac{U}{L}$	$\frac{\psi}{2\pi}$
1	0.0004	0.0066
2	0.0017	0.0132
3	0.0037	0.0198
4	0.0066	0.0262
5	0.0102	0.0326
6	0.0144	0.0389
7	0.0193	0.0450
8	0.0248	0.0509
9	0.0307	0.0567
10	0.0371	0.0623

(c) Simply Support-Sliding

3. 3. Case III: Beam with Clamped-sliding Boundary Condition under End Point Moment

The behavior of a beam with clamped-sliding boundary condition subjected to four normalized endpoint moment cases is depicted in Figure 9 (a). The results are compared to the finite element results. Figure 9 (b) shows the normalized endpoint slopes versus non dimensional arc length for each load case.

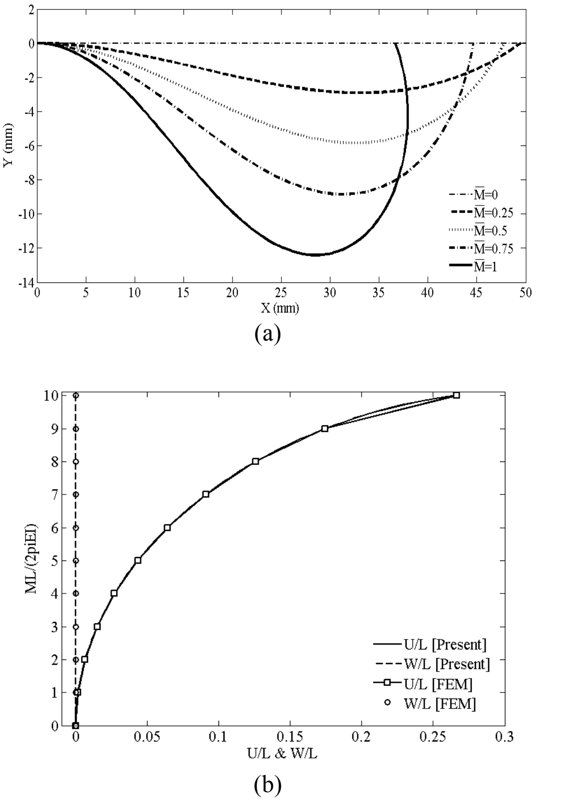


Figure 9. Beam of case III with clamped-sliding boundary condition: (a) Trajectory position and (b) non dimensional load- displacement.

3. 4. Case IV: Non-prismatic Beam with Clamped-sliding Boundary Condition under Non-uniform Distributed Loading

In the final study a general case of research is presented. A non- prismatic beam with clamped-sliding boundary condition subjected to a ramp loading is shown in Figure 10 (a). Out of plane thickness is assumed to be uniform and equals to 1 mm ($b = 1mm$), in plane thickness is varied from h_1=0.6 mm at u=0 to h_2=0.4 at u=1, respectively. The load w_0 is equal to 800 N applied at u=0. Figure 10 illustrates the trajectory position and non-dimensional load-displacement of the beam with clamped-sliding boundary condition.

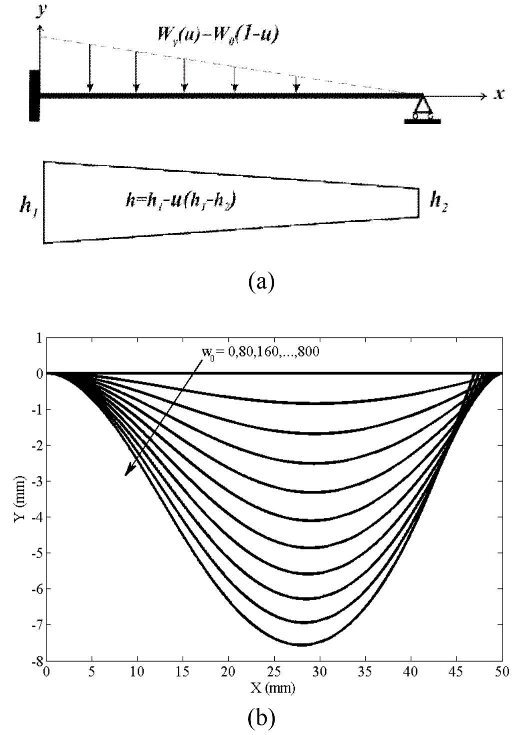


Figure 10. Non prismatic beam of case IV with clamp-sliding boundary condition under distributed ramp load: (a) Trajectory position and (b) non dimensional load- displacement.

TABLE 6. comparison of convergence results for all case studies

	Mean convergence iteration	computation time(s)	Accuracy
Case I (end force)	4	3.3910	1e-8
Case I (end moment)	1	2.3390	1e-8
Case II (Clamp-Free)	5	3.4940	1e-8
Case II (Clamp-sliding)	5	4.1030	1e-8
Case II (simply supported-sliding)	5	3.9620	1e-8
Case III	3	3.1180	1e-8
Case IV	6	4.8360	1e-8

3. 5. Convergency and Computation Time

Table 6 represents the mean convergence iteration, computational time and accuracy of results for all above case studies. The number of both sub-segments and load steps is assumed to be equal to 10. An Intel Pentium 4 Core 2 Duo (2 GHz CPU with 3 GB RAM) hardware has been used to run the program and find the deflected shapes of the beam. Prior to starting the iterations, the Jacobin matrix should be calculated once, however for each load step an approximation will be made using

Quasi- Newton algorithm and the iteration will be continued. Therefore, all of the computational time is spent to calculate the accurate value of the Jacobian matrix which is presented in all examples.

4. DISCUSSION

As data presented in Table 5, the computation time for all studied cases is found to be less than 5 seconds. The lowest computational recorded time corresponds to the Case I which is the cantilever beam under end point load, while the highest one (almost 2 times more) corresponds to Case IV which is the non-prismatic beam with clamped-sliding boundary condition under non-uniform distributed loading. As shown in this table, as the complexity of load case increases or the boundary condition changes towards cancelation of more degrees of freedoms, the computation time increases. This table also shows that an increase in number of sub-segments reduces the convergence rate and increases computational time. However, the solution still converges to a reasonably good accuracy as the number of load steps is increased. Reduction in number of load steps leads to divergence of results. This fact again shows that the nonlinear behavior depends on the loading history. Decreasing the value of ε increases the accuracy of results. However, it is less effective than increasing the number of sub-segments or load steps.

Some assumptions have been made to drive the governing Equation (9) including neglecting the bending effects and shear deformation. Nevertheless, the results are of good accuracy and agreed to those of FEM, as shown in Figures 4-9. It may be seen in Figures 4 and 6 that changes in the shape due to the distributed load is less as compared to the concentrated load. Also, as expected from Figures 7 and 8, the deflection will be increased (in the order of 10 percent) if one of boundary conditions is changed from clamped to simply supported. Figure 9 once gain validates the present approach as the slope at the clamped and simply supported ends matches with the boundary conditions. From Figure 10, it can be seen that the effect of the cross section on the deflection is also significant. This is due to the fact the area moment of inertia is more sensitive to in-plane thickness than out of plane thickness.

5. CONCLUSION

In this paper, large deflection of compliant beams of variable thickness made of non-homogenous material under combined load and multiple boundary conditions has been investigated. The problem was modeled mathematically, and the governing equations were derived and solved numerically using a shooting method. The large deflection analysis of compliant beams in the form of generalized case e.g. combined load, variable thickness, non-homogenous material and multiple boundary conditions has been investigated. Several examples are presented and a comparison was performed with FEM results. The presented method features high degree of accuracy with a considerably good convergence rate and is capable of handling a wide range of cases. It was also found that the multiple shooting method is not sensitive to initial values. Unlike the other methods the quadratic programming problem does not need to calculate the inverse Jacobian matrix that may introduce singularities. Therefore, the presented study can extensively be used in design of compliant mechanisms as well as a comparative for accuracy assessment of other methods.

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Large Deflection Analysis of Compliant Beams of Variable Thickness and Non-homogenous Material under Combined Load and Multiple Boundary Conditions

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چکیده

این مقاله به مطالعه و تحقیق در خصوص روش جدیدی می‌پردازد که رفتار تیرهای منشوری و غیر منشوری همگن و غیر همگن را در تغییر مکانهای بزرگ مورد بررسی قرار می‌دهد. این بررسی در شرایط مرزی متفاوت و بازگذاری مرکب انجام شده است. تهیه مدل ریاضی این تیرها منجر به تشکیل یک دستگاه معادلات متشکل از شش معادله دیفرانسیل معمولی از مرتبه اول گردید. شرایط غیر خطی هندسی با استفاده از روش پرتابه‌ای چندگانه و برنامه‌ریزی غیر خطی تحلیل گردید. نتایج بدست آمده از حل با روش فوق با نتایج موجود از تحقیقات انجام شده در این زمینه مقایسه گردید. نتایج نشان می‌دهد که این روش را می‌توان برای طراحی مکانیزمهای انعطاف‌پذیر، فنرهای غیرخطی و یا موضوعات مرتبط مورد استفاده قرار داد.

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