# THE ANALYSIS OF A BEAM UNDER MOVING LOADS

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**(Received: March 15, 2010 – Accepted in Revised Form: September 15, 2011)**

**doi:** *10.5829/idosi.ije.2011.24.03b.03*

**Abstract** It is assumed that a beam made of material has a physical nonlinear behavior. This beam is analyzed under the moving concentrated and distributed continuous loads. The vibration equations of motion are derived from the Hamilton's Principle and Euler–Lagrange Equation. In this study, the amplitude of vibration, circular frequency, bending moment, stress and deflection of the beam has been calculated. At the state of concentrated moving load, the obtained analytic solution has been exemplified. The results of this study indicate that when the material of the beam is considered physically nonlinear, there is no critical velocity and the resonance phenomenon doesn't happen.

**Keywords:** Moving Load, Hamilton Principle, Euler- Lagrange Equation, Duffing Equation, Resonance

**چکیده** فرض میشود که یک تیر ساخته شده از مصالح با رفتار غیرخطی فیزیکی تحت بار متمرکز متحرک و گسترده یکنواخت متحرک آنالیز می شود. معادلات دیفرانسیل حرکت ارتعاشی بااستفاده از اصل هامیلتون و معادلات اویلر- لاگرانژ بدست می آید. در این یررسی, دامنه حرکت ارتعاشی, فرکانس طبیعی دورانی, ممان خمشی و خیز تیر محاسبه می شود. در حالت بار متمرکز متحرک حل تحلیلی بصورت مثال عددی ارائه شده و در نتیجه مشخص شده است که وقتی مصالح تیر دارای رفتار غیرخطی فیزیکی است سرعت بحرانی وجود ندارد و پدیده رزونانس اتفاق نمی افتد.

## 1. INTRODUCTION

The study of dynamical effect of moving loads at highway and railroad bridges has a history of more than one and a half century. The collapse of Jester Bridge in England in 1847 encouraged both the theoretical and experimental studies. The catastrophe caused tremendous human losses and created a lot of excitement in civil engineering.

Presently, there are many structures made from materials which are not subject to the Hook's law. Therefore, there is a great tendency to study stress and strain in elements of structures made of physical nonlinear material under various static and dynamic loads. In the linear theory, the property of material is not taken into consideration; however, all of relevant parameters are taken into consideration in the theory of nonlinear. Thus, the physical nonlinear theory at small deformations demonstrates an exact calculation method for the analysis of stress, strain, and other internal forces in structural elements.

In order to represent all the possible states of the material by one mathematical law, the following algebraic function may be introduced:

$$\sigma = \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \alpha_3 \varepsilon^3 + \dots \quad (1)$$

This equation is not different from the general relationship between $\sigma$ and $\varepsilon$.

$$\sigma = f(\varepsilon) \quad (2)$$

Manzhaalovsky employed Empeher's method and proposed the following two-term equations of the parabolic type on the basis of the tests carried out by Glushkov [1].

$$\sigma_1 = \alpha_1 \varepsilon - \beta_1 \varepsilon^2 \quad \textit{for compression} \quad (3)$$

$$\sigma_2 = \alpha_2 \varepsilon - \beta_2 \varepsilon^2 \left(\textit{for tension}\right) \quad (4)$$

where, $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are empire elasticity. Trinomial equations of the parabolic type were suggested for cast iron by Glushkov who proposed the use of a similar relationship for metal wire [1]:

$$\varepsilon = \alpha\sigma + b\sigma^2 + c\sigma^3 \quad (5)$$

In addition to exponential and power laws, hyperbolic relationships were suggested for brittle materials. Accordingly, as introduced the following function for cast iron:

$$\varepsilon = \frac{\sigma}{\alpha - b\sigma} \quad (6)$$

Glushkov [1] has developed a theory for bars and discs made of brittle materials, employed the following hyperbolic laws:

$$\sigma = \frac{\sigma}{A + B\varepsilon} \quad (7)$$

$$\varepsilon = \frac{\sigma}{E}\,\frac{1}{1 - \alpha\sigma + \beta\sigma\sqrt{\sigma}} \quad (8)$$

Finally, the relationship between stress and strain, in the case of physical nonlinear, is presented by Kauderer [2]. As the equation proposed by Kauderer is comprehensive and expresses the relationship between the stress and strain in three dimensional states, we preferred to use the equation for the analysis of the physical nonlinear stress and strain

$$\varepsilon_{ij} = \frac{K(\sigma_0)}{3K}\sigma_0 + \frac{l(t_0^2)}{2G}(\sigma_{ij} - \sigma_0\delta_{ij}) \quad (9)$$

i , j = 1 , 2 , 3

where, $\partial_{ij}$ is Croneker symbols, and $\sigma_0$ is average stress:

$$\sigma_0 = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \quad (10)$$

K and G at small deformation are respectively volume contraction and shear elastic module. The relationship among K, E, and G is indicated through the following equation:

$$E = \frac{9KG}{5K + G} \quad (11)$$

K ($\sigma_\circ$) is average stress function and $l\left(t_0^2\right)$ is shear stress function; it can be indicated through the following equation:

$$K(\sigma_0) = 1 + K_1\sigma_0 + K_2\sigma_0^2 + \ldots = \sum_{n=0}^{\infty} K_n\sigma_0^n$$
$$l(t_0^2) = 1 + l_2 t_0^2 + l_4 t_0^4 + \ldots = \sum_{n=0}^{\infty} l_{2n} t_0^{2n} \quad (12)$$

Researches have demonstrated that K ($\sigma_0$) in physical nonlinear material on anaverage relative deformation is close to the straight line (i.e., K ($\sigma_0$) = 1). Also, the two first sentences of the shear stress function are enough.

$$l(t_0^2) = 1 + l_2 t_0^2 \quad (13)$$

In the above equation, $l_2$ is the physical nonlinear coefficient.

As the Eq. (9) indicates, stress components of $\sigma_Z$, $\sigma_y$, and $\sigma_{xy}$ are created. The simplification of the equation suggests that the stress components are too small in comparison with the $\sigma_Z$. These new components of the stress have less impact on the frequency of vibration and on the other parameters during the vibration of the beam. As a result, the following equation is obtained from the Eq. (9) at a two dimensional surface:

$$\sigma_Z = E\left(\varepsilon_Z - \frac{2}{27} l_2 \frac{E^3}{G^3}\varepsilon_Z^3\right) \quad (14)$$

The purpose of this paper was the analysis of a beam made of physical nonlinear material under the moving concentrated and distributed continuous loads, which was discussed through examples analytically.

## 2. THEORY

**2.1. Analytical solution for moving concentrated load** To study the effect of moving load on the prismatic beam, first, we are discussing moving concentrated load. Thus, it is assumed that the load P moves along the beam (Fig. 1).

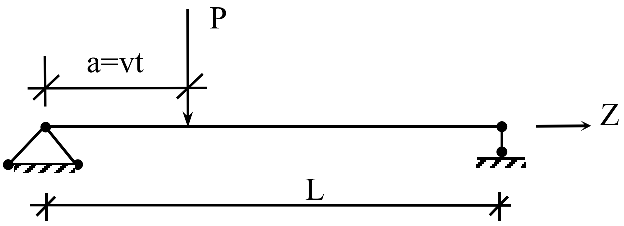


**Figure 1.** Schematic view of a prismatic beam under moving load

To consider the effect of external moving load, it is assumed an equivalent distributed load which depends on z and t as follows [3]:

$$q(z,t)=\sum_{k=1}^{\infty}\frac{2P}{L}\sin\frac{k\pi a}{L}\sin\frac{k\pi z}{L} \tag{15}$$

where, $p(z)=\sin\frac{k\pi z}{L}$, (k = 1,2,3,…) is the head vibration mode of the beam.

Therefore, the potential and kinetic energy of the system will be as follows [4]:

$$\Pi=\int_0^L\left[\frac{1}{2}J_0E\left(\frac{\partial^2 w}{\partial z^2}\right)^2-\frac{1}{54}l_2\frac{E^4}{G^3}J_1\left(\frac{\partial^2 w}{\partial z^2}\right)^4\right]dz \tag{16}$$

$$Ki=\frac{1}{2}\rho F\int_0^L\left(\frac{\partial w}{\partial t}\right)^2 dz \tag{17}$$

Also, the work of the external moving load will be as follows [5]:

$$A=\int_0^L q(z,t).\,w(z,t)dz$$
$$A=\frac{2P}{L}\int_0^L\sum\sin\frac{k\pi a}{L}\sin\frac{k\pi z}{L}\,w(z,t)\,dz \tag{18}$$

The Hamilton's Principle for this beam will be expressed as below [6]:

$$H=\int_{t1}^{t2}(\Pi-A-Ki)dt=\int_{t1}^{t2}\int_0^L\left[\frac{1}{2}J_0E\left(\frac{\partial^2 w}{\partial z^2}\right)^2-\frac{1}{54}l_2\frac{E^4}{G^3}J_1\left(\frac{\partial^2 w}{\partial z^2}\right)^4-\frac{2P}{L}\sum_{k=1}^{\infty}\sin\frac{k\pi a}{L}\sin\frac{k\pi a}{L}w(z,t)-\frac{1}{2}\rho F\left(\frac{\partial w}{\partial t}\right)^2\right]dz.dt \tag{19}$$

where, $J_0=\iint y^2dxdy$ , $J_1=\iint y^4dxdy$

Considering:

$\zeta=\frac{\pi z}{L}$, $\tau=\omega_0 t$, which $\zeta$ , $\tau$ varies respectively from 0 to π and 0 to 2π.

where, $\omega_0$ is circular frequency vibration of the system in linear case and expressing in Hamilton principle (19), the following equations are obtained:

$$H=\frac{L}{\pi\omega_0}\int\int\left\{\frac{1}{2}J_0E\frac{\pi^4}{L^4}\left(\frac{\partial^2 w}{\partial\zeta^2}\right)^2-\frac{1}{54}\frac{E^4}{G^3}J_1\frac{\pi^8}{L^8}\left(\frac{\partial^2 w}{\partial\zeta^2}\right)^4-\frac{2P}{L}\sum_{k=1}^{\infty}\sin k\zeta_0\sin k\zeta\,.\,w(\zeta,\tau)-\frac{1}{2}\rho F\omega_0^2\left(\frac{\partial w}{\partial t}\right)^2\right\}d\zeta.d\tau \tag{20}$$

We assumed that the deformation of the beam would be found from the following expression [7]:

$$w(\zeta,\tau)=p(\zeta)\cdot q(\tau) \tag{21}$$

where, $p(\zeta)$ and $q(\tau)$ are coordinate and generalized functions respectively.

By substitution of expression (21) in expression (20), we obtain the following expressions:

$$H=\frac{L}{\pi\omega_0}\int_0^{2\pi}\int_0^{\pi}\left\{\frac{1}{2}J_0E\frac{\pi^4}{L^4}\left[p''(\zeta)\right]^2\cdot q^2-\frac{1}{54}l_2\frac{E^4}{G^3}J_1\frac{\pi^8}{L^8}p''^4\cdot q^4-\frac{2P}{L}\sum_{k=1}^{\infty}\sin\frac{k\pi vt}{L}\sin k\zeta\cdot p(\zeta)\cdot q(\tau)-\frac{1}{2}\rho F\omega_0^2p^2(\zeta)\cdot q^2\right\}d\zeta\,.\,d\tau \tag{22}$$

The following expression is derived from the calculations indicated above in Eq. (22) [8]:

$$H=\frac{L}{\pi\omega_0}\int_0^{2\pi}\left\{aq^2+bq^4-c\omega_0^2q'^2-d'q\right\}d\tau=\frac{L}{\pi\omega_0}\int_0^{2\pi}Nd\tau \tag{23}$$

where:

$$a = \frac{1}{2} J_0 E \; \frac{\pi^4}{L^4} \int_0^{\pi} p''^2 \, d\zeta$$

$$b = -\frac{1}{54} l_2 \; \frac{E^4}{G^3} \; J_1 \; \frac{\pi^8}{L^8} \int_0^{\pi} p''^4 \, d\zeta \tag{24}$$

$$c = \frac{1}{2} \rho F \int_0^{\pi} p^2(\zeta) \, d\zeta$$

$$d = \frac{\pi P}{L} \; \sin \frac{k \pi v t}{L}$$

For Integral (23), Euler equation gives [9]:

$$\frac{\partial}{\partial \tau} \left( \frac{\partial N}{\partial q} \right) - \frac{\partial N}{\partial q} = 0$$

$$\frac{\partial}{\partial \tau} \left( -2 c \, \omega_0^2 \, q' \right) = -2 c \, \omega_0^2 \, q'' \tag{25}$$

$$\frac{\partial N}{\partial q} = 2 a q + 4 b q^3 - d'$$

$$-2 c \, \omega_0^2 \, q'' - 2 a q - 4 b q^3 + d' = 0$$

$$-2c\,\omega_0^2\, q'' - 2aq \left( 1 + 2 \frac{b}{a} q^2 \right) = d' = \frac{\pi P}{L} \; \sin \frac{k \pi v t}{L} \tag{26}$$

By substitution of $\theta = \frac{k \pi v}{L}$ , $t = \frac{\tau}{\omega_0}$ into Eq. (26) we'll have:

$$\omega_0^2 \, q'' + \frac{a}{c} q \left( 1 + 2 \frac{b}{a} q^2 \right) = d \; \sin \eta \, \tau \tag{27}$$

where, $\eta = \frac{\theta}{\omega_0}$ , $d = \frac{\pi P}{2cL}$

By substitution of $X = \frac{\omega_0^2}{d} q$ into Eq. (27), X is dynamic coefficient. We will have:

$$X'' + X \left( 1 + e X^2 \right) = \sin \eta \, \tau \tag{28}$$

$$\text{where, } e = 2 \frac{b}{a} \, \frac{d^2}{\omega_0^4} \tag{29}$$

To solve the Duffing Eq. (28), we follow the following procedure [10]:

$$X = \sum_{n=1,3,5,\dots}^{\infty} X_n \sin \eta \, \tau = X_1 \sin \eta \, \tau + X_3 \sin 3 \eta \, \tau + X_5 \sin 5 \eta \, \tau + \dots \tag{30}$$

If we substitute the expression (30) into 28, and compare the similar coefficients of $\sin n \, \eta \, \tau$, we will get a lot of cubic nonlinear algebraic equations. To our knowledge, there is no exact solution for these equations. Thus, we employed an approximated method. For this purpose, we applied three constraints Eq. (30) and we assumed that $X_n \gg X_{n+1}$: therefore, in this case, we will have the following system of nonlinear equations:

$$\begin{aligned}
&\left(1 - \eta^2\right) X_1 + e \left( \frac{3}{4} X_1^3 + \frac{3}{2} X_1 X_3 X_5 + \frac{3}{4} X_1^2 X_3 + \frac{3}{2} X_1 X_3^2 + \frac{3}{4} X_3^2 X_5 + \frac{3}{2} X_1 X_5^2 \right) = 1 \\
&\left(1 - 9\eta^2\right) X_3 + e \left( \frac{1}{11} X_1^3 + \frac{3}{4} X_3^3 + \frac{3}{2} X_1 X_3 X_5 + \frac{3}{2} X_1^2 X_3 + \frac{3}{2} X_3 X_5^2 + \frac{3}{4} X_1^2 X_5 \right) = 0 \\
&\left(1 - 25\eta^2\right) X_5 + e \left( \frac{3}{4} X_5^3 + \frac{3}{4} X_1^2 X_3 + \frac{3}{4} X_1 X_3^2 + \frac{3}{2} X_3^2 X_5 + \frac{3}{2} X_1^2 X_5 \right) = 0
\end{aligned} \tag{31}$$

With solving of Eq. (31) by the method of Zeidel, we will have [11]:

$$\begin{aligned}
&\left(1 - \eta^2\right) X_1 + \frac{3}{4} e X_1^3 = 1 \\
&X_3 = \frac{e x_1^3}{4 \left(9 \eta^2 - 1\right)} \\
&X_5 = \frac{3 e x_1 \, X_3 \left(X_1 + X_3\right)}{4 \left(25 \eta^2 - 1\right)}
\end{aligned} \tag{32}$$

When the load is out of the beam, Eq. (27) will be as below in which the system will have free vibration.

$$\frac{dq^2}{d\tau^2} + \frac{a}{\omega^2 c} q \left( 1 + 2 \frac{b}{a} q^2 \right) = 0 \tag{33}$$

Finally, by solving Eq. (33), we found the period of vibration:

$$T = 4 \cdot \frac{\omega}{\sqrt{\frac{a}{c} \left( 1 + \frac{b}{a} Q^2 \right)}} \cdot k(\bar{\theta}) \tag{34}$$

Consequently, the circular frequency is as follows:

$$\omega = \frac{\pi}{2} \sqrt{\frac{a}{c} \left( 1 + \frac{b}{a} Q^2 \right)} \cdot \frac{1}{K\left(\bar{\theta}\right)} \tag{35}$$

where,

$$K(\bar{\theta}) = \frac{\pi}{2}\left[1 + \frac{1}{4}\sin^2\bar{\theta} + \frac{9}{64}\sin^4\bar{\theta} + \frac{25}{256}\sin^6\bar{\theta} + \dots\right]$$

ω is the circular frequency of vibration, Q is amplitude of vibration, and k ($\bar{\theta}$) is a second order elliptic integral. As the above equation indicates, $\frac{b}{a}Q^2$ has a minus sign, which decreases the ω in comparison with $\omega_0$ (as it was indicated in 35).

Based on the equations expressed above, the explanation of the analysis of deflection, bending stress, and bending moment are as follows.

**Deflection:** Deflection is obtained from the Eq. (21).

$$W(z,t) = \frac{d}{\omega^2}\sin\frac{\pi z}{l}\sum_{n=1,3,5} X_n \sin\frac{n\pi a}{l} \tag{36}$$

$$\sum_{n=1,3,5} X_n \sin\frac{n\pi a}{l} = X_1 \sin\frac{\pi a}{l} + X_3 \sin\frac{3\pi a}{l} + X_5 \sin\frac{5\pi a}{l}$$

$$a = Vt\,,\ and\quad \frac{d}{\omega^2} = \frac{Pl^3}{EJ_0\,\pi^3 \int [p''(\zeta)]^2\, d\zeta}$$

As it will be indicated in the tables below, the coefficients $X_3$ and $X_5$ are too small and neglected.

$$W(z,t) = \frac{d}{\omega^2} X_1 \sin\frac{\pi z}{l}\sin\frac{\pi a}{l} \tag{37}$$

**Bending Stress:** Bending Stress is calculated by using Eq. (14) and $\varepsilon_Z = y\frac{\partial^2 w}{\partial z^2}$ , which is called KIRHOF principle.

$$\sigma_Z = E\left[y\frac{\partial^2 w}{\partial z^2} - \frac{2}{27}l_2\frac{E^3}{G^3}\left(y\frac{\partial^2 w}{\partial z^2}\right)^3\right] \tag{38}$$

**Bending Moment:** Bending Moment in every section of the beam is designed by the following equations:

$$M = \iint \sigma_Z y\, dx\, dy = E\iint\left[y\frac{\partial^2 w}{\partial z^2} - \frac{2}{27}l_2\frac{E^3}{G^3}\left(y\frac{\partial^2 w}{\partial z^2}\right)\right]dx\, dy \tag{39}$$

$$M = Ej_0\frac{\partial^2 w}{\partial z^2}\left[1 - \frac{2}{27}l_2\frac{E^3}{G^3}\cdot\frac{J_1}{J_0}\left(\frac{\partial^2 w}{\partial z^2}\right)\right]$$

The expression $\frac{\partial^2 w}{\partial z^2}$ equals the following equation:

$$\frac{\partial^2 w}{\partial z^2} = -\frac{d}{\omega_0^2}\cdot\frac{\pi^2}{l^2}\sin\frac{\pi z}{l}\sin\frac{\pi a}{l}$$

Now the obtained analytic solutions are being applied to the following example. It has been assumed that the material of the beam is copper (Fig. 2).

$J_0 = 1.236\times10^4\, cm^4$

$J_1 = 58.44\times10^4\, cm^6$

$G = 0.46\times10^6\ kg/cm^2$

$E = 1.241\times10^6\ kg/cm^2$

$l_2 = 0.18\times10^6$

$L = 2m\,,\ P = 20\ KN$

$R = 6.615cm$

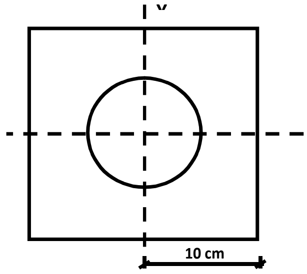


**Figure 2.** Section of beam

Then vibration amplitudes are determined and are shown in Table 1.

**TABLE 1.** Dimensionless vibration amplitudes

| $\eta^2$ | $X^1$ | | | $X_3$ | $X_5$ |
|---|---|---|---|---|---|
| | $X_1^1$ | $X_2^1$ | $X_3^1$ | | |
| 0.0 | -40.2 | -39.83 | 1.00 | -16.11 | -22.04 |
| 0.84 | -21.07 | 10.54 | | 0.1902 | $-1.31\times10^{-3}$ |
| 1 | -11.73 | | | 0.041 | $-1.01\times10^{-4}$ |
| 2 | 1.00 | | | $3.69\times10^{-6}$ | $-1.63\times10^{-11}$ |

Based on Table (1) the diagram of resonance is indicated in Fig. 3.

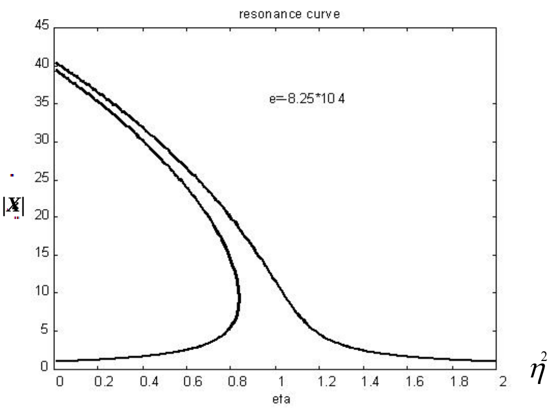


**Figure 3.** Resonance Curve

Deflection, stress, and bending moment in the section of the beam are calculated by using equations (36), (37), and (38) for $\eta^2=1$ *and* $z=a$, which are indicated in Tables 2 and 3.

**TABLE 2.** Calculation of deflection and bending moment at the point of load

| a | $\frac{l}{8}$ | $\frac{l}{4}$ | $3\frac{l}{8}$ | $\frac{l}{2}$ | $5\frac{l}{8}$ | $3\frac{l}{4}$ | $7\frac{l}{8}$ |
|---|---|---|---|---|---|---|---|
| W (cm) | - 0.054 | - 0.1856 | - 0.3468 | - 0.3712 | - 0.316 | - 0.1856 | - 0.054 |
| M (K.N.m) | 2.057 | 6.842 | 11.922 | 14.05 | 11.922 | 6.842 | 2.057 |

**TABLE 3.** Stress in the middle section of the beam

| Y (cm) | 6.61 | 7.9676 | 8.9816 | 10 |
|---|---|---|---|---|
| $\sigma_z(mpa)$ | 49.49 | 58.37 | 66.91 | 70.25 |

Bending stress in the middle section of the beam is indicated in Fig. 4.

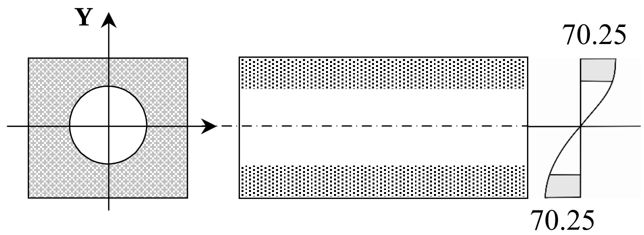


**Figure 4.** The diagram of stress in the middle section of the beam

The Critical velocity for given example is obtained 135.95 m/s Dynamical Coefficients according to velocity is obtained and are written in Table 4.

Dynamic Coefficients at linear state are derived by Kesiliv [12]:

$$\mu=\frac{1}{1-\beta^2}\left[1-\left(\beta\sin\frac{\pi a}{\beta l}\quad\sin\frac{\pi a}{l}\right)\right]$$

where, $a=vt$ , $\beta=\frac{\theta}{\omega}$ , $\theta=\frac{\pi v}{l}$

The Dynamical Coefficient when the load is in the middle of the Span $a=\frac{1}{2}$ is estimated and shown at Fig. 5.

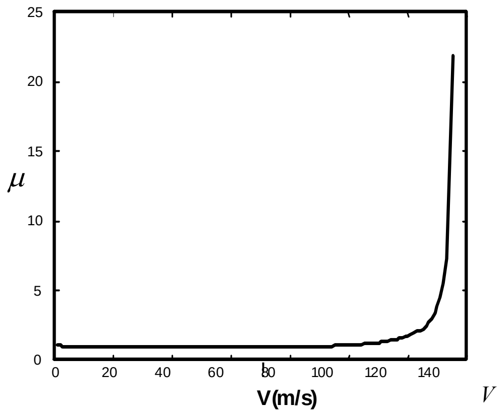


**Figure 5.** Dynamical coefficient-velocity

As shown at the Fig. 5, when $\left(\theta/\omega\right)^2=1$ the critical V equals to 135.95 m/s and resonance happens at linear state. Whenever, at nonlinear state, X has definite value, equals 11.73 in Table 4.

Based on v=50 m/s, (v=180 km/h), deflection, bending moment, and bending stress in the middle of the span are found by the obtained equations for nonlinear state, and for linear state they are found by equations which are given by Kesiliv [12]:

$$y=\frac{p\delta}{1-\beta^2}\left(\sin.\frac{\pi a}{l}-\beta\sin\frac{1}{\beta}\frac{\pi a}{l}\right)$$

$$M=\frac{2pl}{\pi^2}\frac{\sin.\frac{\pi a}{l}}{\left(1-\beta^2\right)}\left(\sin.\frac{\pi a}{l}-\beta\sin.\frac{\pi a}{\beta l}\right)$$

**TABLE 4.** Dynamical Coefficients

| V (m/s) | 0 | 25 | 50 | 75 | 100 | 124.57 | 135.95 |
|---|---|---|---|---|---|---|---|
| $\eta^2=\left(\frac{\theta}{\omega}\right)^2$ | 0 | 0.0338 | 0.1353 | 0.3036 | 0.541 | 0.83895 | 1 |
| \|X\| | 1 | 1.036 | 1.158 | 1.438 | 2.193 | 9.27 | 11.73 |

The numerical resultants have written at Table 5.

**TABLE 5.** A comparison of deflection, stress, and bending moment at linear and nonlinear state

| State | W (cm) | $\sigma$ $(mpa)$ | M (KN.M) |
|---|---|---|---|
| Nonlinear | 0.025 | 7.73 | 9.5 |
| Linear | 0.034 | 10.29 | 12.48 |

**2.2. Analytical solution for moving distributed load** To consider the effect of moving distributed continuous load, it is assumed that the moving distributed continuous load moves along the prismatic beam as it is shown in Fig. 6.

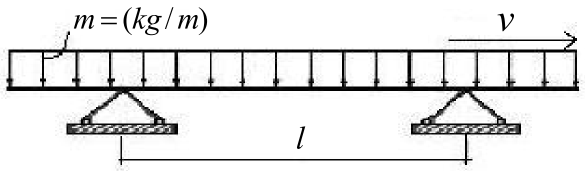


**Figure 6.** Schematic view of a prismatic beam under moving load

The potential and kinetic energy of this system can be written as follows:

$$\Pi = \int_0^l \left[ \frac{1}{2} EI_0 \left( \frac{\partial^2 w}{\partial z^2} \right)^2 - \frac{1}{54} l_2 \frac{E^4}{G^3} I_1 \left( \frac{\partial^2 w}{\partial z^2} \right)^4 \right] dz \quad (40)$$

and:

$$ki = \frac{1}{2} \rho F \int_0^l \left( \frac{\partial w}{\partial t} \right)^2 dz + \frac{1}{2} m \int_0^l \left( \frac{dw}{dt} \right)^2 dz \quad (41)$$

where, E, G, $l_2$, $\rho$, m, and F denote modulus of elasticity, modulus of elasticity in shear, nonlinearity coefficient and density, mass of load per unit length and cross sectional area, respectively. The principle of Hamilton for this beam is as follow:

$$H = \int_{t_1}^{t_2} (\Pi - ki) dt = \int_{t_1}^{t_2} \int_0^l \left[ \frac{1}{2} EI_0 \left( \frac{\partial^2 w}{\partial z^2} \right)^2 - \frac{1}{54} l_2 \frac{E^4}{G^3} I_1 \left( \frac{\partial^2 w}{\partial Z^2} \right)^4 - \frac{1}{2} \rho F \left( \frac{\partial w}{\partial t} \right)^2 - \frac{1}{2} m \left( \frac{dw}{dt} \right)^2 \right] dt dz \quad (42)$$

By considering, $\frac{dw}{dt} = \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial z}$ and Euler equation:

$$\frac{\partial^2}{\partial z^2} \left( \frac{\partial L}{\partial u} \right) - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial k} \right) - \frac{\partial}{\partial z} \left( \frac{\partial L}{\partial i} \right) = 0 \quad (43)$$

where, $u = \frac{\partial^2 w}{\partial z^2}, k = \frac{\partial w}{\partial t}, i = \frac{\partial w}{\partial z}$, by considering Eq. (43) and Eq. (42) and further simplification the differential motion equation can be written as:

$$\frac{\pi^4}{l^4} \frac{\partial^4 w}{\partial \zeta^4} + \frac{\pi^2 m v^2}{l^2 EI_0} \frac{\partial^2 w}{\partial \zeta^2} + \frac{\omega^2 (\rho F + m)}{EI_0} \frac{\partial^2 w}{\partial \tau^2} = \frac{2}{9} l_2 \frac{E^3}{G^3} \frac{I_1}{I_0} \frac{\pi^8}{l^8} \frac{\partial^2 w}{\partial \zeta^2} \left[ 2 \left( \frac{\partial^3 w}{\partial \zeta^3} \right)^2 + \frac{\partial^2 w}{\partial \zeta^2} \frac{\partial^4 w}{\partial \zeta^4} \right] \quad (44)$$

where, $\zeta = \frac{\pi z}{l}, \tau = \omega t$. As solving of the Eq. (44) is difficult the consecutive approximated method is used. Furthermore, to make the solving procedure easier, it is assumed at linear state:

$$w_0 = Q_0 Sink\ \zeta . Cos\ \tau \quad (45)$$

where, $Q_0$ , is amplitude of vibration of beam at linear state.

If Eq. (45) is substituted at right side of Eq. (44) and is simplified, the following equation is derived:

$$\frac{\pi^4}{l^4} \frac{\partial^4 w}{\partial \zeta^4} + \frac{\pi^2 m v^2}{l^2 EI_0} \frac{\partial^2 w}{\partial \zeta^2} + \frac{\omega^2 (\rho F + m)}{EI_0} \frac{\partial^2 w}{\partial \tau^2} = \frac{1}{72} l_2 \frac{E^3}{G^3} \frac{I_1}{I_0} \frac{\pi^8 k^8}{l^8} Q_0^3 (3 Sink\zeta . Cos\tau + Sink\zeta . Cos 3\tau - 9 Sin 3k\zeta . Cos\tau - 3 Sin 3k\zeta . Cos 3\tau) \quad (46)$$

Private solving of Eq. (11) is defined as follow:

$$w(\zeta, \tau) = a_1 Sink\zeta . Cos\tau + a_2 Sink\zeta . Cos 3\tau + a_3 Sin 3k\zeta . Cos\tau + a_4 Sin 3k\zeta . Cos 3\tau \quad (47)$$

where, $a_1$ ,$a_2$ , $a_3$ , $a_4$ are constant coefficients, by substitution Eq. (47) into Eq. (46) and comparing the same coefficients Sin kξ, Cos τ, $a_1$, $a_2$, $a_3$, $a_4$ and ω are obtained, which are as follows [13-15]:

$$\frac{\pi^4 k^4}{\ell^4} a_1 + \frac{\pi^2 k^2 m v^2}{\ell^2 EI_\circ} a_1 - \frac{\omega^2(\rho F + m)}{EI_\circ} = \frac{1}{24} \ell_2 \frac{E^3}{C^3} \frac{I_1}{I_0} \frac{\pi^8 k^8}{\ell^8} Q_0^3 \quad (48)$$

$$\frac{\pi^4 k^4}{\ell^4} a_2 + \frac{\pi^2 k^2 m v^2}{\ell^2 EI_\circ} a_2 - 9\frac{\omega^2(\rho F + m)}{EI_\circ} a_2 = \frac{1}{72} \ell_2 \frac{E^3}{C^3} \frac{I_1}{I_0} \frac{\pi^8 k^8}{\ell^8} Q_0^3 \quad (49)$$

$$81\frac{\pi^4 k^4}{\ell^4} a_3 - 9\frac{\pi^2 k^2 m v^2}{\ell^2 EI_\circ} a_3 - \frac{\omega^2(\rho F + m)}{EI_\circ} a_3 = -\frac{1}{8} \ell_2 \frac{E^3}{C^3} \frac{I_1}{I_0} \frac{\pi^8 k^8}{\ell^8} Q_0^3 \quad (50)$$

$$81\frac{\pi^4 k^4}{\ell^4} a_4 - 9\frac{\pi^2 k^2 m v^2}{\ell^2 EI_\circ} a_4 - 9\frac{\omega^2(\rho F + m)}{EI_\circ} a_4 = -\frac{1}{24} \ell_2 \frac{E^3}{C^3} \frac{I_1}{I_0} \frac{\pi^8 k^8}{\ell^8} Q_0^3 \quad (51)$$

Thus, circular frequency of system ω is obtained:

$$\omega^2 = \frac{EI_0}{(\rho F + m)} . \frac{\pi^2 k^2}{l^2} \left(\frac{\pi^2 k^2}{l^2} - \frac{mv^2}{EI_0}\right) \left(1 - \frac{1}{24} l_2 \frac{E^4}{G^3} I_1 \frac{\pi^6 k^6}{l^4} Q_0^2 \frac{1}{\pi^2 k^2 EI_0 - mv^2 l^2}\right) \quad (52)$$

The critical velocity is derived by considering the criteria ω = 0.0

$$V_{cr} = \frac{\pi k}{l} \sqrt{\frac{EI_0}{m}\left(1 - \frac{1}{24} l_2 \frac{E^3}{G^3} \frac{I}{I_0} \frac{\pi^4 k^4}{l^4} Q_0^2\right)} \quad (53)$$

If $l_2$=0.0 the critical velocity is obtained at linear state, $V_{cr} = \frac{\pi k}{l}\sqrt{\frac{EI_0}{m}}$

If at Eq. (52), m= 0.0 that is the load is out of beam:

$$\omega = \frac{\pi^2 k^2}{l^2} \sqrt{\frac{EI_0}{\rho F}\left(1 - \frac{1}{24} l_2 \frac{E^3}{G^3} \frac{I_1}{I_0} \frac{\pi^4 k^4}{l^4} Q_0^2\right)} \quad (54)$$

ω is the circular frequency at free vibration at physical nonlinear . It is seen from Eq. (52) the circular frequency of system depends on nonlinearity of material and the velocity of load.

To understand the analysis obtained here an example is presented in this section. For the assuming beam which is shown in Fig. (1), the parameters used in studied equation are as follows:

E =2.1× $10^5$Mpa
G = 0.87×$10^5$Mpa
$l_2$ = .085×$10^6$
q = mg =20 KN/m (unit weight of load)
P = ρfg = 0.66 KN/m (unit weight of beam)
$I_0$=1.51×$10^{-3}$$m^4$
$I_1$ = 1.3×$10^{-4}$$m^6$
L=12m

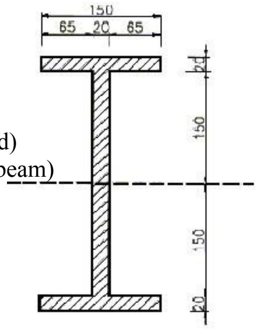


**Figure 7.** Section of beam

Circular frequency is obtained from Eq. (52) and plotted in Fig. 8.

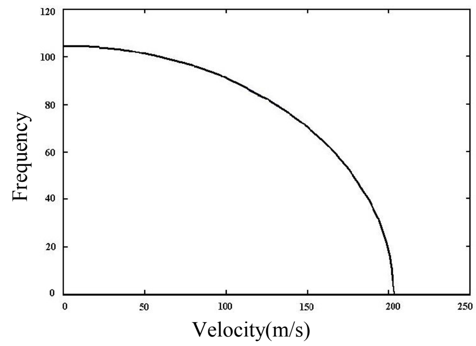


**Figure 8.** Frequency – Velocity Curve

The critical velocity is obtained from Eq. (53): Vcr =103.044m/sec.

The dynamic coefficient (Yd/Ys) at the middle point of beam is derived from Eq.(52) and plotted in Fig. 9.

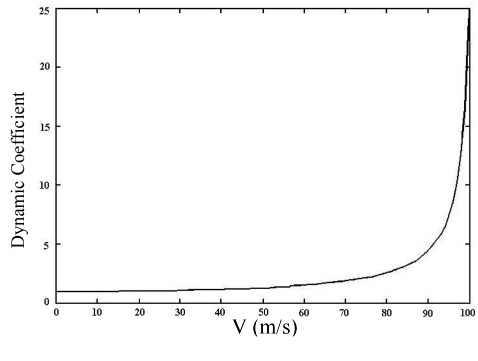


**Figure 9.** Dynamic Coefecient- Velocity Curve

Diagram of motion at interval of a period is shown in Fig. 10 (for v = 30m/, 50m/s)

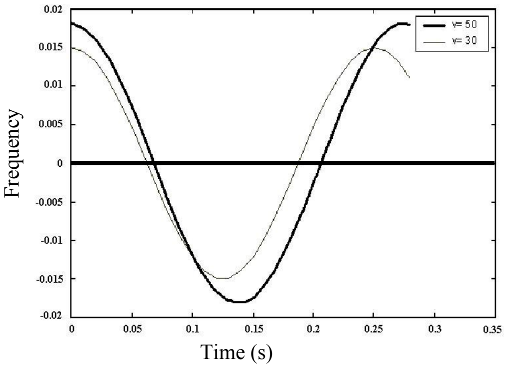


**Figure 10.** The time history of the oscillation at the middle section of the beam during one cycle

The bending moment at the middle point (a = L/2) is shown in Fig. 11.

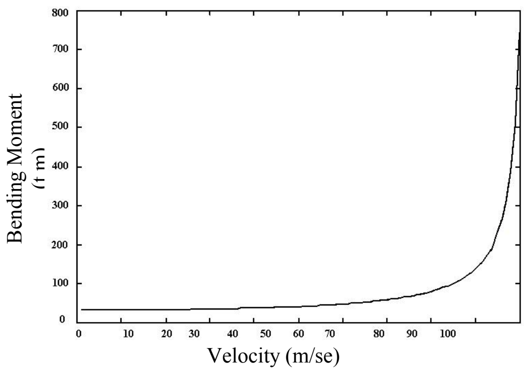


**Figure 11.** Bending moment – Velocity Curve

## 3. CONCLUSION

The effect of material nonlinearity on the response parameters of bridge under concentrated and distributed moving loads are investigated analytically. The Hamiltonian principles and Euler's equations employed to found the nonlinear vibration equation of the system. The Fourier series is used to decompose the deflection as a multiplication of functions in time and space. The resulting equation in time is the well known Duffing's equation. Solving the Duffing equation by perturbation method the response parameters of the system is evaluated. In the case of concentrated moving load and linear material, theoretically with increasing the speed of the moving load resonance might happen. However considering the material nonlinearity, resonance doesn't happen, and the internal forces will have definite values. Taking into account the material nonlinearity the internal forces for velocities blew critical velocity reduces as much as 10-15 percent in comparison with the linear case. Using the results dynamic amplification factors is calculated for the system. Increasing the material nonlinearity, results in decreasing in the value of vibration amplitude. In the case of distributed continuous moving loads using analytical solution, vibration frequency and the dynamic amplification factors and bending moment are evaluated for different velocities, blew critical velocity. Analysis shows that the more is the speed of the moving load, the more is the amplitude of the vibration.

**Notation**

A = external work of the moving load
$\alpha$ = the distance of the concentrated load from the support
E = module of elasticity
F = an area of the section
G = sheer elastic module
H = Hamilton Principle
$J_0$ = moment of inertia
$J_1 = \iint x^4 dxdy$
K = volume contraction
$K(\sigma_0)$ = average stress function
Ki = kinetic energy
L = the span of the beam
$l(t_0^2)$ = sheer stress function
$l_2$ = nonlinear coefficient
$p(\zeta)$ = coordinate function
P = concentrated load
p(z) = the head vibration made of the beam
$q(\tau)$ = generalized function
q(z , t) = equation distributed load
w (z , t) = deformation of the beam
X = dimensionless amplitude (dynamic coefficient)
Z = a distance of any point from the support
$\sigma$ = stress
$\sigma_0$ = average stress
$\sigma_X$ , $\sigma_Y$ , $\sigma_Z$ = original stress
$\varepsilon$ = strain
$\varepsilon_{ij}$ = three – dimensional strain
$\rho$ = density

$\pi$ = potential energy

$\omega_0$ = circular frequency

$\theta$ = circular frequency of load

V = velocity

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