CALCULATION OF THE BUCKLING LOAD AND EIGENFREQUENCIES FOR PLANAR TRUSS STRUCTURES WITH MULTI-SYMMETRY

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Abstract In this paper, an efficient method is presented for calculating the buckling load and eigenfrequencies of the planar truss structures having double symmetry axes. In this method, considering the axes of symmetry, the region in which structural system is situated is divided into four subregions, namely upper, lower, left and right subregions. The stiffness matrix of the entire system is then formed, and using the existing direct symmetry and reverse symmetry, the relationships between the entries of the matrix are established. Examples are included to illustrate the process of the presented method.

Keywords Symmetry, multi-symmetry, truss structures, decomposition, Eigen solution, graph

چکیده در این مقاله ناحیه‌ای که برای سیستم سازه‌ای در نظر گرفته شده است به چهار زیر ناحیه تقسیم می‌شود. بالا، پایین، چپ و راست. پس از تشکیل ماتریس سختی با استفاده از تقارنهای مستقیم و معکوس رابطه بین درایه‌های ماتریس سختی استخراج می‌گردد. مثالها جهت شرح و توضیح این روش بیان شده است.

1. INTRODUCTION

Symmetry has been widely used in science and engineering using group theory [1-5]. Many eigenvalue problems arise in many scientific and engineering problems such as free vibration, and forced vibration and stability of structures [6-8]. While the basic mathematical ideas are independent of the size of matrices, the numerical determination of eigenvalues and eigenvectors becomes more complicated as the dimensions of matrices increase. Special methods are beneficial for efficient solution of such problems, especially when their corresponding matrices are highly sparse.

Graph theoretical methods are developed for decomposing and healing the graph models of structures, in order to calculate the eigenvalues of matrices and graph matrices with special patterns [9-11]. In these methods, the eigenvectors corresponding to such patterns for the symmetry of Form I, Form II and Form III are studied, and the applications to vibrating mass-spring systems and frame structures are developed in [12] and [13], respectively. These forms are also applied to calculating the buckling load of symmetric frames using linear algebra and canonical forms [14].

Consider a structural system with two translational degrees of freedom (DOFs) per node which has two axes of symmetry. Suppose each DOF is parallel to one of the axes and is perpendicular to the other axis. For the following three cases, one can find matrices in canonical forms, and using the symmetry relationships twice, developed previously [10, 15], one can find 4 submatrices. The union of the eigenvalues for these 4 submatrices results in the eigenvalues of the original matrix.

As mentioned before, various symmetries have been previously developed. In these symmetries which will be presented in Section 2, a matrix is decomposed into two submatrices S and T and the eigenvalues of these submatrices results in the eigenvalues of the main matrix.

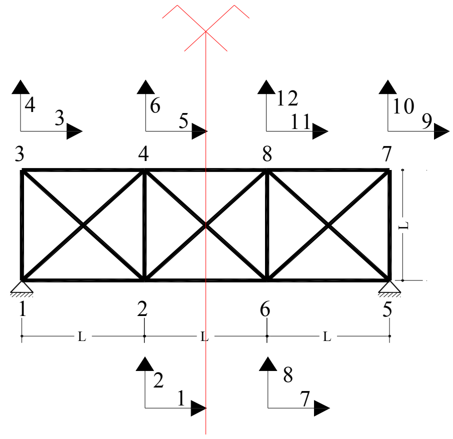
In this paper, the region in which the structural system is situated is divided into upper, lower, left and right subregions. The stiffness matrix of the entire system is formed and then using the existing direct and reverse symmetries, relationships between the entries of the matrix are established.

2. SINGLE SYMMETRIES A AND B

As described in [15], symmetry of structural systems with each node having two DOFs can be studied in two general forms A and B. These forms are briefly described in the following subsections.

2.1. Symmetry of Form A (modified Form II symmetry) For trusses with axes of symmetry not passing through nodes with active DOFs, we have the Form A symmetry, as shown in Fig. 1(a). The main reason for not being able to employ the previously developed forms of symmetry for calculating the buckling and eigenfrequencies load of truss structures is due to the existence of oblique cross members. These members affect the entries of the stiffness and geometric stiffness matrices and change the sign of some entries. Separation of the horizontal and vertical DOFs as shown in Fig. 1(b) results in stiffness matrices of the symmetric trusses for the case where the axis of symmetry does not pass through the nodes with active DOFs, as Figure 1.

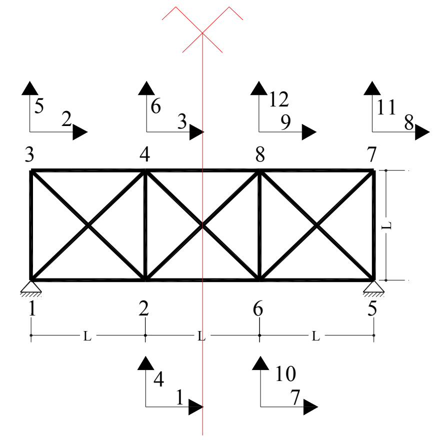
First, the nodes in the left-hand side (LHS) of the symmetry axis are numbered, followed by the numbering of the nodes in the right-hand side (RHS). Now the horizontal DOFs (along x-axis) are first numbered and then the vertical DOFs (in y-direction) are numbered for the LHS. A similar numbering is then performed for the DOFs of the RHS.



(a)

Modification of numbering for the DOFs

$\Downarrow$



(b)

Figure 1. Modified numbering of the DOFs (Form A)

Pattern of the weighted block adjacency matrix M is as follows:

$$
\begin{array}{cc}
 & \begin{array}{cccc} \multicolumn{2}{c}{\text{LHS}} & \multicolumn{2}{c}{\text{RHS}} \\ H & V & H & V \end{array} \\
M = & \left[\begin{array}{cccc} A & C & D & F \\ C & B & F & E \\ D & -F & A & -C \\ -F & E & -C & B \end{array}\right]
\end{array}
\begin{array}{l} \left.\begin{array}{l} H \\ V \end{array}\right\} \text{LHS} \\ \left.\begin{array}{l} H \\ V \end{array}\right\} \text{RHS} \end{array}
$$

Conditions for symmetry:

All the submatrices are symmetric, except F which is anti-symmetric.

$$A^T = A \quad B^T = B \quad C^T = C \quad D^T = D \quad F^T = -F$$

Here $F^T = -F$ corresponding to the effect of the horizontal DOFs of the LHS nodes on the vertical DOFs of the RHS, and vice versa.

Performing the row and column permutations, the matrix M can be transformed into the following Schur's form:

$$M = \begin{bmatrix} A+D & C-F & D & F \\ C+F & B-E & F & E \\ 0 & 0 & A-D & -C-F \\ 0 & 0 & F-C & B+E \end{bmatrix}$$

Thus

$$Det[M] = \underbrace{Det\begin{bmatrix} A+D & C-F \\ C+F & B-E \end{bmatrix}}_{S} \times \underbrace{Det\begin{bmatrix} A-D & -C-F \\ -C+F & B+E \end{bmatrix}}_{T}$$

Therefore, the eigenvalues of M can be obtained as:

$$\lambda(M) = \lambda(S) \cup \lambda(T)$$

It should be noted that S and T are both symmetric, because F is anti-symmetric and the remaining submatrices are symmetric. The above relationships provide the basis of the algebraic method of this paper for trusses having an odd number of bays.

2.2. Symmetry of Form B (modified Form III symmetry) For trusses with axes of symmetry passing through nodes with active DOFs, one will have the Form B symmetry, as shown in Fig. 2. First, the nodes in the LHS of the symmetry axis are numbered followed by the numbering of the nodes in the RHS, and then the central nodes on the axis of symmetry are numbered. Now the horizontal DOFs (along x-axis) are first numbered and then the vertical DOFs (in y-direction) are numbered for the LHS. A similar numbering is then performed for the DOFs of the RHS. Finally, the horizontal DOFs (in x-direction) are numbered followed by the vertical DOFs (in y-direction) for the central nodes on the axis of symmetry.

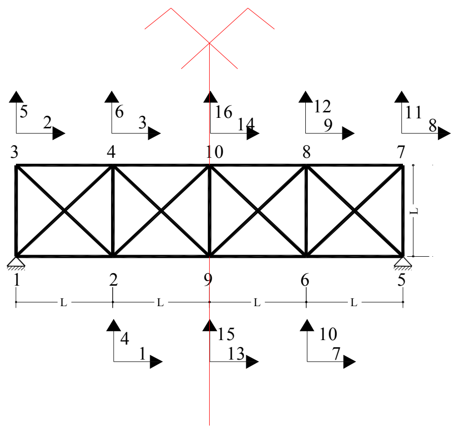


Figure 2. A symmetric truss with axis of symmetry passing through central nodes

Pattern of the matrix M is as follows:

$$M = \left[\begin{array}{cc:cc:cc} A & C & D & F & G & I \\ C & B & F & E & I & H \\ \hdashline D & -F & A & -C & G & -I \\ -F & E & -C & B & -I & H \\ \hdashline G^T & I^T & G^T & -I^T & J & L \\ I^T & H^T & -I^T & H^T & L & K \end{array}\right]$$

Nodes on the LHS of the axis | Nodes on the RHS of the axis | Nodes on the axis of symmetry

Column permutations

$$\left[\begin{array}{cc:cc:cc} A & C & G & I & D & F \\ B & B & H & G & F & E \\ \hdashline D & -F & G & -I & A & -C \\ -F & E & -I & H & -C & B \\ \hdashline G^T & I^T & J & 0 & G^T & -J^T \\ J^T & H^T & 0 & K & -J^T & H^T \end{array}\right]$$

Exchange of rows

$$\left[\begin{array}{cc:cc:cc} A & C & G & I & D & F \\ C & B & I & H & F & E \\ \hdashline G^T & I^T & J & 0 & G^T & -I^T \\ I^T & H^T & 0 & K & -I^T & H^T \\ \hdashline D & -F & G & -I & A & -C \\ -F & E & -I & H & -C & B \end{array}\right]$$

$$\xrightarrow[C_2 = C_2 - C_6]{C_1 = C_1 - C_5} \left[\begin{array}{cc:cc:cc} A+D & C-F & G & I & D & F \\ C+F & B-E & I & H & F & E \\ \hdashline 2G^T & 2I^T & J & 0 & G^T & -I^T \\ 0 & 0 & 0 & K & -I^T & H^T \\ \hdashline A+D & -F+C & G & -I & A & -C \\ -F-C & E-B & -I & H & -C & B \end{array}\right]$$

Now the following Schur's form is obtained as

$$\xrightarrow[R_6 = R_6 + R_2]{R_5 = R_5 - R_1} \left[\begin{array}{cc:cccc} A+D & C-F & G & I & D & F \\ C+F & B-E & I & H & F & E \\ 2G^T & 2I^T & J & 0 & G^T & -I^T \\ \hdashline 0 & 0 & 0 & K & -I^T & H^T \\ 0 & 0 & 0 & -2I & A-D & -C-F \\ 0 & 0 & 0 & 2H & -C+F & B+E \end{array}\right]$$

Therefore

$$Det\left[M\right] =$$

$$Det \underbrace{\begin{bmatrix} A+D & C-F & G \\ C+F & B-E & I \\ 2G^T & 2I^T & J \end{bmatrix}}_{S} \times \underbrace{\begin{bmatrix} A-D & -C-F & -2I \\ -C+F & B+E & 2H \\ -I^T & H^T & K \end{bmatrix}}_{T}$$

Thus

$$\lambda(M) = \lambda(S) \cup \lambda(T)$$

Matrix L is always a null matrix due to the symmetry. One may move the nodes on the axis of symmetry in the y direction, these nodes should not move in x direction.

The matrices A, B, C, D, and E are symmetric, and F is anti-symmetric. These submatrices are $n \times n$, with n being the number of free nodes in each side of the axis of symmetry. I, H, and G are $n \times m$ submatrices, where m is the number of node on the axis and L, J, and K are $m \times m$ submatrices. L is replaced by the null matrix 0.

3. DOUBLE SYMMETRIES

For structural systems with two DOFs per node having two axes of symmetry, the following cases may arise:

Case 1. The symmetry is of Type A with respect to both axes, i.e. there exist no active DOFs on either of the axes. In this case, it is sufficient to perform Type A symmetry and reordering the rows and columns of the matrices S and T, again Form A symmetries are created, and the operations corresponding to this symmetry are performed.

Case 2. The symmetry is of Type A in one direction and of Type B in the other direction, i.e. there exist active DOFs just in one axis of symmetry, e.g. only on horizontal axis. In this case first Form A symmetry is performed, and then the order of rows and columns S and T are changed to create Form A symmetry. Then the operations corresponding to Form A symmetry is applied to either of the matrices S and T to produce Form B symmetries.

Case 3. The symmetry is of Type B in both directions, i.e. there exist active DOFs on both axes. In this case, first Form B symmetry is performed, and then the order of rows and columns at S and T are changed to create a new form of form B symmetry, which is actually a generalized form of form B symmetry. Then the operations corresponding to this new form B symmetry is applied to either of the matrices S and T. The new submatrices will contain the eigenvalues of the main matrix.

In this section, three types of symmetry are defined. The structural matrices of the considered systems are decomposed and transformed into three canonical forms. For each case the buckling load and eigenfrequencies of a symmetric system is obtained as the least of the buckling load and eigenfrequencies of the constructed submatrices.

3.1. Case 1 Symmetry In this case, there exists no active DOF on the symmetry axes. The symmetry axes subdivide the space in which the structure is situated into four subregions. Suppose there exists m active DOFs in the direction 1 and n active DOFs in direction 2. Now the numbering of two typical selected nodes from the upper-left subregion, and those of the corresponding symmetric nodes in the other 3 subregions are shown in Fig. 3.

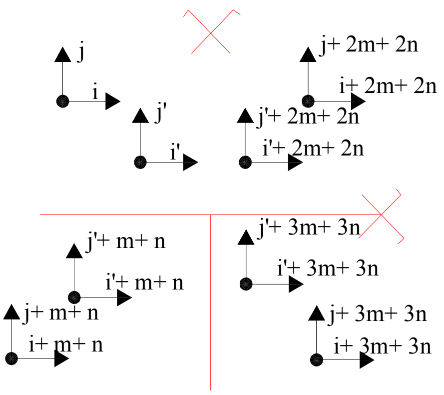


Figure 3. General form of DOFs in Case 1 symmetry

For this purpose, first all the DOFs in direction 1 at the upper part of the LHS of the vertical axis of symmetry are numbered followed by the DOFs at the direction 2 at this part, Fig. 3. Then numbering is performed for the corresponding DOFs at the lower part of LHS of the vertical axis of symmetry, upper part of RHS of the vertical axis of symmetry and at last lower part of RHS of the vertical axis of symmetry respectively and in a similar order to upper part of the LHS of the vertical axis of symmetry.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ A_{12}^T & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ A_{13}^T & A_{23}^T & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ A_{14}^T & A_{24}^T & A_{34}^T & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} \\ A_{15}^T & A_{25}^T & A_{35}^T & A_{45}^T & A_{55} & A_{56} & A_{57} & A_{58} \\ A_{16}^T & A_{26}^T & A_{36}^T & A_{46}^T & A_{56}^T & A_{66} & A_{67} & A_{68} \\ A_{17}^T & A_{27}^T & A_{37}^T & A_{47}^T & A_{57}^T & A_{67}^T & A_{77} & A_{78} \\ A_{18}^T & A_{28}^T & A_{38}^T & A_{48}^T & A_{58}^T & A_{68}^T & A_{78}^T & A_{88} \end{bmatrix}$$

It should be noted that not all the DOFs of a node need to be active, and the following relationships are not based on this assumption.

Now using the symmetry one can establish the following relationships between each pairs of the DOFs in the four subregions:

$$K_{i,i'} = K_{i+m+n,i'+m+n} = K_{i+2m+2n,i'+2m+2n} = K_{i+3m+2n,i'+3m+3n}$$
$$\Rightarrow A_{11} = A_{33} = A_{55} = A_{77}$$

$$K_{j,j'} = K_{j+m+n,j'+m+n} = K_{j+2m+2n,j'+2m+2n} = K_{i+3m+3n,j'+3m+3n}$$
$$\Rightarrow A_{22} = A_{44} = A_{66} = A_{88}$$

$$K_{i.j'} = -K_{i+m+n,j'+m+n} = -K_{i+2m+2n,j'+2m+2n} = K_{i+3m+3n,j'+3m+3n}$$
$$\Rightarrow A_{12} = -A_{34} = -A_{56} = A_{78}$$

$$K_{i,i'+m+n} = K_{i+2m+2n,i'+3m+3n} \Rightarrow A_{13} = A_{57}$$
$$K_{j,j'+m+n} = K_{j+2m+2n,j'+3m+3n} \Rightarrow A_{24} = A_{68}$$
$$K_{i,j'+m+n} = -K_{i+2m+2n,j'+3m+3n} \Rightarrow A_{14} = -A_{58}$$
$$K_{j',i+m+n} = -K_{j'+2m+2n,i+3m+3n} \Rightarrow A_{23} = -A_{67}$$
$$K_{i,i'+2m+2n} = K_{i+m+n,i'+3m+3n} \Rightarrow A_{15} = A_{37}$$
$$K_{j,j'+2m+2n} = K_{j+m+n,j'+3m+3n} \Rightarrow A_{26} = A_{48}$$
$$K_{i,j'+2m+2n} = -K_{i+m+n,j'+3m+3n} \Rightarrow A_{16} = -A_{38}$$
$$K_{j,i'+2m+2n} = -K_{j+m+n,i'+3m+3n} \Rightarrow A_{25} = -A_{47}$$
$$K_{i.i'+3m+3n} = K_{i+2m+2n,i'+m+n} \Rightarrow A_{17} = A_{35}$$
$$K_{j,j'+3m+3n} = K_{j+2m+2n,j'+m+n} \Rightarrow A_{28} = A_{46}$$
$$K_{i,j'+3m+3n} = -K_{i+2m+2n,j'+m+n} \Rightarrow A_{18} = -A_{36}$$
$$K_{j,i'+3m+3n} = -K_{j+2m+2n,i'+m+n} \Rightarrow A_{27} = -A_{45}$$
$$K_{j,i'+2m+2n} = -K_{j+2m+2n,i'} \Rightarrow A_{16} = -A_{25}^T$$
$$K_{j,i'+3m+3n} = K_{j+3m+3n,i'} \Rightarrow A_{18} = A_{27}^T$$
$$K_{j+2m+2n,i'+3m+3n} = -K_{i'+2m+2n,j+3m+3n} \Rightarrow A_{58} = -A_{67}^T$$
$$K_{i,j'+m+n} = -K_{j',i+m+n} \Rightarrow A_{14} = -A_{23}^T$$
$$K_{i,i'+m+n} = K_{i',i+m+n} \Rightarrow A_{13} = A_{13}^T$$
$$K_{j,j'+m+n} = K_{j',j+m+n} \Rightarrow A_{24} = A_{24}^T$$
$$K_{i,i'+2m+2n} = K_{i',i+2m+2n} \Rightarrow A_{15} = A_{15}^T$$
$$K_{j,j'+2m+2n} = K_{j',j+2m+2n} \Rightarrow A_{26} = A_{26}^T$$
$$K_{i,i'+3m+3n} = K_{i',i+3m+3n} \Rightarrow A_{17} = A_{17}^T$$
$$K_{j,j'+3m+3n} = K_{j',j+3m+3n} \Rightarrow A_{28} = A_{28}^T$$

The above six relationship show that apart from the submatrices on the diagonal, the submatrices **D**, **E**, **G**, **H**, **J** and **K** are also symmetric.

Considering the above relationships, the pattern of the stiffness matrix of the structure can be re-written in the following form:

$$F_{22}=\left[\begin{array}{cc:cc:cc:cc} A & C & D & F & G & I & J & L \\ C^T & B & -F^T & E & -I^T & H & L^T & K \\ \hdashline D & -F & A & -C & J & -L & G & -I \\ F^T & E & -C^T & B & -L^T & K & I^T & H \\ \hdashline G & -I & J & -L & A & -C & D & -F \\ I^T & H & -L^T & K & -C^T & B & F^T & E \\ \hdashline J & L & G & I & D & F & A & C \\ L^T & K & -I^T & H & -F^T & E & C^T & B \end{array}\right]$$

The above relationships also hold for other parameters such as mass and damping. In other words, apart from the stiffness matrix having the above form, the mass and damping and geometric stiffness matrices have also the same patterns. Similar point is also applicable for subsequent relationships.

Permuting the rows and columns we have

$$F_{22}=\left[\begin{array}{cc:cc:cc:cc} A & D & C & F & G & J & I & L \\ D & A & -F & -C & J & G & -L & -I \\ \hdashline C^T & -F^T & B & E & -I^T & L^T & H & K \\ F^T & -C^T & E & B & -L^T & I^T & K & H \\ \hdashline G & J & -I & -L & A & D & -C & -F \\ J & G & L & I & D & A & F & C \\ \hdashline I^T & -L^T & H & K & -C^T & F^T & B & E \\ L^T & -I^T & K & H & -F^T & C^T & E & B \end{array}\right]$$

The above matrix has the Form A symmetry, and therefore using the equations of Section (2-1) one can decompose it into the submatrices S and T as:

$$S=\begin{bmatrix} A+G & D+J & C-I & F-L \\ D+J & A+G & -F+L & -C+I \\ C^T-I^T & -F^T+L^T & B-H & E-K \\ F^T-L^T & -C^T+I^T & E-K & B-H \end{bmatrix}$$

$$T=\begin{bmatrix} A-G & D-J & -C-I & -F-L \\ D-J & A-G & F+L & C+I \\ -C^T-I^T & F^T+L^T & B+H & E+K \\ -F^T-L^T & C^T+I^T & E+K & B+H \end{bmatrix}$$

Again, the permutation of rows and columns of the above matrix with Form A symmetry leads to the following matrices:

$$S'=\begin{bmatrix} A+G & C-I & D+J & F-L \\ C^T-I^T & B-H & -F^T+L^T & E-K \\ D+J & -F+L & A+G & -C+I \\ F^T-L^T & E-K & -C^T+I^T & B-H \end{bmatrix}$$

$$T'=\begin{bmatrix} A-G & -C-I & D-J & -F-L \\ -C^T-I^T & B+H & F^T+L^T & E+K \\ D-J & F+L & A-G & C+I \\ -F^T-L^T & E+K & C^T+I^T & B+H \end{bmatrix}$$

Now utilizing the equations of Section (2-1), the submatrices S and T for the above matrices can be written as:

$$SS'=\begin{bmatrix} A+G+D+J & C-I-F+L \\ C^T-I^T-F^T+L & B-H-E+K \end{bmatrix}$$

$$ST'=\begin{bmatrix} A-G+D-J & -C-I+F+L \\ -C^T-I^T+F^T+L^T & B+H-E-K \end{bmatrix}$$

$$TS'=\begin{bmatrix} A+G-D-J & -C+I-F+L \\ -C^T+I^T-F^T+L^T & B-H+E-K \end{bmatrix}$$

$$TT'=\begin{bmatrix} A-G-D+J & C+I+F+L \\ C^T+I^T+F^T+L^T & B+H+E+K \end{bmatrix}$$

The union of the above four submatrices forms the eigenvalue of the entire matrix as:

$$\lambda(F_{22})=\lambda(SS')\cup\lambda(ST')\cup\lambda(TS')\cup\lambda(TT')$$

Example: Consider an indeterminate truss as shown in Fig. 4. For this truss: $E=2.07\times10^7\ kN/m^2$, $I=100\,cm^4$, $r=7800\ kg/m^3$ and $A=10\,cm^2$.

The buckling load and frequencies of the truss are calculated as follows:

$$P_{cr}=[185890]\,kN \quad P_{cr(SS')}=[480000]\,kN$$

$$P_{cr(TS')}=[185890]\,kN \quad P_{cr(TT')}=[246600]\,kN$$

$$P_{cr(ST')}=[1740400]\,kN$$

$$\omega=\left[82.8, 82.8, 182.47, 182.47, 244.41, 244.41, 283.5, 283.5\right] rad/sec$$

$$\omega_{SS'}=\left[82.8, 244.4\right] \quad \omega_{TS'}=\left[182.47, 182.47\right]$$

$$\omega_{ST'}=\left[283.52, 283.52\right] \quad \omega_{TT'}=\left[82.8, 244.4\right]$$

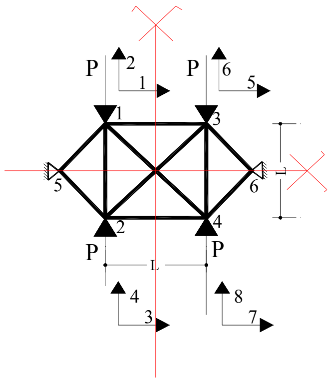


Figure 4. An indeterminate planar truss with Case 1 symmetry

3.2. Case 2 symmetry In this case, there are active DOFs on one of the symmetry axes (here the horizontal axis of symmetry). Similar to the previous case, there exists m active DOFs in the direction 1 and n active DOFs in direction 2. Also in the RHS there exists m DOFs in the horizontal direction and n DOFs in the vertical direction. Similar situation is present for the LHS.

Now in addition to the previous case, where we had 4 subregions (Fig. 3), two nodes in the LHS of the horizontal axis together with the corresponding nodes in the RHS of this axis are considered. In this case, the numbering of the DOFs is performed as shown in Fig. 5.

For this purpose, first all the DOFs in direction 1 at the upper part of the LHS of the vertical axis of symmetry (which there is no active DOF on it) are numbered followed by the DOFs at the direction 2 at this part, Fig. 5. Then numbering is performed for the corresponding DOFs at the lower part of LHS of the vertical axis of symmetry, upper part of RHS of the vertical axis of symmetry and lower part of RHS of the vertical axis of symmetry respectively and in a similar order to upper part of the LHS of the vertical axis of symmetry. Then all the DOFs in direction 1 of nodes on left hand part of horizontal axis of symmetry are numbered followed by the DOFs in direction 2 of these nodes. Then numbering is performed for the corresponding DOFs of nodes on the right hand part of horizontal axis of symmetry in a similar order to DOFs of nodes on the left hand part of horizontal axis of symmetry.

Considering the above numbering scheme, the $\mathbf{F}_{22}$ matrix will be present in the overall stiffness matrix and therefore only the relationships between the complementary entries of the stiffness matrix should be formed.

$$F_{23}=\left[\begin{array}{cccccccc:cccc}
 & & & & & & & & A_{1,9} & A_{1,10} & A_{1,11} & A_{1,12} \\
 & F_{22} & & & & & & & A_{2,9} & A_{2,10} & A_{2,11} & A_{2,12} \\
 & & & & & & & & A_{3,9} & A_{3,10} & A_{3,11} & A_{3,12} \\
 & & & & & & & & A_{4,9} & A_{4,10} & A_{4,11} & A_{4,12} \\
 & & & & & & & & A_{5,9} & A_{5,10} & A_{5,11} & A_{5,12} \\
 & & & & & & & & A_{6,9} & A_{6,10} & A_{6,11} & A_{6,12} \\
 & & & & & & & & A_{7,9} & A_{7,10} & A_{7,11} & A_{7,12} \\
 & & & & & & & & A_{8,9} & A_{8,10} & A_{8,11} & A_{8,12} \\
\hdashline
A^T_{1,9} & A^T_{2,9} & A^T_{3,9} & A^T_{4,9} & A^T_{5,9} & A^T_{6,9} & A^T_{7,9} & A^T_{8,9} & A_{9,9} & A_{9,10} & A_{9,11} & A_{9,12} \\
A^T_{1,10} & A^T_{2,10} & A^T_{3,10} & A^T_{4,10} & A^T_{5,10} & A^T_{6,10} & A^T_{7,10} & A^T_{8,10} & A^T_{9,10} & A_{10,10} & A_{10,11} & A_{10,12} \\
A^T_{1,11} & A^T_{2,11} & A^T_{3,11} & A^T_{4,11} & A^T_{5,11} & A^T_{6,11} & A^T_{7,11} & A^T_{8,11} & A^T_{9,11} & A^T_{10,11} & A_{11,11} & A_{11,12} \\
A^T_{1,12} & A^T_{2,12} & A^T_{3,12} & A^T_{4,12} & A^T_{5,12} & A^T_{6,12} & A^T_{7,12} & A^T_{8,12} & A^T_{9,12} & A^T_{10,12} & A^T_{11,12} & A^T_{12,12}
\end{array}\right]$$

Now using the symmetry one can establish the following relationships between the stiffness of each pairs of the DOFs in the four subregions, and the left and right part of the horizontal axis of symmetry

$$K_{i,i''}=K_{i+2m+2n,i''+m'+n'} \quad \Rightarrow \quad A_{1,9}=A_{5,11}$$

$$K_{i,j''}=-K_{i+2m+2n,j''+m'+n'} \quad \Rightarrow \quad A_{1,10}=-A_{5,12}$$

$$K_{j,i''}=-K_{j+2m+2n,j''+m'+n'} \quad \Rightarrow \quad A_{2,9}=-A_{6,11}$$

$$K_{j,j''}=K_{j+2m+2n,j''+m'+n'} \quad \Rightarrow \quad A_{2,10}=A_{6,12}$$

$$K_{i,i''+m'+n'}=K_{i+2m+2n,i''} \quad \Rightarrow \quad A_{1,11}=A_{5,9}$$

$$K_{j,j''+m'+n'}=-K_{j+2m+2n,j''} \quad \Rightarrow \quad A_{2,12}=A_{6,10}$$

$$K_{i,j''+m'+n'}=-K_{j+2m+2n,i''} \quad \Rightarrow \quad A_{1,12}=-A_{5,10}$$

$$K_{j,i''+m'+n'}=K_{i+2m+2n,i''} \quad \Rightarrow \quad A_{2,11}=-A_{6,9}$$

$$K_{i+m+n,i''}=K_{i+3m+2n,i''+m'+n'} \quad \Rightarrow \quad A_{3,9}=A_{7,11}$$

$$K_{i+m+n,j''}=-K_{i+3m+3n,j''+m'+n'} \quad \Rightarrow \quad A_{3,10}=-A_{7,12}$$

$$K_{j+m+n,i''}=-K_{j+3m+3n,i''+m'+n'} \quad \Rightarrow \quad A_{4,9}=-A_{8,11}$$

$$K_{i+m+n,i''+m'+n'}=K_{i+3m+3n,i''} \quad \Rightarrow \quad A_{3,11}=A_{7,9}$$

$$K_{i+m+n,j''+m'+n'}=-K_{i+3m+3n,j''} \quad \Rightarrow \quad A_{3,12}=-A_{7,10}$$

$$K_{j+m+n,i''+m'+n'}=-K_{j+3m+3n,i''} \quad \Rightarrow \quad A_{4,11}=-A_{8,9}$$

$$K_{j+m+n,j''+m'+n'}=K_{j+3m+3n,j''} \quad \Rightarrow \quad A_{4,12}=A_{8,10}$$

$$K_{i',i'''} = K_{i''+m'+n',i'''+m'+n'} \quad \Rightarrow \quad A_{9,9} = A_{11,11}$$

$$K_{j'',j'''} = K_{j''+m'+n',j'''+m'+n'} \quad \Rightarrow \quad A_{10,10} = A_{12,12}$$

$$K_{i'',j'''} = -K_{i''+m'+n',j'''+m'+n'} = 0 \quad \Rightarrow \quad A_{9,10} = A_{11,12} = 0$$

$$K_{i'',j'''+m'+n'} = K_{j'',i'''+m'+n'} = 0 \quad \Rightarrow \quad A_{9,12} = A_{10,11} = 0$$

$$K_{i,i''} = K_{i+m+n,i''} \quad \Rightarrow \quad A_{1,9} = A_{3,9}$$

$$K_{j,j''} = K_{j+m+n,j''} \quad \Rightarrow \quad A_{2,10} = A_{4,10}$$

$$K_{i,j''} = -K_{i+m+n,j''} \quad \Rightarrow \quad A_{1,10} = -A_{3,10}$$

$$K_{j,i''} = -K_{j+m+n,i''} \quad \Rightarrow \quad A_{2,9} = -A_{4,9}$$

$$K_{i+2m+2n,i''+m'+n'} = K_{i+3m+3n,i''+m'+n'} \quad \Rightarrow \quad A_{5,11} = A_{7,11}$$

$$K_{i+2m+2n,j''+m'+n'} = -K_{i+3m+3n,j''+m'+n'} \quad \Rightarrow \quad A_{5,12} = -A_{7,12}$$

$$K_{j+2m+2n,i''+m'+n'} = -k_{j+3m+3n,i''+m'+n'} \quad \Rightarrow \quad A_{6,11} = -A_{8,11}$$

$$K_{j+2m+2n,j''+m'+n'} = K_{j+3m+3n,j''+m'+n'} \quad \Rightarrow \quad A_{6,12} = A_{8,12}$$

$$K_{j'',j'''+m'+n'} = K_{j'',j'''+m'+n'} \quad \Rightarrow \quad A_{9,12} = A_{9,12}^T$$

$$K_{i'',i'''+m'+n'} = K_{i''',i''+m'+n'} \quad \Rightarrow \quad A_{9,10} = A_{9,10}^T$$

In this way the following relationships hold:

$$\begin{cases} A_{1,11} = A_{3,11} \\ A_{1,12} = -A_{3,12} \\ A_{2,11} = -A_{4,11} \\ A_{2,12} = A_{4,12} \end{cases} \qquad \begin{cases} A_{5,9} = A_{7,9} \\ A_{5,10} = -A_{7,10} \\ A_{6,9} = -A_{8,9} \\ A_{6,10} = A_{8,10} \end{cases}$$

Therefore, in the case of symmetry, the general pattern of the overall stiffness matrix can be rewritten in the following two forms:

$$F_{23} = \left[\begin{array}{cc:cc:cc:cc:cc:cc} A & C & D & F & G & I & J & L & M & O & Q & S \\ C^T & B & -F^T & E & -I^T & H & L^T & K & P & N & T & R \\ \hdashline D & -F & A & -C & J & -L & G & -I & M & -O & Q & -S \\ F^T & E & -C^T & B & -L^T & K & I^T & H & -P & N & -T & R \\ \hdashline G & -I & J & -L & A & -C & D & -F & Q & -S & M & -O \\ I^T & H & -L^T & K & -C^T & B & F^T & E & -T & R & -P & N \\ \hdashline J & L & G & I & D & F & A & C & Q & S & M & O \\ L^T & K & -I^T & H & -F^T & E & C^T & B & T & R & P & N \\ \hdashline M^T & P^T & M^T & -P^T & Q^T & -T^T & Q^T & T^T & U & 0 & W & 0 \\ O^T & N^T & -O^T & N^T & -S^T & R^T & S^T & R^T & 0 & V & 0 & X \\ \hdashline Q^T & T^T & Q^T & -T^T & M^T & -P^T & M^T & P^T & W & 0 & U & 0 \\ S^T & R^T & -S^T & R^T & -O^T & N^T & O^T & N^T & 0 & X & 0 & V \end{array}\right]$$

By permuting the rows and columns we have

$$F_{23} = \left[\begin{array}{cc:cc:cc:cc:cc:cc} A & D & M & C & F & O & G & J & Q & I & L & S \\ D & A & M & -F & -C & -O & J & G & Q & -L & -I & -S \\ \hdashline M^T & M^T & U & P^T & -P^T & 0 & Q^T & Q^T & W & -T^T & T^T & 0 \\ C^T & -F^T & P & B & E & N & -I^T & L^T & T & H & K & R \\ \hdashline F^T & -C^T & -P & E & B & N & -L^T & I^T & -T & K & H & R \\ O^T & -O^T & 0 & N^T & N^T & V & -S^T & S^T & 0 & R^T & R^T & X \\ \hdashline G & J & Q & -I & -L & -S & A & D & M & -C & -F & -O \\ J & G & Q & L & I & S & D & A & M & F & C & O \\ \hdashline Q^T & Q^T & W & T^T & -T^T & 0 & M^T & M^T & U & -P^T & P^T & 0 \\ I^T & -L^T & -T & H & K & R & -C^T & F^T & -P & B & E & N \\ \hdashline L^T & -I^T & T & K & H & R & F^T & C^T & P & E & B & N \\ S^T & -S^T & 0 & R^T & R^T & X & -O^T & O^T & 0 & N^T & N^T & V \end{array}\right]$$

The above matrix has the Form A symmetry and thus using the relationships established in Section 2-1, this matrix can be decomposed into S and T as follows:

$$S = \left[\begin{array}{ccc:ccc} A+G & D+J & M+Q & C-I & F-L & O-S \\ D+J & A+G & M+Q & -F+L & -C+I & -O+S \\ M^T+Q^T & M^T+Q^T & U+W & P^T+T^T & -P^T-T^T & 0 \\ \hdashline C^T-I^T & -F^T+L^T & P+T & B-H & E-K & N-R \\ F^T-L^T & -C^T+I^T & -P-T & E-K & B-H & N-R \\ O^T-S^T & -O^T+S^T & 0 & N^T-R^T & N^T-R^T & V-X \end{array}\right]$$

$$T = \left[\begin{array}{ccc:ccc} A-G & D-J & M-Q & -C-I & -F-L & -O-S \\ D-J & A-G & M-Q & F+L & C+I & O+S \\ M^T-Q^T & M^T-Q^T & U-W & -P^T+T^T & P^T-T^T & 0 \\ \hdashline -C^T-I^T & F^T+L^T & -P+T & B+H & E+K & N+R \\ -F^T-L^T & C^T+I^T & P-T & E+K & B+H & N+R \\ -O^T-S^T & O^T+S^T & 0 & N^T+R^T & N^T+R^T & V+X \end{array}\right]$$

Again by permutation of rows and column of the above matrices, we obtain matrices with Form B symmetry.

$$S' = \left[\begin{array}{cc:cc:cc} A+G & C-I & D+J & F-L & M+Q & O-S \\ C^T-I^T & B-H & -F^T+L^T & E-K & P+T & N-R \\ \hdashline D+J & -F+L & A+G & -C+I & M+Q & -O+S \\ F^T-L^T & E-K & -C^T+I^T & B-H & -P-T & N-R \\ \hdashline M^T+Q^T & P^T+T^T & M^T+Q^T & -P^T-T^T & U+W & 0 \\ O^T-S^T & N^T-R^T & -O^T+S^T & N^T-R^T & 0 & V-X \end{array}\right]$$

$$T' = \left[\begin{array}{cc:cc:cc} A-G & -C-I & D-J & -F-L & M-Q & -O-S \\ -C^T-I^T & B+H & F^T+L^T & E+K & -P+T & N+R \\ \hdashline D-J & F+L & A-G & C+I & M-Q & O+S \\ -F^T-L^T & E+K & C^T+I^T & B+H & P-T & N+R \\ \hdashline M^T-Q^T & -P^T+T^T & M^T-Q^T & P^T-T^T & U-W & 0 \\ -O^T-S^T & N^T+R^T & O^T+S^T & N^T+R^T & 0 & V+X \end{array}\right]$$

Now using the relationships of Section 2-2, the matrices S and T for the above matrices are extracted.

$$SS' = \begin{bmatrix} A+G+D+J & C-I-F+L & M+Q \\ C^T-I^T-F^T+L^T & B-H-E+K & P+T \\ 2M^T+2Q^T & 2P^T+2T^T & U+W \end{bmatrix}$$

$$TS' = \begin{bmatrix} A+G-D-J & -C+I-F+L & -2O+2S \\ -C^T+I^T-F^T+L^T & B-H+E-K & 2N-2R \\ -O^T+S^T & N^T-R^T & V-X \end{bmatrix}$$

$$ST' = \begin{bmatrix} A-G+D-J & -C-I+F+L & M-Q \\ -C^T-I^T+F^T+L^T & B+H-E-K & -P+T \\ 2M^T-2Q^T & -2P^T+2T^T & U-W \end{bmatrix}$$

$$TT' = \begin{bmatrix} A-G-D+J & C+I+F+L & 2O+2S \\ C^T+I^T+F^T+L^T & B+H+E+K & 2N+2R \\ O^T+S^T & N^T+R^T & V+X \end{bmatrix}$$

The above matrix has the Form B symmetry and therefore

$$\lambda(F_{23}) = \lambda(SS') \cup \lambda(ST') \cup \lambda(TS') \cup \lambda(TT')$$

Example: Consider an indeterminate truss as shown in Fig. 6. For this truss we have

$E = 2.07 \times 10^7\ kN/m^2$, $I = 100\,cm^4$,

$\rho = 7800\ kg/m^3$ and $A = 10\,cm^2$.

The buckling load and eigenfrequencies of the truss is calculated as follows:

$P_{cr} = [126000]\,kN \qquad P_{cr(SS')} = [407600]\,kN$

$P_{cr(TS')} = [126000]\,kN \qquad P_{cr(ST')} = [511100]\,kN$

$P_{cr(TT')} = [172900]\,kN$

$w = [58.137, 116.36, 117.047, 186.35, 197.23, 197.23, 218.65, 262.18$
$320.39, 323.47, 326.19, 353.77]\,rad/sec$

$w_{SS'} = [320.39, 117.047, 186.35] \quad w_{TS'} = [326.19, 197.23, 116.36]$

$w_{ST'} = [353.77, 262.18, 197.23] \quad w_{TT'} = [58.137, 218.65, 323.43]$

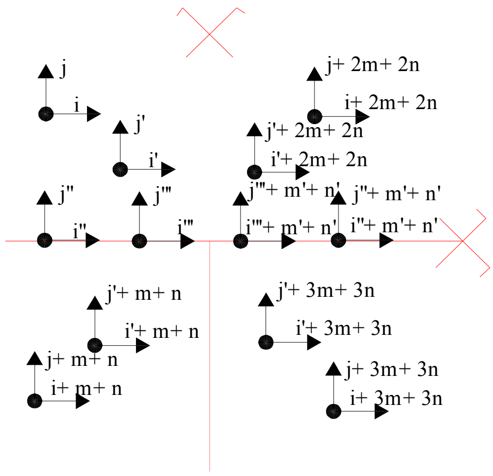


Figure 5. General form of DOFs in Case 2 symmetry.

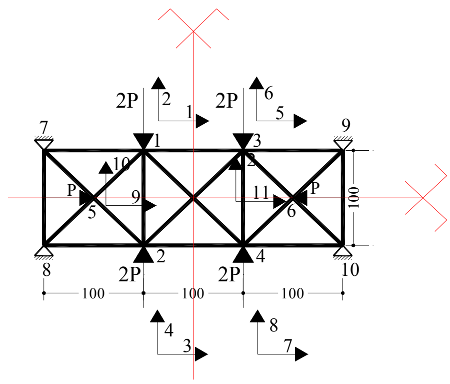


Figure 6. An indeterminate truss with Case 2 symmetry

3.3. Case 3 Symmetry In this case, there are nodes with active DOFs on both axes of symmetry. Suppose in each 4 subregions we have *m* active DOFs in direction 1 and *n* active DOFs in the direction 2. Also in the RHS of the horizontal axis of symmetry, there are *m*' DOFs in the horizontal direction and *n*' DOFs in vertical direction. Similarly, in the LHS and in the upper part of the vertical axis of symmetry there are *m*'' DOFs in the horizontal direction and n'' DOFs in the vertical direction. Similar situation is present in the lower part. Now, apart from the nodes which were selected in the previous case (Fig. 5), two nodes are selected in the upper part of the vertical axis of symmetry and the corresponding DOFs in the lower part of the axis. Here, the numbering of the DOFs of the nodes is performed as shown in Fig. 5.

For this purpose, first all the DOFs in direction 1 at the upper part of the LHS of the vertical axis of symmetry (where there is no active DOF on it) are numbered, followed by the DOFs at the direction 2 at this part, Fig. 7. Then numbering is performed for the corresponding DOFs at the lower part of LHS of the vertical axis of symmetry, upper part of RHS of the vertical axis of symmetry and lower part of RHS of the vertical axis of symmetry, respectively. In a a similar manner the DOFs of the upper part of the LHS of the vertical axis of symmetry are numbered. Then all the DOFs in direction 1 of nodes on left hand part of horizontal axis of symmetry are numbered, followed by the DOFs in direction 2 of these nodes. Then numbering is carried out for the corresponding DOFs of nodes on the right hand part of horizontal axis of symmetry in a similar order to the DOFs of the nodes on the left hand part of horizontal axis of symmetry. Finally, all the DOFs in direction 1 of nodes on upper part of vertical axis of symmetry are numbered followed by the DOFs in direction 2 of these nodes. Then numbering is performed for the corresponding DOFs of nodes on the lower part of vertical axis of symmetry in a similar order to the DOFs of the nodes on the upper part of vertical axis of symmetry. Considering this numbering, the matrix F23 is present as before and it is only necessary to obtain the relationships corresponding to the complementary part of the stiffness matrix.

$$F_{33}=\left[\begin{array}{cccccccccccc|cccc}
\multicolumn{12}{c|}{} & A_{1,13} & A_{1,14} & A_{1,15} & A_{1,16}\\
\multicolumn{12}{c|}{} & A_{2,13} & A_{2,14} & A_{2,15} & A_{2,16}\\
\multicolumn{12}{c|}{} & A_{3,13} & A_{3,14} & A_{3,15} & A_{3,16}\\
\multicolumn{12}{c|}{} & A_{4,13} & A_{4,14} & A_{4,15} & A_{4,16}\\
\multicolumn{12}{c|}{F_{23}} & A_{5,13} & A_{5,14} & A_{5,15} & A_{5,16}\\
\multicolumn{12}{c|}{} & A_{6,13} & A_{6,14} & A_{6,15} & A_{6,16}\\
\multicolumn{12}{c|}{} & A_{7,13} & A_{7,14} & A_{7,15} & A_{7,16}\\
\multicolumn{12}{c|}{} & A_{8,13} & A_{8,14} & A_{8,15} & A_{8,16}\\
\multicolumn{12}{c|}{} & A_{9,13} & A_{9,14} & A_{9,15} & A_{9,16}\\
\multicolumn{12}{c|}{} & A_{10,13} & A_{10,14} & A_{10,15} & A_{10,16}\\
\multicolumn{12}{c|}{} & A_{11,13} & A_{11,14} & A_{11,15} & A_{11,16}\\
\multicolumn{12}{c|}{} & A_{12,13} & A_{12,14} & A_{12,15} & A_{12,16}\\
\hline
A^T_{1,13} & A^T_{2,13} & A^T_{3,13} & A^T_{4,13} & A^T_{5,13} & A^T_{6,13} & A^T_{7,13} & A^T_{8,13} & A^T_{9,13} & A^T_{10,13} & A^T_{11,13} & A^T_{12,13} & A_{13,13} & A_{13,14} & A_{13,15} & A_{13,16}\\
A^T_{1,14} & A^T_{2,14} & A^T_{3,14} & A^T_{4,14} & A^T_{5,14} & A^T_{6,14} & A^T_{7,14} & A^T_{8,14} & A^T_{9,14} & A^T_{10,14} & A^T_{11,14} & A^T_{12,14} & A^T_{13,14} & A_{14,14} & A_{14,15} & A_{14,16}\\
A^T_{1,15} & A^T_{2,15} & A^T_{3,15} & A^T_{4,15} & A^T_{5,15} & A^T_{6,15} & A^T_{7,15} & A^T_{8,15} & A^T_{9,15} & A^T_{10,15} & A^T_{11,15} & A^T_{12,15} & A^T_{13,15} & A^T_{14,15} & A_{15,15} & A_{15,16}\\
A^T_{1,16} & A^T_{2,16} & A^T_{3,16} & A^T_{4,16} & A^T_{5,16} & A^T_{6,16} & A^T_{7,16} & A^T_{8,16} & A^T_{9,16} & A^T_{10,16} & A^T_{11,16} & A^T_{12,16} & A^T_{13,16} & A^T_{14,16} & A^T_{15,16} & A_{16,16}
\end{array}\right]$$

Now using the symmetry, one can establish the following relationships between the stiffnesses of each pair of DOFs in four subregions and the complementary relationships between the DOFs above and under the horizontal axis of symmetry, and the upper and lower part of the vertical axis of symmetry.

$$K_{i,k} = K_{i+m+n,k+m''+n''} \Rightarrow A_{1,13} = A_{3,15}$$

$$K_{i,l} = -K_{i+m+n,l+m''+n''} \Rightarrow A_{1,14} = -A_{3,16}$$

$$K_{j,k} = -K_{j+m+n,k+m''+n''} \Rightarrow A_{2,13} = -A_{4,15}$$

$$K_{j,l} = K_{j+m+n,k+m''+n''} \Rightarrow A_{2,14} = A_{4,16}$$

$$K_{i,k} = K_{i+2m+2n,k} \Rightarrow A_{1,13} = A_{5,13}$$

$$K_{i,l} = -K_{i+2m+2n,l} \Rightarrow A_{1,14} = -A_{5,14}$$

$$K_{j,k} = -K_{j+2m+2n,k} \Rightarrow A_{2,13} = -A_{6,13}$$

$$K_{j,l} = K_{j+2m+2n,l} \Rightarrow A_{2,14} = A_{6,14}$$

$$K_{i,k} = K_{i+3m+3n,k+m''+n''} \Rightarrow A_{1,13} = A_{7,15}$$

$$K_{i,l} = K_{i+3m+3n,l+m''+n''} \Rightarrow A_{1,14} = A_{7,16}$$

$$K_{j,k} = K_{j+3m+3n,k+m''+n''} \Rightarrow A_{2,13} = A_{8,15}$$

$$K_{j,l} = K_{j+3m+3n,l+m''+n''} \Rightarrow A_{2,14} = A_{8,16}$$

Similarly

$$
\begin{cases}
A_{1,15} = A_{3,13} & A_{1,15} = A_{5,15} & A_{1,15} = A_{7,13} \\
A_{1,16} = -A_{3,14} & A_{1,16} = -A_{5,16} & A_{1,16} = A_{7,14} \\
A_{2,15} = -A_{4,13} & A_{2,15} = -A_{6,15} & A_{2,15} = A_{8,13} \\
A_{2,16} = A_{4,14} & A_{2,16} = A_{6,16} & A_{2,16} = A_{8,14}
\end{cases}
$$

$$
\begin{aligned}
K_{i'',k} &= K_{i''+m'+n',k} && \Rightarrow && A_{9,13} = A_{11,13} \\
K_{i'',l} &= -K_{i''+m'+n',l} && \Rightarrow && A_{9,14} = -A_{11,14} \\
K_{j'',k} &= -K_{j''+m'+n',k} && \Rightarrow && A_{10,13} = -A_{12,13} \\
K_{j'',l} &= K_{j''+m'+n',l} && \Rightarrow && A_{10,14} = A_{12,14} \\
K_{i'',k} &= K_{i'',k+m''+n''} && \Rightarrow && A_{9,13} = A_{9,15} \\
K_{i'',l} &= -K_{i'',l+m''+n''} && \Rightarrow && A_{9,14} = -A_{9,16} \\
K_{j'',k} &= -K_{j'',k+m''+n''} && \Rightarrow && A_{10,13} = -A_{10,15} \\
K_{j'',l} &= K_{j'',l+m''+n''} && \Rightarrow && A_{10,14} = A_{10,16}
\end{aligned}
$$

In a similar way,

$$
\begin{cases}
A_{9,13} = A_{11,15} \\
A_{9,14} = A_{11,16} \\
A_{10,13} = A_{12,15} \\
A_{10,14} = A_{12,16}
\end{cases}
$$

$$
\begin{aligned}
K_{k,k'} &= K_{k+m''+n'',k'+m''+n''} && \Rightarrow && A_{13,13} = A_{15,15}, \quad A_{14,14} = A_{16,16} \\
K_{k,l'} &= -K_{k+m''+n'',l'+m''+n''} = 0 && \Rightarrow && A_{13,14} = A_{15,16} = 0 \\
K_{k,l'+m''+n''} &= K_{l,k'+m''+n''} = 0 && \Rightarrow && A_{13,16} = A_{14,15} = 0 \\
K_{k,k'+m''+n''} &= K_{k,k+m''+n''} = && \Rightarrow && A_{13,15} = A_{13,15}^T \\
K_{l,l'+m'+n'} &= K_{l',l+m''+n''} && \Rightarrow && A_{14,16} = A_{14,16}^T
\end{aligned}
$$

Therefore, in the case of having symmetry, the general pattern of the overall stiffness matrix can be rewritten in the following form:

$$
F_{33} = \left[\begin{array}{c:cccc}
 & AA & CC & EE & GG \\
 & DD & BB & HH & FF \\
 & EE & -GG & AA & -CC \\
 & -HH & FF & -DD & BB \\
 & AA & -CC & EE & -GG \\
F_{23} & -DD & BB & -HH & FF \\
 & EE & GG & AA & CC \\
 & HH & FF & DD & BB \\
 & II & KK & II & -KK \\
 & LL & JJ & -LL & JJ \\
 & II & -KK & II & KK \\
 & -LL & JJ & LL & JJ \\
\hdashline
\begin{array}{cccccccccccc}
AA^T & DD^T & EE^T & -HH^T & AA^T & -DD^T & EE^T & HH^T & II^T & LL^T & II^T & -LL^T \\
CC^T & BB^T & -GG^T & FF^T & -CC^T & BB^T & GG^T & FF^T & KK^T & JJ^T & -KK^T & JJ^T \\
EE^T & HH^T & AA^T & -DD^T & EE^T & -HH^T & AA^T & DD^T & II^T & -LL^T & II^T & LL^T \\
GG^T & FF^T & -CC^T & BB^T & -GG^T & FF^T & CC^T & BB^T & -KK^T & JJ^T & KK^T & JJ^T
\end{array}
& \begin{array}{c} MM \\ 0 \\ OO \\ 0 \end{array} & \begin{array}{c} 0 \\ NN \\ 0 \\ PP \end{array} & \begin{array}{c} OO \\ 0 \\ MM \\ 0 \end{array} & \begin{array}{c} 0 \\ PP \\ 0 \\ NN \end{array}
\end{array}\right]
$$

After permuting of rows and columns, this matrix will have the Form B symmetry, and therefore the submatrices S and T are constructed using the relationships of Section 2-2 as follows:

$$
S = \left[\begin{array}{ccc:ccc:cc}
A+G & D+J & M+Q & C-I & F-L & O-S & AA & EE \\
D+J & A+G & M+Q & -F+L & -C+I & -O+S & EE & AA \\
M^T+Q^T & M^T+Q^T & U+W & P^T+T^T & -P^T-T^T & 0 & II & II \\
\hdashline
C^T-I^T & -F^T+L^T & P+T & B-H & E-K & N-R & DD & HH \\
F^T-L^T & -C^T+I^T & -P-T & E-K & B-H & N-R & -HH & -DD \\
O^T-S^T & -O^T+S^T & 0 & N^T-R^T & N^T-R^T & V-X & LL & -LL \\
\hdashline
2AA^T & 2EE^T & 2II^T & 2DD^T & -2HH^T & -2LL^T & MM & OO \\
2EE^T & 2AA^T & 2II^T & 2HH^T & -2DD^T & -2LL^T & OO & MM
\end{array}\right]
$$

Again permuting the rows and columns of the above matrices, new canonical forms are constructed. These forms are generalized symmetric Form B being called “Form C” in here. This was not present in the matrices of structural systems with one axis of symmetry.

$$
\left[\begin{array}{ccc|ccc|cc}
A+G & AA & C-I & D+J & EE & F-L & M+Q & O-S \\
2AA^T & MM & 2DD^T & 2EE^T & OO & -2HH^T & 2II^T & 2LL^T \\
C^T-I^T & DD & B-H & -F^T+L^T & HH & E-K & P+T & N-R \\
\hline
D+J & EE & -F+L & A+G & AA & -C+I & M+Q & -O+S \\
2EE^T & OO & 2HH^T & 2AA^T & MM & -2DD^T & 2II^T & -2LL^T \\
F^T-L^T & -HH & E-K & -C^T+I^T & -DD & B-H & -P-T & N-R \\
\hline
M^T+Q^T & II & P^T+T^T & M^T+Q^T & II & -P^T-T^T & U+W & 0 \\
O^T-S^T & LL & N^T-R^T & -O^T+S^T & -LL & N^T-R^T & 0 & V-X
\end{array}\right]
$$

The above matrix has the Form C symmetry which is the generalized form of the Form B symmetry. Consider a square matrix in the following form:

$$
\begin{bmatrix}
A & C & E & G & I & K \\
D & B & H & F & L & J \\
E & -G & A & -C & I & -K \\
-H & F & -D & B & -L & J \\
M & O & M & -O & Q & 0 \\
P & N & -P & N & 0 & R
\end{bmatrix}
$$

in which A-R are also matrices (submatrices). By permuting the rows and columns one will have:

$$
\xrightarrow[C_2 = C_2 - C_6]{C_1 = C_1 + C_5}
\begin{bmatrix}
A & C & I & K & E & G \\
D & B & L & J & H & F \\
M & O & Q & 0 & M & -O \\
P & N & 0 & R & -P & N \\
E & -G & I & -K & A & -C \\
-H & F & -L & J & -D & B
\end{bmatrix}
$$

$$\downarrow$$

$$
\xrightarrow[R_6 = R_6 + R_1]{R_5 = R_5 - R_1}
\left[\begin{array}{cc|cc|cc}
A+E & C-G & I & K & E & Q \\
D+H & B-F & L & J & H & F \\
\hline
2M & 2O & Q & 0 & M & -O \\
0 & 0 & 0 & R & -P & N \\
\hline
A+E & C-G & I & -K & A & -C \\
-D-H & F-B & -L & J & -D & B
\end{array}\right]
$$

$$
\left[\begin{array}{ccc|ccc}
A+E & C-G & I & K & E & G \\
D+H & B-F & L & J & H & F \\
2M & 2O & Q & 0 & M & -O \\
\hline
0 & 0 & 0 & R & -P & N \\
0 & 0 & 0 & -2K & A-E & -C-G \\
0 & 0 & 0 & 2J & -D+H & B+F
\end{array}\right]
$$

$$\downarrow$$

$$
SS = \left[\begin{array}{c|cc}
A+E & C-G & I \\
D+H & B-F & L \\
\hline
2M & 2O & Q
\end{array}\right]
$$

$$\downarrow$$

$$
SS = \left[\begin{array}{ccc|c}
A+G+D+J & AA+EE & C-I-F+L & M+Q \\
2AA^T+2EE^T & MM+OO & 2DD^T+2HH^T & 2II^T \\
C^T-I^T-F^T+L^T & DD+HH & B-H-E+K & P+T \\
\hline
2M^T+2Q^T & 2II & 2P^T+2T^T & U+W
\end{array}\right]
$$

$$\downarrow$$

$$
TS = \left[\begin{array}{cc|c}
A-E & -C-G & -2K \\
-D+H & B+F & 2J \\
\hline
-P & N & R
\end{array}\right]
$$

$$\downarrow$$

$$
TS = \left[\begin{array}{ccc|c}
A+G-D-J & AA-EE & -C+I-F+L & -2O+2S \\
2AA^T-2EE^T & MM-OO & -2DD^T+2HH^T & -4LL^T \\
-C^T+I^T-F^T+L^T & -DD+HH & B-H+E-K & 2N-2R \\
\hline
-O^T+S^T & -LL & N^T-R^T & V-X
\end{array}\right]
$$

$$
\begin{bmatrix}
A-G & -C-I & -2CC & D-J & -F-L & -2GG & M-Q & -O-S \\
-C^T-I^T & B+H & 2BB & F^T+L^T & E+K & 2FF & -P+T & N+R \\
-CC^T & BB^T & NN & GG^T & FF^T & PP & -KK^T & JJ^T \\
D-J & F+L & 2GG & A-G & C+I & 2CC & M-Q & O+S \\
-F^T-L^T & E+K & 2FF & C^T+I^T & B+H & 2BB & P-T & N+R \\
-GG^T & FF^T & PP & CC^T & BB^T & NN & KK^T & JJ^T \\
M^T-Q^T & -P^T+T^T & -2KK & M^T-Q^T & P^T-T^T & 2KK & U-W & 0 \\
-O^T-S^T & N^T+R^T & 2JJ & O^T+S^T & N^T+R^T & 2JJ & 0 & V+X
\end{bmatrix}
$$

The above matrix has the Form C symmetry.

$$
\begin{bmatrix}
A & C & E & G & I & K \\
D & B & H & F & L & J \\
E & -G & A & -C & I & -K \\
-H & F & -D & B & -L & J \\
M & O & M & -O & Q & 0 \\
P & N & -P & N & 0 & R
\end{bmatrix}
$$

Permuting the rows and columns, one obtains:

$$
\begin{bmatrix}
A & C & I & K & E & G \\
D & B & L & J & H & F \\
M & O & Q & 0 & M & -O \\
P & N & 0 & R & -P & N \\
E & -G & I & -K & A & -C \\
-H & F & -L & J & -D & B
\end{bmatrix}
$$

$$
\xrightarrow[C_2 = C_2 - C_6]{C_1 = C_1 + C_5}
\left[\begin{array}{cc|cc|cc}
A+E & C-G & I & K & E & Q \\
D+H & B-F & L & J & H & F \\
\hline
2M & 2O & Q & 0 & M & -O \\
0 & 0 & 0 & R & P & N \\
\hline
A+E & C-G & I & -K & A & -C \\
-D-H & F-B & -L & J & D & B
\end{array}\right]
$$

$$
\xrightarrow[R_6 = R_6 + R_1]{R_5 = R_5 - R_1}
\left[\begin{array}{ccc|ccc}
A+E & C-G & I & K & E & G \\
D+H & B-F & L & J & H & F \\
2M & 2O & Q & 0 & M & -O \\
\hline
0 & 0 & 0 & R & -P & N \\
0 & 0 & 0 & -2K & A-E & -C-G \\
0 & 0 & 0 & 2J & -D+H & B+F
\end{array}\right]
$$

$$
ST = \left[\begin{array}{cc|c}
A+E & C-G & I \\
D+H & B-F & L \\
\hline
2M & 2O & Q
\end{array}\right]
$$

$$
ST = \left[\begin{array}{ccc|c}
A-G+D-J & -C-I+F+L & -2CC+2GG & M-Q \\
-C^T-I^T+F^T+L^T & B+H-E-K & 2BB-2FF & -P+T \\
-CC^T+GG^T & BB^T-FF^T & NN-PP & -KK^T \\
\hline
2M^T-2Q^T & 2P^T+2T^T & -4KK & U-W
\end{array}\right]
$$

$$
TT = \left[\begin{array}{cc|c}
A-E & -C-G & -2K \\
-D+H & B+F & 2J \\
\hline
-P & N & R
\end{array}\right]
$$

$$
TT = \left[\begin{array}{ccc|c}
A-G-D+J & C+I+F+L & 2CC+2GG & 2O+2S \\
C^T+I^T+F^T+L^T & B+H+E+K & 2BB+2FF & 2N+2R \\
CC^T+GG^T & BB^T+FF^T & NN+PP & 2JJ^T \\
\hline
O^T+S^T & N^T+R^T & 2JJ & V+X
\end{array}\right]
$$

The union of the eigenvalues of these four matrices results in the eigenvalues of the main stiffness matrix.

$$
\lambda(F_{22}) = \lambda(SS') \,\overline{\cup}\, \lambda(ST') \,\overline{\cup}\, \lambda(TS') \,\overline{\cup}\, \lambda(TT')
$$

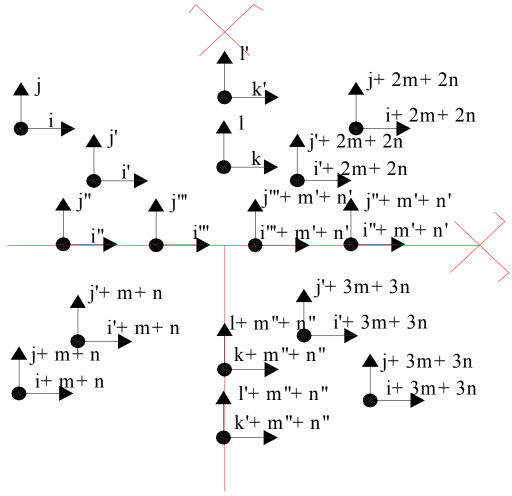


Figure 7. General form of DOFs in Case 3 symmetry.

Example: Consider an indeterminate truss as shown in Fig. 8. For this truss,

$E = 2.07 \times 10^7 \ kN / m^2$, $I = 100\,cm^4$, $\rho = 7800 \ kg / m^3$ and $A = 10\,cm^2$.

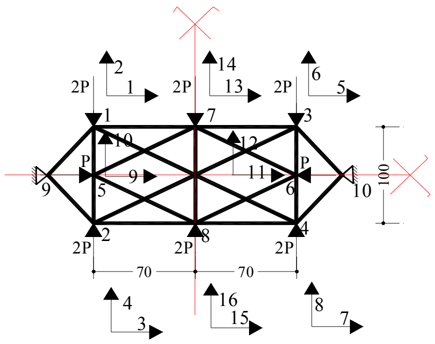


Figure 8. An indeterminate truss with Case 3 symmetry

The buckling load and eigenfrequencies of the truss is calculated as follows:

$P_{SS} = [58100]\,kN \quad P_{TS} = [81900]\,kN \quad P_{ST} = [76600]\,kN$

$P_{TT} = [75600]\,kN \quad P_{cr} = [58100]\,kN$

$$w_{SS} = [348.62, 90.72, 218.33, 189.60]$$

$$w_{TS} = [410.26, 362.78, 187.29, 104.88]$$

$$w_{ST} = [315.69, 207.86, 274.63, 260.77]$$

$$w_{TT} = [330.99, 198.22, 39.37, 133.56]$$

$$w = w_{SS} \text{È} w_{TS} \text{È} w_{ST} \text{È} w_{TT} = [348.62, 90.72, 218.33, 189.60, 410.26, 362.78, 187.29, 104.88, 315.69, 207.86, 274.63, 260.77, 330.99, 198.22, 39.37, 133.56]\ rad / sec$$

4. CONCLUSIONS

Unlike some the previous canonical forms which decompose the structural matrices into two submatrices, in the present method the matrices are decomposed into 4 submatrices enabling the calculation of eigenvalues by employing submatrices of smaller dimensions. Therefore, the computational time is also decreased. Naturally, if the submatrices have further symmetry, additional decomposition of submatrices becomes feasible, leading to further efficiency of the method. Though the examples of this paper are selected from truss structures; however, a similar approach can be utilized for calculating the buckling load of the frame structures. Naturally, the present method can also be used in the free vibration analysis of the different types of skeletal structures.

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