RESERCH NOTE

DIFFUSION PROCESS FOR G/G/R MACHINING SYSTEM WITH SPARES, BALKING AND RENEGING

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Abstract This paper deals with the G/G/R machining system consisting of M operating machines as well as S cold standbys. The concepts of balking and reneging are incorporated which make our model more versatile to deal with real time systems. The broken-down machines are sent to repair facilities consisting of R permanent repairmen. The failure times and repair times are generally identical and independent distributed random variables. The failed machines are repaired according to FIFO rules. By using a diffusion process, the steady-state probability density function for the queue size is obtained in terms of the first two moments of inter-arrival times and repair times of machines. Some expressions in explicit form that characterize the system performance are also mentioned.

Key Words G/G/R Queue, Machine Repair, Standby, Diffusion, Balking, Reneging

چکیده در این مقاله یک سیستم G/G/R جهت ماشین کاری شامل M ماشین عملیاتی و S ماشین رزرو سرد مد نظر می‌باشد. از مفاهیم توده ای (انتظار در محل توقف) و برگشتی (منصرف شدن) برای نزدیک شدن مدل پیشنهادی به شرایط سیستم های واقعی استفاده شده است. ماشین های خراب شده به بخش تعمیرات فرستاده می‌شوند که شامل R تعمیرکار دائمی است. زمانهای خرابی و تعمیر به صورت متغیرهای تصادفی با توزیع مشابه (یکسان) و مستقل می‌باشند. ماشین های خراب شده مطابق با قاعده FIFO تعمیر می‌شوند. با استفاده از فرایند انتشار، تابع چگالی احتمال پایدار (ثابت) برای اندازه صف بر حسب دو گشتاور اول زمانهای بین ورود و خرابی ماشین ها بدست می‌آیند. برخی توضیحات که عملکرد سیستم را توصیف می‌کنند، ارایه شده است.

1. INTRODUCTION

In many manufacturing and production systems, it is noted that the queue of failed units is formed when repair facilities for failed machines are inadequate. The efficiency of any machining system can be improved by facilitating spare parts as well as skilled repairmen. Such types of queuing systems are treated as machine interference systems. In the case of the non-Markovian process, the use of diffusion approximation techniques is easily recommended in the case of many failed units.

Numerous authors have made successful efforts considering different frame- works for the real time congestion problems. Iglehart and Lemoine [1] studied the diffusion approximation for the repairmen problem consisting of two-repair facilities with/without provision of spares. A diffusion approximation technique for the GI/G/1 queuing system with finite capacity was considered by Kimura et al. [2]. Jain and Sharma [3] investigated the G/G/r machine interference problems with cold standby machines via diffusion approximation technique. Sivazlian and Wang [4] studied the G/G/R machine repair problem with warm standbys using diffusion approximation and obtained system characteristics and economic analysis of the system. Diffusion approximation for queuing systems with non-exhaustive cyclic service was considered by Yanagiya and Takahashi [5].

Diffusion process with reflecting boundaries for the $G^*/G/m$ machine repair model having spares was studied by Jain [6]. G/G/m machine repair problems based on diffusion approximation associated with elementary return and instantaneous return boundary policies was proposed by Lee et al. [7]. Jain [8] used the diffusion approximation technique for (m, M) machine repair problems with spares and dependent rates. G/G/r machine repair problems incorporating spares and additional repairmen were examined by Jain et al. [9]. Jain et al. [10] discussed multi-repairmen machining system with spares and balking via a diffusion process. Recently, Jain [11] considered a diffusion process for the $G^X/G/m$ queuing system with balking and reneging.

In this paper, we analyze a multi-repairmen machining system with spare part support by including the balking and reneging characteristics.

The rest of the paper is arranged into the following sections. Section 2 appends assumptions and notations used in the system. The analysis based on the diffusion process is presented in section 3. Further sections, 4 and 5, provide queue size distribution and special cases, respectively. The system performances are facilitated in section 6. In final section 7, the conclusion and future scope of the work done are outlined.

2. MODEL DESCRIPTION

Consider the G/G/R machining system with M operating and S cold standby units. The failed machines are repaired by a repair facility having R repairmen in order of its breakdowns. The balking and reneging behavior of the machines are also taken into consideration. Some assumptions made for the formulation of the model are given below:

1. If an operating machine breaks down, the standby machine replaces it.
2. The life time and repair time of units are generally, independent and identically distributed with mean $1/\lambda$ and $1/\mu$, respectively. The square coefficient variation of life time and repair time are denoted by C_a^2 and C_s^2, respectively.
3. The failed machines are renewed by skilled repairmen, provided it is available in order of their breakdowns.
4. When all the repairmen are busy, the machines are assumed to join the queue with the probability:

$$\frac{1}{n-R+2}$$

5. The failed machines may renege exponentially with parameter ν in case when all repairmen are busy. The arrival rate and repair rate are given by

$$\lambda_n = \begin{cases} M\lambda, & n < R \\ \dfrac{M\lambda}{n-R+2}, & R \le n < S \\ \dfrac{(N-n)\lambda}{n-R+2}, & S \le n \le N \end{cases} \quad (1)$$

and

$$\mu_n = \begin{cases} n\mu, & n < R \\ R\mu + (n-R)\nu, & R \le n \le N \end{cases} \quad (2)$$

Where

N = M+S.

The differential equation satisfied by the diffusion process is given by Cox and Miller, [12]

$$\frac{1}{2}\frac{d^2}{dx^2}\{a(x).b(x)\} - \frac{d}{dx}\{b(x).p(x)\} = 0 \quad (3)$$

where b(x) and a(x) are infinitesimal mean and infinitesimal variance, respectively.

Using reflecting barrier at x = 0, the Equation 3 provides

$$p(x) = \frac{k}{a(x)} \exp\left\{2\int_0^x \frac{b(x)}{a(x)}dx\right\} \quad (4)$$

where k is an arbitrary constant.

With the help of Equations 1 and 2, we propose

$$b(x) = \begin{cases} M\lambda - x\mu, & x < R \\ \dfrac{M\lambda}{x-R+2} - [R\mu + (x-R)\nu], & R \le x < S \\ \dfrac{(N-x)\lambda}{x-R+2} - [R\mu + (x-R)\nu], & S \le x \le N \end{cases} \quad (5)$$

and

$$a(x)=\begin{cases} M\lambda C_a^2 + x\mu C_s^2, & x<R \\ \frac{M\lambda}{x-R+2}C_a^2+[R\mu+(x-R)\nu]C_s^2, & R\le x<S \\ \frac{(N-x)\lambda}{x-R+2}C_a^2+[R\mu+(x-R)\nu]C_s^2, & S\le x\le N \end{cases} \quad (6)$$

3. MATHEMATICAL ANALYSIS

The following three cases are considered to provide probability density function of failed machines in the system

Case I: x < R

By using Equations 5 and 6, we find

$$\frac{b(x)}{a(x)}=\left\{\frac{M\lambda - x\mu}{M\lambda C_a^2 + x\mu C_s^2}\right\} \quad (7)$$

Now substituting the value from Equation 7 in Equation 4 and integrating we have

$$p_1(x)=k_1\left(M\lambda C_a^2 + x\mu C_s^2\right)^{\left\{\frac{2M\lambda}{\mu C_s^4}\left(C_a^2+C_s^2\right)-1\right\}}.\ \exp\left(\frac{-2x}{C_s^2}\right)\equiv k_1 g_1(x) \quad (8)$$

Case II: R ≤ x < S

In this case Equations 5 and 6 give

$$\frac{b(x)}{a(x)}=\left\{\frac{\frac{M\lambda}{x-R+2}-[R\mu+(x-R)\nu]}{\frac{M\lambda}{x-R+2}C_a^2+[R\mu+(x-R)\nu]C_s^2}\right\} \quad (9)$$

Again using Equation 9, Equation 4 yields

$$p_2(x)=k_2\left(x^2-\frac{\alpha}{\nu}x+\frac{M\lambda C_a^2+\beta C_s^2}{\nu C_s^2}\right)^{-1}$$

$$\exp\left\{-\frac{2}{C_s^2}x+\frac{4M\lambda(C_a^2+C_s^2)}{C_s^4\sqrt{4\nu^2(M\lambda C_a^2+\beta C_s^2)-\nu\alpha^2 C_s^2}}\tan^{-1}\left(\frac{2\nu x-\alpha}{\sqrt{4\nu^2(M\lambda C_a^2+\beta C_s^2)-\nu\alpha^2 C_s^2}}\right)\right\}$$

$$\equiv k_2 g_2(x) \quad (10)$$

where

$$\alpha=-\{R(\mu-2\nu)+2\nu\},\quad \beta=R(2-R)(\mu-\nu)$$

Case III: S≤x≤N

Now Equations 5 and 6 provide

$$\frac{b(x)}{a(x)}=\left\{\frac{\frac{(N-x)\lambda}{x-R+2}-[R\mu+(x-R)\nu]}{\frac{(N-x)\lambda}{x-R+2}C_a^2+[R\mu+(x-R)\nu]C_s^2}\right\} \quad (11)$$

Again substituting values from Equations 11 in 4 and integrating, we get

$$p_3(x)=k_3\left(x^2-\frac{(\alpha+\lambda)}{\nu}x+\frac{N\lambda C_a^2+\beta C_s^2}{\nu C_s^2}\right)^{-\left(1+\frac{4\lambda}{\nu C_s^2}\right)}$$

$$\exp\left\{-\frac{2}{C_s^2}x+\frac{4\lambda\left[N\nu(C_a^2+C_s^2)-2(\alpha+\lambda)C_s^2\right]}{\nu C_s^4\sqrt{4\nu^2(N\lambda C_a^2+\beta C_s^2)-\nu(\alpha+\lambda)^2C_s^2}}\tan^{-1}\left(\frac{2\nu x-(\alpha+\lambda)}{\sqrt{4\nu^2(N\lambda C_a^2+\beta C_s^2)-\nu(\alpha+\lambda)^2C_s^2}}\right)\right\}$$

$$\equiv k_3 g_3(x) \quad (12)$$

$$p(x)=\begin{cases} p_1(x)\equiv k_1 g_1(x), & 0\le x<R \\ p_2(x)\equiv k_2 g_2(x), & R\le x<S \\ p_3(x)\equiv k_3 g_3(x), & S\le x<N \end{cases} \qquad (13)$$

The continuity of p(x) at x = R and x = S can be employed to eliminate constants k_2 and k_3. Now $p_2(x)$ becomes

$$p_2(x)=k_1\frac{g_1(R)}{g_2(R)}.\,g_2(x) \qquad (14)$$

Also $p_3(x)$ is obtained as

$$p_3(x)=k_1\,\frac{g_1(R)}{g_2(R)}\cdot\frac{g_2(S)}{g_3(S)}\cdot g_3(x) \qquad (15)$$

To obtain the value of k_1, the normalizing condition is used which is given by

$$\int_0^R p_1(x)\,dx+\int_R^S p_2(x)\,dx+\int_S^N p_3(x)\,dx=1$$

so that

$$k_1=\left\{\int_0^R g_1(x)\,dx+\int_R^S\frac{g_1(R)}{g_2(R)}.\,g_2(x)\,dx+\int_S^N\frac{g_1(R)}{g_2(R)}.\frac{g_2(S)}{g_3(S)}.\,g_3(x)\,dx\right\}^{-1} \qquad (16)$$

4. QUEUE SIZE DISTRIBUTION

One of the following ways to discretiz p(x) into $\hat{p}_n$, the steady state probability of n customers being in the system can be applied.

(1) $\hat{p}_n=p(x)$

(2) $\hat{p}_n=\int_{n-1}^{n}p(x)\,dx$ (17)

(3) $\hat{p}_n=\int_{n-0.5}^{n+0.5}p(x)\,dx$

(4) $\hat{p}_n=\int_{n}^{n+1}p(x)\,dx$

5. SOME SPECIAL CASES

(a) If we assume $C_a^2=1$, then Equations 8, 10 and 12 reduce to a M/G/R machining system with balking and reneging. Now we get

(i)

$$p_1(x)=k_1\left(M\lambda+x\mu C_s^2\right)^{\left\{\frac{2M\lambda}{\mu C_s^4}\left(1+C_s^2\right)-1\right\}}.\exp\left(\frac{-2x}{C_s^2}\right)$$

(ii)

$$p_2(x)=k_2\left(x^2-\frac{\alpha}{\nu}x+\frac{M\lambda+\beta C_s^2}{\nu C_s^2}\right)^{-1}$$

$$\exp\left\{-\frac{2}{C_s^2}x+\frac{4M\lambda(1+C_s^2)}{C_s^4\sqrt{4\nu^2(M\lambda+\beta C_s^2)-\nu\alpha^2C_s^2}}\tan^{-1}\left(\frac{2\nu x-\alpha}{\sqrt{4\nu^2(M\lambda+\beta C_s^2)-\nu\alpha^2C_s^2}}\right)\right\}$$

(iii)

$$p_3(x)=k_3\left(x^2-\frac{(\alpha+\lambda)}{\nu}x+\frac{N\lambda+\beta C_s^2}{\nu C_s^2}\right)^{-\left(1+\frac{4\lambda}{\nu C_s^2}\right)}$$

$$\exp\left\{-\frac{2}{C_s^2}x+\frac{4\lambda\left[N\nu(1+C_s^2)-2(\alpha+\lambda)C_s^2\right]}{\nu C_s^4\sqrt{4\nu^2(N\lambda+\beta C_s^2)-\nu(\alpha+\lambda)^2C_s^2}}\tan^{-1}\left(\frac{2\nu x-(\alpha+\lambda)}{\sqrt{4\nu^2(N\lambda+\beta C_s^2)-\nu(\alpha+\lambda)^2C_s^2}}\right)\right\}$$

On taking $C_a^2 = 1$ and $C_s^2 = 1$, the Equations 8, 10 and 12 provide results for M/M/R machining system with balking and reneging. Then we have

1. $p_1(x)=k_1(M\lambda+x\mu)^{\left\{\frac{4M\lambda}{\mu}-1\right\}}.\exp(-2x)$

2. $p_2(x)=k_2\left(x^2-\frac{\alpha}{\nu}x+\frac{M\lambda+\beta}{\nu}\right)^{-1}.$

$$\exp\left\{-2x+\frac{8M\lambda}{\sqrt{4\nu^2(M\lambda+\beta)-\nu\alpha^2}}.\tan^{-1}\left(\frac{2\nu x-\alpha}{\sqrt{4\nu^2(M\lambda+\beta)-\nu\alpha^2}}\right)\right\}$$

3. $p_3(x)=k_3\left(x^2-\frac{(\alpha+\lambda)}{\nu}x+\frac{N\lambda+\beta}{\nu}\right)^{-\left(1+\frac{4\lambda}{\nu}\right)}$

$$\exp\left\{-2x+\frac{8\lambda\left[N\nu-(\alpha+\lambda)\right]}{\nu\sqrt{4\nu^2(N\lambda+\beta)-\nu(\alpha+\lambda)^2}}.\tan^{-1}\left(\frac{2\nu x-(\alpha+\lambda)}{\sqrt{4\nu^2(N\lambda+\beta)-\nu(\alpha+\lambda)^2}}\right)\right\}$$

6. SOME PERFORMANCE MEASURES

Applying discretizing scheme $\hat{p}_n=\int_{n-0.5}^{n+0.5}p(x)dx$ and using Equations 8, 10, 12 and 17, we find the average number of failed machines in the system as

$$E(N)=\sum_{n=1}^{N}n.\hat{p}_n$$

$$=\sum_{n=1}^{N}n\int_{n-0.5}^{n+0.5}p(x)\,dx$$

$$=\sum_{n=1}^{R-1}n\int_{n-0.5}^{n+0.5}p_1(x)\,dx+R\left\{\int_{R-0.5}^{R}p_1(x)dx+\int_{R}^{R+0.5}p_2(x)dx\right\}$$

$$+\sum_{n=R+1}^{S-1}n\int_{n-0.5}^{n+0.5}p_2(x)\,dx+$$

$$S\left\{\int_{S-0.5}^{S}p_2(x)\,dx+\int_{S}^{S+0.5}p_3(x)\,dx\right\}$$

$$+\sum_{n=S+1}^{N-1}n\int_{n-0.5}^{n+0.5}p_3(x)dx+N\int_{N-0.5}^{N}p_3(x)dx \tag{18}$$

Average queue length of failed machines is obtained as

$$L_q=\sum_{n=R}^{N}(n-R)\,\hat{p}_n$$

$$=\sum_{n=R+1}^{S-1}(n-R)\int_{n-0.5}^{n+0.5}p_2(x)\,dx+(S-R)$$

$$\left\{\int_{S-0.5}^{S}p_2(x)\,dx+\int_{S}^{S+0.5}p_3(x)\,dx\right\}$$

$$+\sum_{n=S+1}^{N-1}(n-R)\int_{n-0.5}^{n+0.5}p_3(x)dx+ \tag{19}$$

$$(N-R)\int_{N-0.5}^{N}p_3(x)dx$$

7. DISCUSSION

We have investigated the queue size distribution of a G/G/R machining system with cold standbys via. a diffusion approximation technique. The balking and reneging behavior of the failed machines are incorporated. The repair facility is recommended so that failed machines can be repaired at proper times and after getting repaired may become operational. Several performance measures viz. queue size distribution, average number of machines in the system and average queue length are obtained. For general systems, when exact results are difficult to find, the suggested diffusion approximation method may be helpful to predict performance indices in particular when the traffic load is heavy.

8. REFERENCES

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