EXACT ELASTICITY SOLUTIONS FOR THICK-WALLED FG SPHERICAL PRESSURE VESSELS WITH LINEARLY AND EXPONENTIALLY VARYING PROPERTIES

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Abstract In this paper, exact closed-form solutions for displacement and stress components of thick-walled functionally graded (FG) spherical pressure vessels are presented. To this aim, linear variation of properties, as an important case of the known power-law function model is used to describe the FG material distribution in thickness direction. Unlike the pervious studies, the vessels can have arbitrary inner to outer stiffness ratio without changing the function variation of FGM. After that, a closed-form solution is presented for displacement and stress components based on exponential model for variation of properties in radial direction. The accuracy of the present analyses is verified with known results. Finally, the effects of non-homogeneity and different values of inner to outer stiffness ratios on the displacement and stress distribution are discussed in detail. It can be seen that for FG vessels subjected to internal pressure, the variation of radial stress in radial direction becomes linear as the inner stiffness becomes five times higher than outer one. When the inner stiffness is half of the outer one, the distribution of the circumferential stress becomes uniform. For the case in which the external pressure is applied, as the inner to outer shear modulus becomes lower than 1/5, the value of the maximum radial stress is higher than external pressure.

Keywords Thick-Walled Pressure Vessels, Functionally Graded Materials, Linearly-Varying Properties, Exponentially-Varying Properties

چکیده در این مقاله، برای جابجایی و میدان تنش مخازن جدار ضخیم کروی ساخته شده از مواد هدفمند پاسخ بسته عرضه شده است. بدین‌منظور، تغییرات خطی خواص، به‌عنوان مهم‌ترین حالت خاص از مدل معروف تابع توانی، برای تشریح توزیع مواد سازنده در راستای ضخامت استفاده شده است. در این مطالعه برخلاف مطالعات مشابه، بدون محدودیت تغییر حالت خطی توزیع مواد، نسبت سفتی سطح داخل مخزن به سطح خارج می‌تواند مقادیر متفاوتی داشته باشد. سپس، برپایهٔ مدل تابع نمایی تغییرات خواص در راستای شعاعی، برای مؤلفه‌های میدان تنش و جابجایی پاسخ بسته عرضه شده و صحت تحلیل‌ها با نتایج موجود بررسی شده است. در پایان، تأثیرات ناهمگنی و مقادیر مختلف نسبت سفتی سطوح مخزن بر توزیع تنش و جابه‌جایی بررسی شده است. نتایج نشان داد که در مخازن FG کروی تحت فشار داخلی، هرگاه نسبت سفتی سطح داخلی به سطح خارجی حدوداً برابر ۵ باشد، توزیع تنش شعاعی در راستای شعاعی تقریباً خطی است. هنگامی‌که سفتی سطح داخلی تقریباً نصف سفتی سطح خارج باشد، توزیع تنش حلقوی یکنواخت است. برای حالتی که تنها فشار خارجی وجود داشته باشد، هرگاه نسبت مدول برشی سطح داخل به سطح خارج کمتر از ۱/۵ باشد، مقدار بیشترین تنش شعاعی در داخل جدارهٔ مخزن، بیشتر از مقدار فشار خارجی خواهد بود.

1. INTRODUCTION

Functionally graded materials (FGMs) are non-homogenous composites with continuous variation of the constituents from one surface of the material to the other. Such a material was first introduced in Japan [1]. FGMs were first used as a thermal shield in industries. New applications have become possible using FGM such as energy conversion [2], dental and orthopaedic implants [3,4], thermogenerators and sensors [5] and joining dissimilar materials [6]. Another important usage of FGM is wear resistant coatings and the covering of mechanical parts such as gears, cams, roller bearing and machine tools

[7,8]. Functionally graded materials are made by combining different materials using powder metallurgy methods.

There are many researches in literature that consider static analysis of functionally graded cylindrical and spherical vessels based on numerical approaches such as finite element method [9-11]. However, some semi-analytical solutions are found for thick-walled FG pressure vessels. You et al [12] presented a static analysis for thick-walled FG spherical pressure vessels. Stress analysis of thick-walled FG cylinders was carried out by Tutuncu [13]. These analyses are based on exponential model for property variation in thickness direction by using the power series solution. It is noticeable that in most of analyses carried out in the field of FGM, the variation of the material properties has been modeled as exponential [14-19] or power-law function [20-25]. Based on power-law model, for all values of FGM power, mechanical properties ratio of the structure surfaces can have any arbitrary value. There are a few analytical solutions for FG hollow cylinders [26,27]. An analytical solution for static analysis of thick-walled cylindrical and spherical FG pressure vessels based on power-law model has been investigated by Tutuncu, et al [28]. Eslami, et al [29] studied thermal and mechanical stresses in FG thick spheres. However, in their analyses an incomplete power-law model is used for variation of Young's modulus as $E = E_0 r^{\beta}$. The limitation of this model is that for specific value of FGM power β, the inner to outer Young's modulus ratio can not have arbitrary value. For example, if the properties function is supposed to vary linearly (e.g. $\beta = 1$), it causes to have inner to outer Young's modulus ratio equal to radius ratio (e.g. $E_o/E_i = R_0/R_i$). However, no closed-form solutions were found for thick-walled FG spherical vessels based on exponential FG model and also based on power-law function without the mentioned limitation.

In this article, in order to vanish the limitation of incomplete power-law model, an exact analytical solution is presented for thick-walled functionally graded spherical pressure vessels with linearly-varying properties in radial direction. In this closed-form solution, the inner and outer surface properties can have arbitrary value. Besides, a closed-form solution has been obtained for thick-walled FG spherical pressure vessels based on exponential model for material distribution in radial direction. The inner to outer material property ratio can have arbitrary value. Therefore, the effects of applying different materials in designing of pressure vessels can be considered on the displacement and stress components. Finally, the results are verified by other known results. Meanwhile, the effects of different parameters on the solution are discussed and the solutions based on different FGM models are compared.

2. PROBLEM ANALYSIS

In order to elastostatics analysis of spherically symmetric thick-walled functionally graded spherical pressure vessels, only one equilibrium equation exists as follow

$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0 \quad ; \quad \sigma_\theta = \sigma_\varphi \tag{1}$$

Supposing infinitesimal deformation, the strain-displacement relations in spherical coordinate system are

$$\varepsilon_r = \frac{du_r}{dr} \quad ; \quad \varepsilon_\theta = \varepsilon_\varphi = \frac{u_r}{r} \tag{2}$$

For linear behavior of material, the stress-strain relations in spherically symmetric state for a non-homogenous isotropic material are written as

$$\begin{aligned} \sigma_r &= \frac{2\mu(r)}{(1-2\nu)}\left[(1-\nu)\,\varepsilon_r + 2\nu\varepsilon_\theta\right] \\ \sigma_\theta &= \frac{2\mu(r)}{(1-2\nu)}\left[\varepsilon_\theta + \nu\varepsilon_r\right] \end{aligned} \tag{3}$$

Where $\mu(r)$ is the radius dependant shear modulus and ν is the Poisson's ratio. With regard to the small variation of Poisson's ratio, it is usually consider constant in analysis of functionally graded materials (e.g. [22-26]).

Case 1.

Linearly-varying properties In this case, it is assumed that the variation of mechanical properties is linear through the thickness. Therefore, variation of shear modulus of a functionally graded material is modeled as

$$\mu(r) = A + Br \tag{4}$$

Supposing that the shear modulus of inner and outer surfaces are μ_i and μ_o, respectively, the constant coefficients A and B are calculated as

$$A = \frac{\mu_i R_o - \mu_o R_i}{R_o - R_i} \; ; \; B = \frac{\mu_o - \mu_i}{R_o - R_i} \tag{5}$$

Using above equation, the Equation 4 can be rewritten as

$$\mu(r)/\mu_o = 1 - (1 - \mu_i/\mu_o)\left(\frac{1 - r/R_o}{1 - R_i/R_o}\right) \tag{6}$$

Substituting Equations 2 and 3 into Equation 1 yields an ordinary differential equation as

$$\frac{d^2u_r}{dr^2} + \frac{2}{r}\frac{du_r}{dr} - \frac{2}{r^2}u_r + \frac{\mu'(r)}{\mu(r)}\left(\frac{du_r}{dr} + \frac{2\nu}{1-\nu}\frac{u_r}{r}\right) = 0 \tag{7}$$

Substituting Equation 4 into above equation, the following equation yields

$$\frac{d^2u_r}{dr^2} + \left(\frac{B}{A+Br} + \frac{2}{r}\right)\frac{du_r}{dr} + \frac{1}{r}\left(\frac{2\nu}{1-\nu}\frac{B}{A+Br} - \frac{2}{r}\right)u_r = 0 \tag{8}$$

The differential Equation 8 can be solved in term of Hypergeometric function as

$$u_r = C_{1L}.r^{-(1+N)}H\left([-(1-N), 2+N], [1+2N], -\frac{A}{B}\frac{1}{r}\right) + C_{2L}.r^{-(1-N)}H\left([-(1+N), 2-N], [1-2N], -\frac{A}{B}\frac{1}{r}\right) \tag{9}$$

Where H is the Hypergeometric function in which is defined by power series as [30]

$$H([a,b],[c],x) = 1 + \sum_{i=0}^{\infty} \frac{ab(a+1)(b+1)...(a+i)(b+i)}{i!\,c(c+1)...(c+i)} x^{i+1} \tag{10}$$

and the constant coefficient N is

$$N = \sqrt{\frac{3-5\nu}{1-\nu}} \tag{11}$$

Substituting Equation 9 into strain-displacement relations (2) and using stress-strain relations (3), the radial and circumferential stresses are obtained as

$$\sigma_r = -\frac{2(A+Br)}{(1-2\nu)r^2}\left\{C_{1L}r^{-N}\left((1-\nu)f_1(r) + [1-3\nu+N(1-\nu)]f_3(r)\right) + C_{2L}r^{N}\left((1-\nu)f_2(r) + [1-3\nu-N(1-\nu)]f_4(r)\right)\right\} \tag{12}$$

$$\sigma_\theta = \sigma_\varphi = -\frac{2(A+Br)}{(1-2\nu)r^2}\left\{C_{1L}r^{-N}\left(\nu f_1(r) - (1-\nu-\nu N)f_3(r)\right) + C_{2L}r^{N}\left(\nu f_2(r) - (1-\nu+\nu N)f_3(r)\right)\right\} \tag{13}$$

Where $f_1(r)$ through $f_4(r)$ functions are defined as following

$$\begin{Bmatrix} f_1(r) \\ f_2(r) \end{Bmatrix} = \frac{A}{B}\frac{(2\pm N)(1\mp N)}{(1\pm 2N)}\frac{1}{r}H\left([\pm N, 3\pm N], [2\pm 2N], -\frac{A}{B}\frac{1}{r}\right) \tag{14}$$

$$\begin{Bmatrix} f_3(r) \\ f_4(r) \end{Bmatrix} = H\left([2\pm N, -1\pm N], [1\pm 2N], -\frac{A}{B}\frac{1}{r}\right)$$

In Equations 9, 12 and 13, the unknown coefficients C_{1L} and C_{2L} can be determined by applying the boundary conditions of inner and outer surfaces. For a spherical vessel subjected to both internal hydrostatic pressure P_i and external hydrostatic pressure P_o, the boundary conditions can be written as

$$\begin{aligned}\sigma_r(r=R_i) &= -P_i\\ \sigma_r(r=R_o) &= P_o\end{aligned} \tag{15}$$

Satisfying the above boundary conditions, yields

$$\begin{aligned}C_{1L} &= \frac{(1-2\nu)\Delta^{-1}}{2(A+BR_i)(A+BR_o)}\{(A+BR_o)R_i{}^2R_o{}^N\\ &\Big((1-\nu)f_2(R_o)+[1-3\nu-N(1-\nu)]f_4(R_o)\Big)P_i-(A+BR_i)\\ &R_o{}^2R_i{}^N\Big((1-\nu)f_2(R_i)+[1-3\nu-N(1-\nu)]f_4(R_i)\Big)P_o\}\\ C_{2L} &= -\frac{(1-2\nu)\Delta^{-1}}{2(A+BR_i)(A+BR_o)}\{(A+BR_o)R_i{}^2R_o{}^{-N}\\ &\Big((1-\nu)f_1(R_o)+[1-3\nu+N(1-\nu)]f_3(R_o)\Big)P_i-(A+BR_i)\\ &R_o{}^2R_i{}^{-N}\Big((1-\nu)f_1(R_i)+[1-3\nu+N(1-\nu)]f_3(R_i)\Big)P_o\}\end{aligned} \tag{16}$$

Where the parameter Δ is defined as

$$\begin{aligned}\Delta =\ & R_i{}^{-N}R_o{}^N\Big((1-\nu)f_1(R_i)+[1-3\nu+N(1-\nu)]f_3(R_i)\Big)\times\\ &\Big((1-\nu)f_2(R_o)+[1-3\nu-N(1-\nu)]f_4(R_o)\Big)-\\ &R_i{}^NR_o{}^{-N}\Big((1-\nu)f_2(R_i)+[1-3\nu-N(1-\nu)]f_4(R_i)\Big)\times\\ &\Big((1-\nu)f_1(R_o)+[1-3\nu+N(1-\nu)]f_3(R_o)\Big)\end{aligned} \tag{17}$$

Case 2.

Exponentially-varying properties In order to consider the effect of different FGM models on displacement and stress distribution of a spherical FG vessel, it is assumed that the variation of functionally graded shear modulus obeys the exponential model as

$$\mu(r) = c\exp(\Gamma r) \tag{18}$$

Where c and Γ are two coefficients. Supposing that the shear modulus of inner and outer surfaces are μ_i and μ_o, respectively, the constants c and Γ are obtained as

$$c=\mu_o\left(\frac{\mu_i}{\mu_o}\right)^{\frac{R_o}{R_o-R_i}} \quad ; \quad \Gamma=\frac{\ln(\mu_o/\mu_i)}{R_o-R_i} \tag{19}$$

Substituting the above relations into Equation 18 yields the following function for variation of shear modulus in radial direction of vessel

$$\mu(r)/\mu_o=\left(\frac{\mu_i}{\mu_o}\right)^{\frac{1-(r/R_o)}{1-(R_i/R_o)}} \tag{20}$$

Replacing Equation 20 into differential Equation 7, the following differential equation with variable coefficients can be obtained

$$\frac{d^2u_r}{dr^2}+\frac{1}{r}(\Gamma r+2)\frac{du_r}{dr}+\frac{2}{r^2}\left(\frac{\nu}{1-\nu}\Gamma r-1\right)u_r=0 \tag{21}$$

The solution of this equation is

$$u_r=\frac{\exp(-\Gamma r/2)}{r}\begin{bmatrix}C_{1L}W_M\left(-\frac{1-3\nu}{1-\nu},\frac{3}{2},\Gamma r\right)+\\ C_{2L}W_W\left(-\frac{1-3\nu}{1-\nu},\frac{3}{2},\Gamma r\right)\end{bmatrix} \tag{22}$$

In which W_M and W_W are Whittaker functions [31]. Substituting relation (22) into Equations 2 and 3 yields

$$\sigma_r=\frac{2c}{1-2\nu}\frac{\exp(\Gamma r/2)}{r^2}\left\{C_{1E}(1+\nu)g_1(r)-C_{2E}(1-\nu)g_3(r)\right\} \tag{23}$$

$$\begin{aligned}\sigma_\theta=\sigma_\varphi &= \frac{2c}{(1-\nu)(1-2\nu)}\frac{\exp(\Gamma r/2)}{r^2}\\ &\Big\{C_{1E}(1+\nu)\big[\nu g_1(r)+(1-2\nu)g_2(r)\big]+\\ &C_{2E}\big[-\nu(1-\nu)g_3(r)+(1+\nu)(1-2\nu)g_4(r)\big]\Big\}\end{aligned} \tag{24}$$

Where

$$\left\{\begin{matrix} g_1(r) \\ g_2(r) \end{matrix}\right\} = W_M\left[-\frac{1}{2}\left(\frac{1-5\nu}{1-\nu} \mp 1\right), \frac{3}{2}, \Gamma r\right]$$
$$\left\{\begin{matrix} g_3(r) \\ g_4(r) \end{matrix}\right\} = W_W\left[-\frac{1}{2}\left(\frac{1-5\nu}{1-\nu} \mp 1\right), \frac{3}{2}, \Gamma r\right] \quad (25)$$

It is assumed that the vessel is subjected to internal hydrostatic pressure P_i and external hydrostatic pressure P_o, simultaneously. Thus, the boundary conditions will be presented by relation (15). Satisfying the mentioned boundary conditions result in the following relation for unknown coefficients as

$$C_{1E} = \frac{(1-2\nu)}{2(1+\nu)c\exp\left(\frac{\Gamma}{2}(R_i+R_o)\right)} \left(\frac{R_i{}^2\exp(\Gamma R_o/2)\, g_3(R_o)\, P_i - R_o{}^2\exp(\Gamma R_i/2)\, g_3(R_i)\, P_o}{g_3(R_i)\, g_1(R_o) - g_3(R_o)\, g_1(R_i)}\right)$$
$$C_{2E} = \frac{(1-2\nu)}{2(1-\nu)c\exp\left(\frac{\Gamma}{2}(R_i+R_o)\right)} \left(\frac{R_i{}^2\exp(\Gamma R_o/2)\, g_1(R_o)\, P_i - R_o{}^2\exp(\Gamma R_i/2)\, g_1(R_i)\, P_o}{g_3(R_i)\, g_1(R_o) - g_3(R_o)\, g_1(R_i)}\right) \quad (26)$$

Substituting C_{1E} and C_{2E} into relations (22-24) will result in relations for displacement and radial and circumferential stress distribution.

3. VERIFYING THE SOLUTION

In order to verify the accuracy of the present solutions, results of two specific problems are compared with known results in literature. The first problem is stress analysis of a homogeneous thick-walled pressure vessel (e.g. $\mu_i/\mu_o \to 1$). Distribution of radial and circumferential stresses for a homogeneous thick-walled spherical pressure vessels are as follows [32]

$$\sigma_r = -\frac{(R_o/r)^3 - 1}{(R_o/R_i)^3 - 1}P_i \;\; ; \sigma_\theta = \sigma_\varphi = \frac{(R_o/r)^3 + 2}{2[(R_o/R_i)^3 - 1]}P_i \quad (27)$$

In Figure 1, a comparison between stress distribution of a spherical homogeneous vessel based on reference [32] and both linear and exponential model of present analysis is shown.

In second problem, a comparison has been carried out with the results of reference [28]. Based on the relation (4), the ratio of the inner to outer elasticity modulus are as follow

$$\frac{E(R_i)}{E(R_o)} = \frac{A + BR_i}{A + BR_o} \quad (28)$$

However the variation of the mechanical properties in reference [28] was assumed as

$$\frac{E(R_i)}{E(R_o)} = \left(\frac{R_i}{R_o}\right)^\beta \quad (29)$$

It can be seen that as the constants A of relation (29) and β of relation (30) are equal to zero and unity respectively, the present model and the model of reference (28) becomes the same. In Figure 2 the comparison is presented for this special case. The superscript H denotes the homogeneous state.

As it is shown in Figures 1 and 2, the present results have a good agreement with those in literature.

4. PARAMETRIC STUDY AND DISCUSSION

To present numerical results, some examples have been considered. In all of them, the outer to inner radius ratio (R_o/R_i) of 2 is used. All figures are depicted for five different inner to outer shear modulus (μ_i/μ_o) as 1/5, 1/2, 1, 2 and 5. In these figures, the solid line corresponds to the isotropic homogeneous material.

As it is mentioned, in the first case, it is assumed that the variation of properties for functionally graded material is linear and in the second case, it is supposed that variation of properties is as an exponential function.

In Figures 3a,b the variation of nondimensional shear modulus across the radial direction based on linear function and exponential models are depicted,

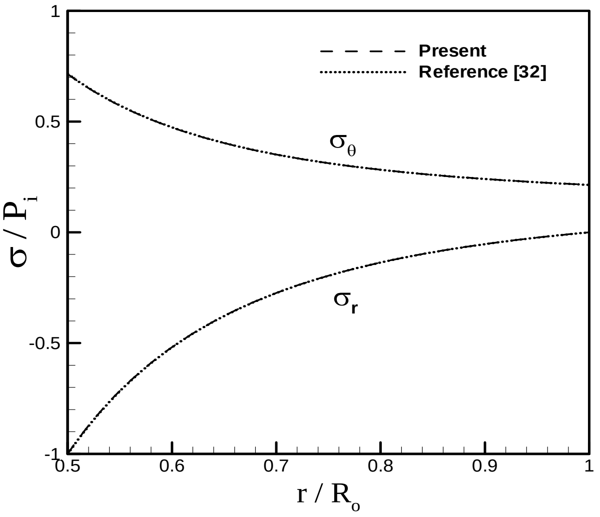


Figure 1. A comparison of nondimensional normal stresses for a homogeneous thick-walled spherical vessel subjected to internal pressure P_i .

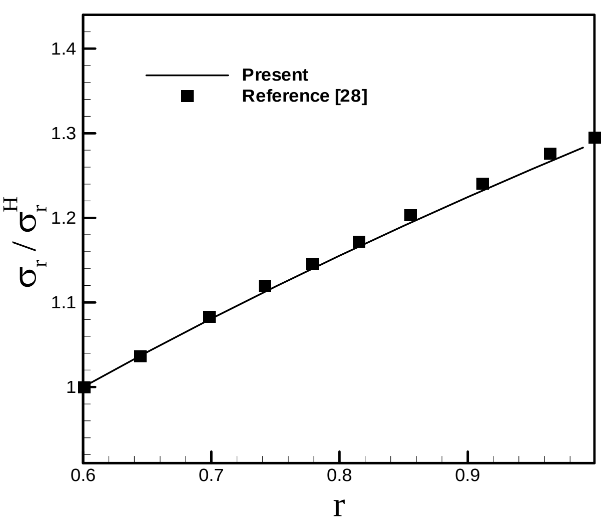


Figure 2. A comparison of nondimensional radial stress for a specific case of FG thick-walled spherical vessel subjected to internal pressure.

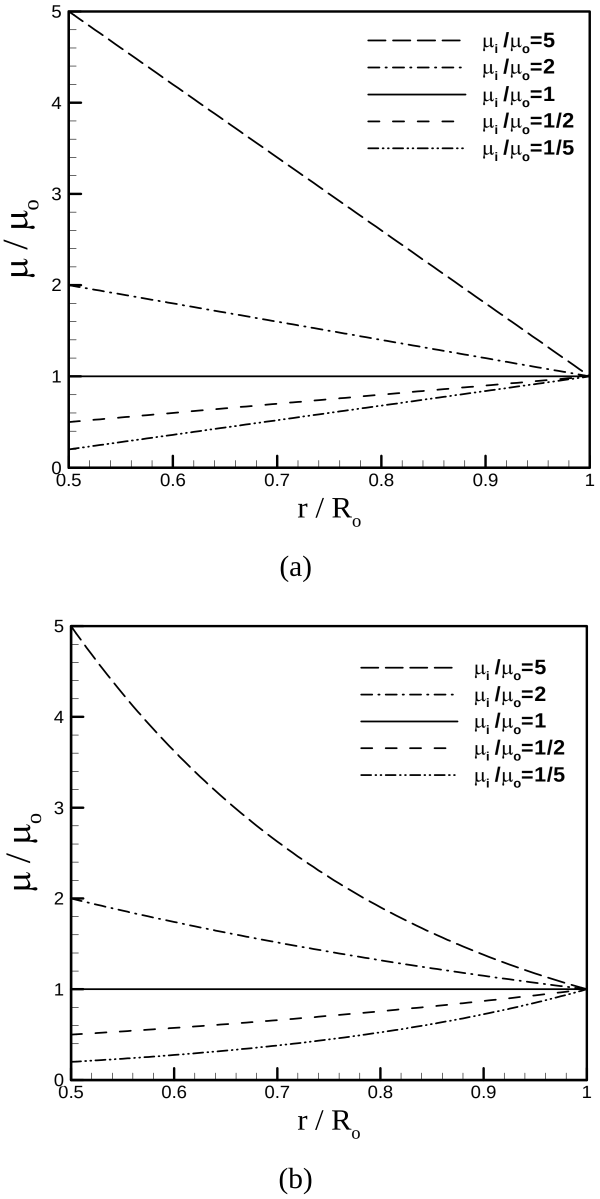
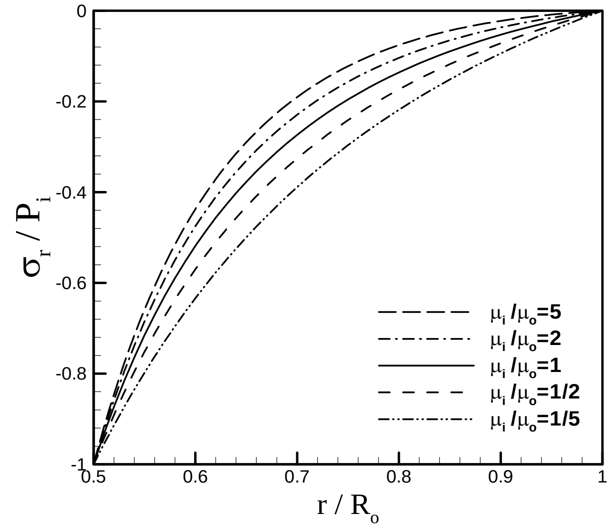


Figure 3. (a) Nondimensional shear modulus across the radial direction for linear variation model and (b) Nondimensional shear modulus across the radial direction for exponential model.

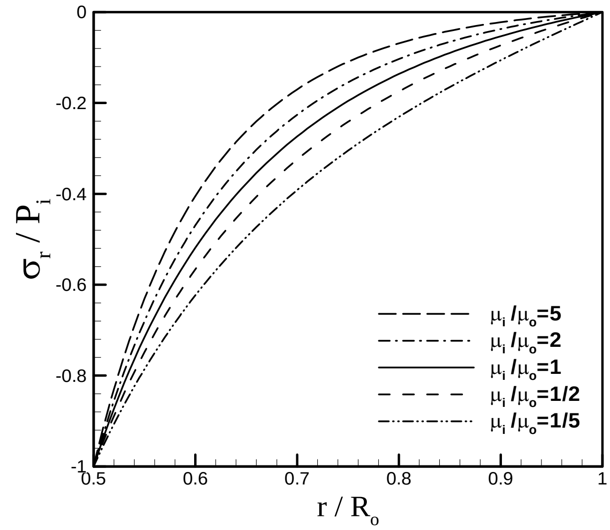
respectively. It can be seen that for different inner and outer shear modulus ratio, the variation of shear modulus stay linear.

4.1. Example 1.

4.1.1. A FG spherical Vessel subjected to internal pressure P_i In Figures 4a,b the variation of nondimensional radial stress through the radial direction of a thick-walled spherical vessel based on linear function and exponential model are shown, respectively. Focusing on these figures, it can be seen that by increasing the inner to outer shear modulus ratio, the radial stress distribution tends to be more nonlinear. However, decreasing this ratio from unity, the state of nonlinearity

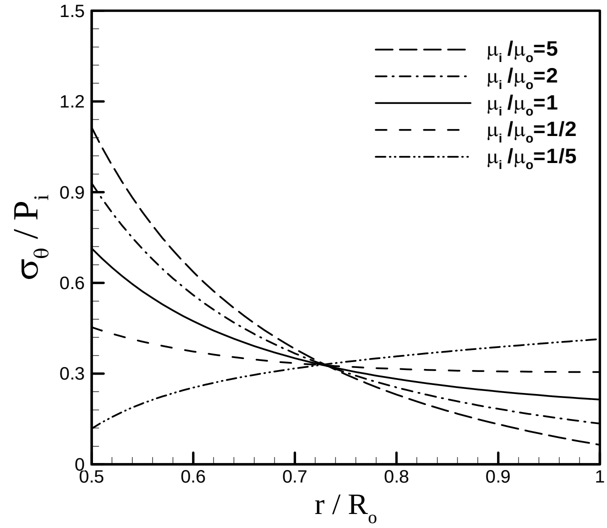


(a)

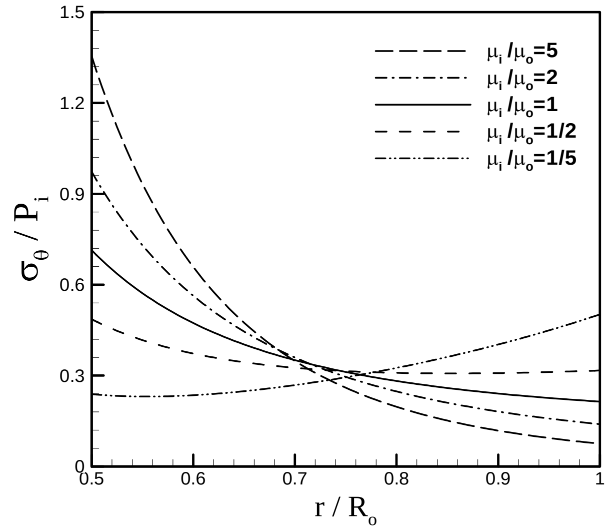


(b)

Figure 4. (a) Nondimensional radial stress across the radial direction for a thick-walled spherical vessel subjected to internal pressure P_i based on linear variation model and (b) Nondimensional radial stress across the radial direction for a thick-walled spherical vessel subjected to internal pressure P_i based on exponential model.



(a)



(b)

Figure 5. (a) Nondimensional circumferential stress across the radial direction for a thick-walled spherical vessel subjected to internal pressure P_i based on linear variation model and (b) Nondimensional circumferential stress across the radial direction for a thick-walled spherical vessel subjected to internal pressure P_i based on exponential model.

decreases with respect to homogeneous one. In other words, when the stiffness of the outer surface gets five times higher than inner one, the radial stress distribution nearly varies linear.

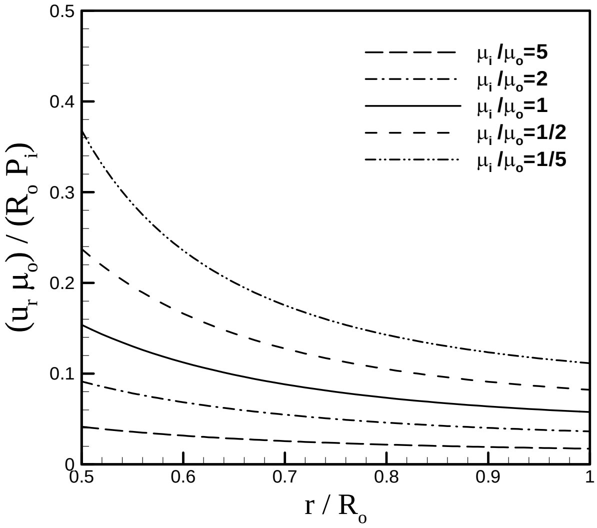
In Figures 5a,b the variation of nondimensional circumferential stress through the radial direction based on linear and exponential function models are shown, respectively. As it can be found in Figures 5a,b by increasing the inner to outer stiffness ratio, the value of circumferential stress at inner surface increases. In this case, the value of circumferential stress decreases as the points close to outer surface. Another important point that can be found from these figures is that the difference between circumferential stress of surfaces decreases with decreasing of stiffness of inner to outer surface. For inner to outer shear modulus ratio of 1/2, the circumferential stress distribution across the thickness of the vessel is approximately uniform. Whenever the stiffness of inner to outer surface is lower than this value, the value of circumferential stress in the inner surface is lower than the value at outer surface.

The variation of nondimensional displacement through the radial direction based on linear and exponential function models are shown in Figures 6a,b respectively. As it can be seen in these figures, by increasing the stiffness from inner to outer surfaces, the value of radial displacement decreases and its nonlinearity increases in radial direction.

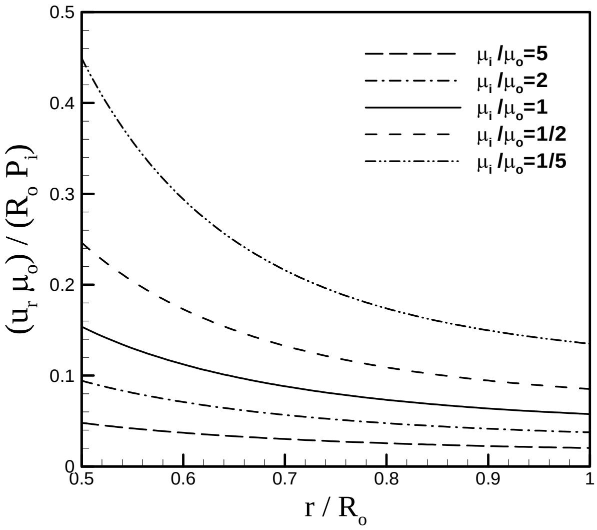
4.2. Example 2.

4.2.1. A FG spherical vessel subjected to both internal pressure P_i and external pressure P_o

In order to consider the effects of applying both internal and external pressures on the surfaces of thick walled spherical vessel, in this example the results of a FG spherical vessel subjected to both internal and external pressures are presented. It is assumed that the intensity of the internal and external pressure is the same. In Figures 7a,b the variation of the radial stress in radial direction are shown for the linear and exponential FGM model respectively. It can be seen that for this loading conditions, the radial stress in radial direction is uniform and equal to the external pressure when the vessel is made of homogeneous material. For FG vessels, although internal and external pressure is the same, the variation of the radial stress in radial direction is not uniform. Also, it can be said that as the inner shear modulus is higher than outer one, the compression stress is more significant than the surface pressure. On the other hand, the maximum of compression stress located in a point other than the surfaces. However, as the outer shear modulus is higher than inner one, the compression stress in the wall of the vessel is lower than the surface pressure. In this case, there is a point in the vessel that the stress is minimum.



(a)



(b)

Figure 6. (a) Nondimensional radial displacement across the radial direction for a thick-walled spherical vessel subjected to internal pressure P_i based on linear variation model and (b) Nondimensional radial displacement across the radial direction for a thick-walled spherical vessel subjected to internal pressure P_i based on exponential model.

In Figures 8a,b the variation of the circumferential stress of spherical vessel in radial direction are depicted. It can be seen that only for the homogenous material, the distribution of the circumferential

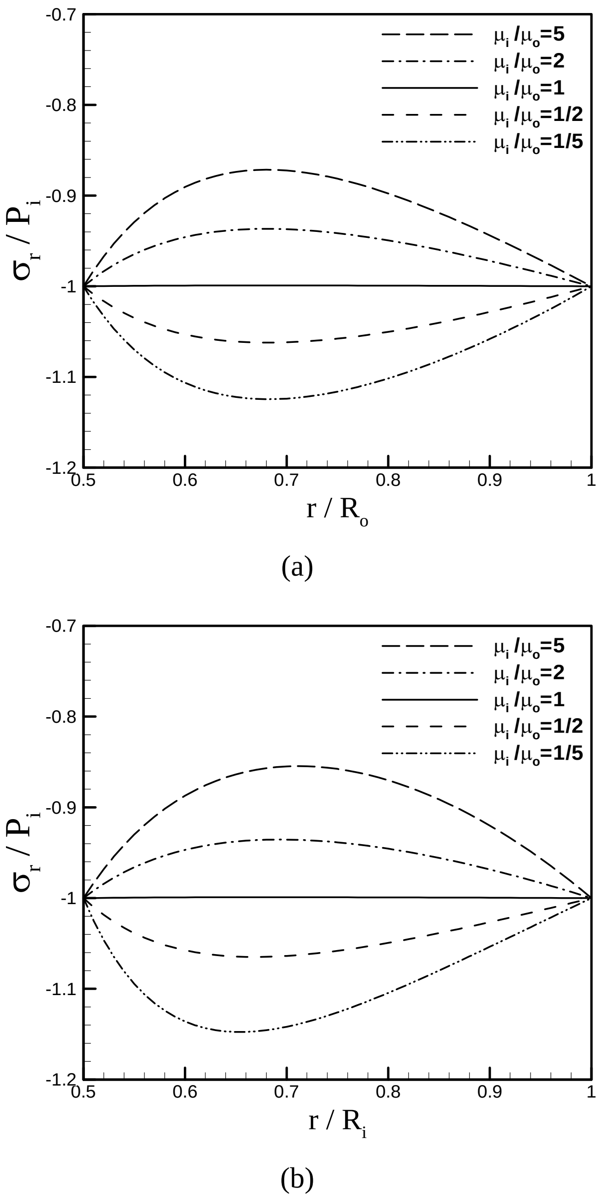


Figure 7. (a) Nondimensional radial stress across the radial direction for a thick-walled spherical vessel subjected to internal and external pressure P_i based on linear variation model and (b) Nondimensional radial stress across the radial direction for a thick-walled spherical vessel subjected to internal and external pressure P_i based on exponential model.

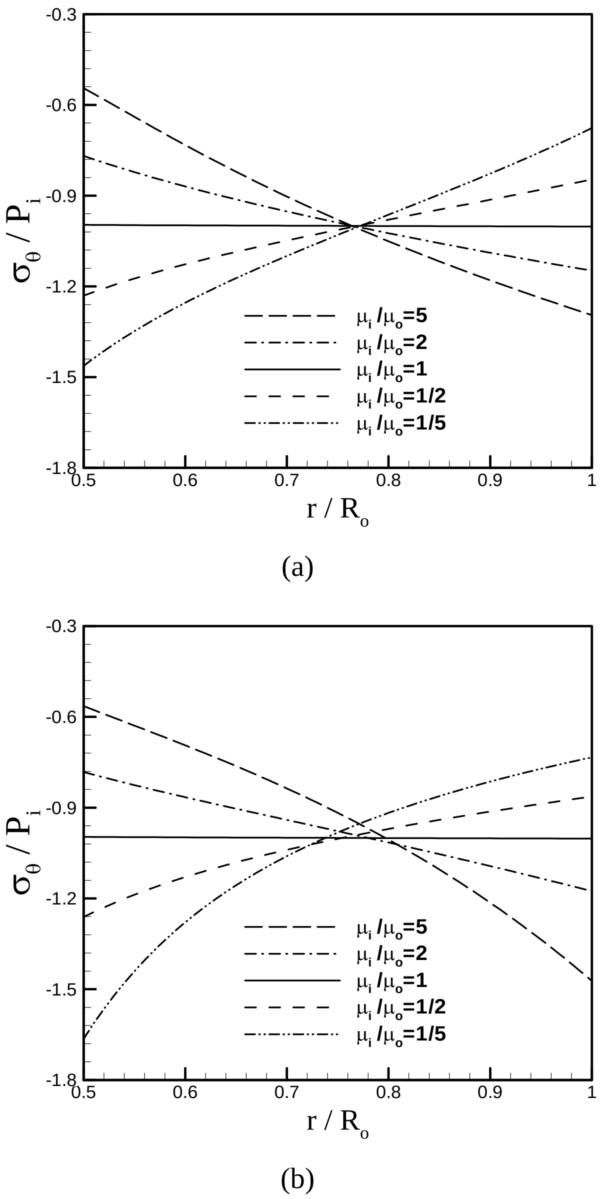


Figure 8. (a) Nondimensional circumferential stress across the radial direction for a thick-walled spherical vessel subjected to internal and external pressure P_i based on linear variation model and (b) Nondimensional circumferential stress across the radial direction for a thick-walled spherical vessel subjected to internal and external pressure P_i based on exponential model.

stress is uniform. Also, for FG spherical vessels with this kind of surface conditions, the variation of the circumferential stress in radial direction is nearly linear. This state of variation is less in exponential variation of the FGM. It can be seen that for different values of shear ratio, the circumferential stress in the middle of the wall is nearly equal to the surface pressure.

4.3. Example 3.

4.3.1. A FG spherical vessel subjected to external pressure P_o In this example, the results of a FG spherical vessel subjected to external pressure on the outer surface is presented.

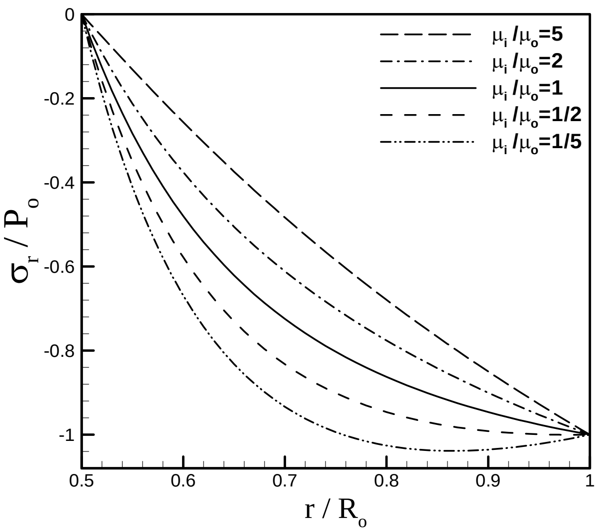
In Figures 9a,b the variation of the radial stress in radial direction are depicted for various shear modulus ratios. It can be seen that as the inner to outer shear modulus ratio approaches 5, the variation of the radial stress becomes linear and for lower shear ratios, the variation is nonlinear. Another important issue that can be concluded is when the inner to outer shear modulus ratio is lower than 1/5, the value of the radial stress in some points of the vessel is higher than external pressure. This value is the maximum of the radial stress.

In Figures 10a,b the variation of the circumferential stress is depicted for both linear and exponential models, respectively. In both models, the variation of the circumferential stress is uniform for inner to outer shear ratio close to 2. For higher shear ratios the value of the circumferential stress in inner surface is lower than that of the outer surface and vice versa.

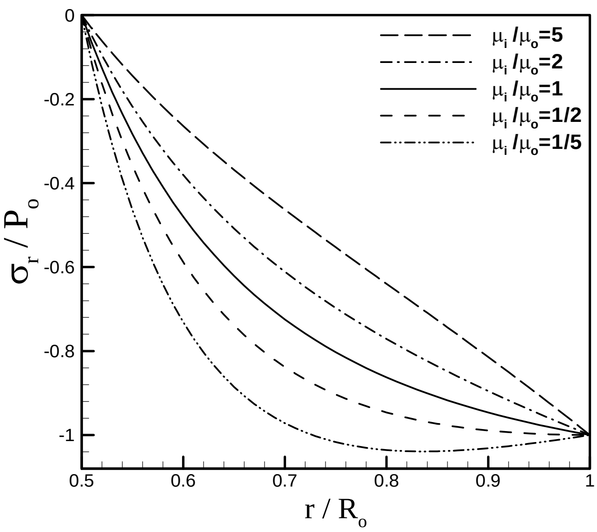
5. CONCLUSION

In a functionally graded material, the property of material varies continuously from point to point. By this characteristic, appropriate mechanical properties are achievable. Employing such a material is commonly used in building the thick-walled pressure vessels.

In this paper exact elasticity solutions have been presented to study the static analysis of thick-walled spherical vessels subjected to internal and external hydrostatic pressure. The closed-form solutions have been obtained for displacement and stress fields based on two common FGM models. With regard to use of different compound of materials in FG vessels, at first, the variation function of FGM has been assumed to be linear with respect to thickness direction. By this proposed function, inner to outer stiffness ratio becomes independent of the variation function of properties. After that, it has been supposed that variation function of FGM obeys exponential model. Some important results of this analysis



(a)



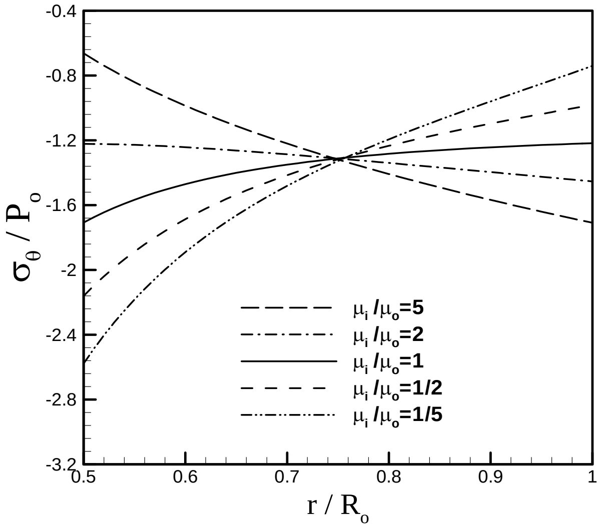
(b)

Figure 9. (a) Nondimensional radial stress across the radial direction for a thick-walled spherical vessel subjected to external pressure P_o based on linear variation model and (b) Nondimensional radial stress across the radial direction for a thick-walled spherical vessel subjected to external pressure P_o based on exponential model.

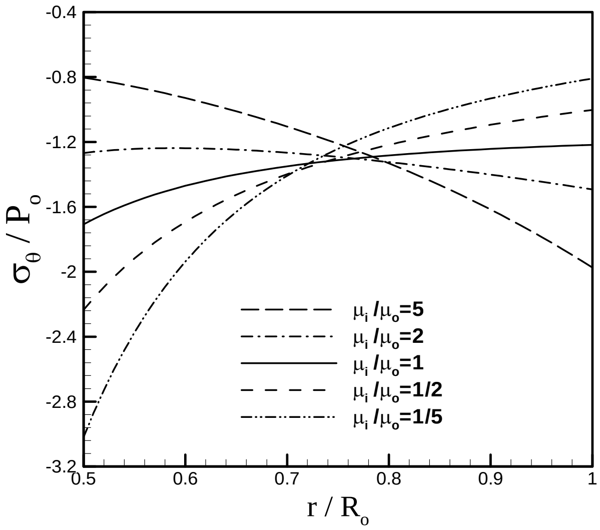
have been obtained as follow:

5.1. Only Internal Pressure Applies

- When the stiffness of the outer surface gets



(a)



(b)

Figure 10. (a) Nondimensional circumferential stress across the radial direction for a thick-walled spherical vessel subjected to external pressure P_o based on linear variation model and (b) Nondimensional circumferential stress across the radial direction for a thick-walled spherical vessel subjected to external pressure P_o based on exponential model.

five times higher than inner one, the radial stress distribution nearly varies linear and by increasing the inner to outer shear modulus ratio, the radial stress distribution tends to be more nonlinear.

- For inner to outer shear modulus ratio of 1/2, the circumferential stress distribution across the thickness of the vessel is approximately uniform and by increasing the inner to outer stiffness ratio, the value of circumferential stress at inner surface increases.

5.2. The Same Internal and External Pressure Applies Simultaneously

- As the shear modulus of the inner surface is higher than that of the outer surface, the value of the compression stress in the vessel is higher than the surface pressure. Conversely, the value of the compression stress in the vessel is lower than the surface pressure when the inner shear modulus is lower than the outer one.
- The variation of the circumferential stress in radial direction is nearly linear and for arbitrary shear modulus ratios, the value of this stress at the middle of the wall of the vessel becomes equal to the surface pressure.

5.3. Only External Pressure Applies

- For the inner to outer shear ratios of 5, the variation of the radial stress is linear and with decreasing this ratio the distribution becomes nonlinear.
- When the inner to outer shear modulus is less than 1/5, in some point of the wall of the vessel, the value of the radial stress is higher than external pressure.
- For inner to outer shear ratios close to 2, the variation of the circumferential stress is uniform.

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