AXISYMMETRIC STAGNATION-POINT FLOW OF A VISCOUS FLUID ON A MOVING CYLINDER WITH TIME-DEPENDENT AXIAL VELOCITY

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Abstract The unsteady viscous flow in the vicinity of an axisymmetric stagnation point of an infinite moving cylinder with time-dependent axial velocity is investigated. The impinging free stream is steady with a strain rate *k*. An exact solution of the Navier-Stokes equations is derived in this problem. A reduction of these equations is obtained by use of appropriate transformations. The general self-similar solution is obtained when the axial velocity of the cylinder varies as specified time-dependent functions. In particular, the cylinder may move with different velocity patterns. For completeness, sample semi-similar solutions of the unsteady Navier-Stokes equations have been obtained numerically using a finite-difference scheme. These solutions are presented for special cases when the time-dependent axial velocity of the cylinder is a step-function, a ramp, and a non-linear function. All the solutions above are presented for Reynolds numbers, $\mathrm{Re}=ka^2/2\upsilon$, ranging from 0.1 to 100 where a is cylinder radius and υ is kinematic viscosity of the fluid. Shear stresses corresponding to all the cases increase with the Reynolds number. The maximum value of the shear stress increases with increasing oscillation frequency and amplitude. An interesting result is obtained in which a cylinder moving with certain axial velocity function and at particular value of Reynolds number is axially stress-free.

Key Words Axisymmetric Flow, Stagnation Flow, Time-Dependent Axial Movement, Exact Solution

چکیده در این مقاله جریان لزج سکون متقارن بر روی یک سیلندر با سرعت محوری وابسته به زمان مورد بررسی قرار می گیرد. در این مساله حل دقیق معادلات ناویر – استوکس به دست می آیند. حل عمومی تشابهی مساله برای سرعت محوری سیلندر بر حسب توابع مختلف زمان حاصل می شود. همچنین نمونه ای از حل نیمه تشابهی معادلات ناویر – استوکس با استفاده از روشهای تفاضلی حل و ارائه می شوند. تنش برشی در تمام موارد با افزایش عدد رینولدز افزایش می یابد. نتیجه جالب این است که سیلندر در حال حرکت با سرعت محوری خاص و در عدد رینولدز معلوم بدون اصطکاک خواهد بود. این حلها برای اعداد رینولدز بین ۰/۱ تا ۱۰۰ ارائه می شوند. عدد رینولدز از رابطه $\mathrm{Re}=ka^2/2\upsilon$ بدست می آید که در آن a شعاع سیلندر و υ لزجت سینماتیکی سیال است.

1. INTRODUCTION

The task of finding exact solutions for Navier-Stokes equations is a difficult one due to nonlinearity of these equations. However, exact solutions of Navier-Stokes equations are available for special cases. Hiemenz [1] has obtained an exact solution of the Navier-Stokes equations for two dimensional stagnation point flow on a flat plate. The analogous axisymmetric stagnation-point flow 2 was investigated by Homann [2]. Results of the problem of stagnation flow against a flat plate for asymmetric cases were presented by Howarth [3] and Davey [4]. Wang [5] was first to find exact solution for the problem of axisymmetric stagnation flow on an infinite stationary circular cylinder. Gorla [6-10], in a series of papers, studied the steady and unsteady flows over a circular cylinder in the vicinity of the stagnation-point for the cases of constant axial movement, and the special case of axial harmonic motion of a non-rotating cylinder. This special case is only for small and high values of frequency parameter using perturbation techniques. Recently, Cunning, Davis, and Weidman [11] have considered the stagnation flow problem on a rotating circular cylinder with constant angular velocity, including the effects of suction and blowing with constant rate. Takhar, Chamkha and Nath [12] have also investigated the unsteady viscous flow in the vicinity of an axisymmetric stagnation point of an infinite circular cylinder when both the cylinder and the free stream velocities vary as a same function of time. Their self-similar solution is only for the case when both the cylinder and the free-stream velocities vary inversely as a linear function of time and by taking an average value for Reynolds number. Also their semi-similar solutions are for the accelerating and decelerating cases of the cylinder movement but with the same type of time dependent function as the free-stream velocity and only for Reynolds numbers up to 10. The study considered by Rahimi [13] presents a systematic solution of Gorla's results for high Prandtl number fluids using an inner-outer expansion of the fluid properties.

The effects of cylinder movement with time-dependent axial velocity in general, especially with different types of harmonic oscillation, which are of interest in certain special manufacturing processes, textile technology, accelerating phases of rocket motors, have not yet been considered.

In the present analysis, the unsteady viscous flow in the vicinity of an axisymmetric stagnation point of an infinite cylinder with time-dependent axial movement is considered. An exact solution of the Navier-Stokes equations is obtained.

The general self-similar solution is obtained when the axial velocity of the cylinder varies in a prescribed manner. The cylinder may perform different types of motion: it may move with constant speed, with exponentially increasing/decreasing axial velocity, with harmonically varying axial speed, or with accelerating/decelerating oscillatory axial speed.

For different forms of azimuthal component of velocity, sample distributions of shear stresses are presented for Reynolds numbers ranging from 0.1 to 100. Particular cases of these results are compared with existing results of Wang [5] and Gorla [7,9], correspondingly. For completeness, some semi-similar solutions of the Navier-Stokes equations are obtained and results for few examples of cylinder axial motion in the form of a step-function, a linear, and a few non-linear function are presented for different values of flow parameters.

2. PROBLEM FORMULATION

Flow is considered in cylindrical coordinates (r, θ, z) with corresponding velocity components (u,v,w), as Figure 1. We consider the laminar unsteady incompressible flow of a viscous fluid in the neighborhood of an axisymmetric stagnation-point of an infinite circular cylinder when it moves axially with a velocity that varies with time. An external axisymmetric radial stagnation flow of strain rate k impinges on the cylinder with radius a and centered on r = 0. References 5-9 give the unsteady Navier-Stokes equations in cylindrical coordinates governing the axisymmetric flow:

Mass:

$$\frac{\partial}{\partial r}(ru)+r\frac{\partial w}{\partial z}=0 \tag{1}$$

Momentum:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\upsilon(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}) \tag{2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\upsilon(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}) \tag{3}$$

Where p, ρ and υ are the fluid pressure, density, and kinematic viscosity. The boundary conditions for velocity field are:

$$r=a: \quad u=0, \quad w=V(t) \tag{4}$$

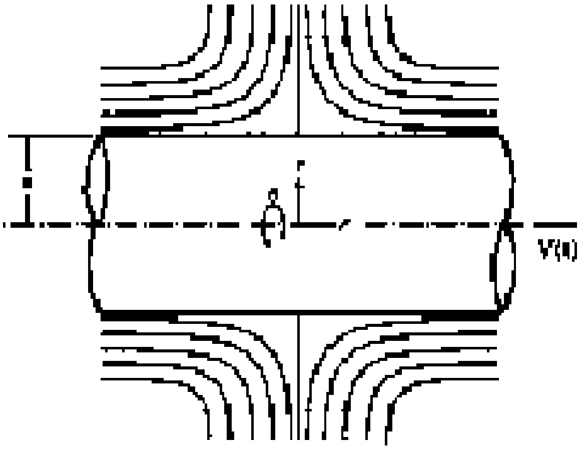

Figure 1. Schematic diagram of an axially moving cylinder under radial stagnation flow in the fixed cylindrical coordinate system, (r,θ,z).

$$r\rightarrow\infty: u=-k(r-\frac{a^2}{r})-\frac{aU_o}{r}, \quad w=2kz \tag{5}$$

in which, Equations 4 are no-slip conditions on the cylinder wall and Relations 5 show that the viscous flow solution approaches the potential flow solution as $r\rightarrow\infty$ [10].

A reduction of the Navier-Stokes equations is obtained by the following coordinate separation of the velocity field:

$$\begin{aligned} &u=-k\frac{a}{\sqrt{\eta}}f(\eta,\tau) \\ &w=2kf'(\eta,\tau)z+H(\eta,\tau), \qquad P=\rho k^2 a^2 P \end{aligned} \tag{6}$$

where $\tau=2kt$ and $\eta=(r/a)^2$ are dimensionless time and radial variables and prime denotes differentiation with respect to η Transformations 6 satisfy Equation 1 automatically and their insertion into Equations 2-3 yield a coupled system of differential Equations in terms of $f(\eta,\tau)$ and $H(\eta,\tau)$ and an expression for the pressure:

$$\eta f'''+f''+Re\left[1-(f')^2+ff''-\frac{\partial f'}{\partial\tau}\right]=0 \tag{7}$$

$$\eta H''+H'+Re\left[fH'-f'H-\frac{\partial H}{\partial\tau}\right]=0 \tag{8}$$

$$P-P_o=-\left[\frac{f^2}{2\eta}+\frac{1}{Re}f'+2(\frac{z}{a})^2\right] \tag{9}$$

In these Equations, primes indicate differentiation with respect to η and $Re=ka^2/2\upsilon$ is the Reynolds number. From Conditions 4 and 5, the boundary conditions for (7) and (8) are as follows:

$$\begin{aligned} &\eta=1: \quad f=0, \quad f'=0, \quad H=V(\tau) \\ &\eta\rightarrow\infty: \quad f'=1, \quad H=0 \end{aligned} \tag{10}$$

Here, Equations 7 and 8 are for different forms of $V(\tau)$ functions and have been solved numerically with Re as parameter.

Note that none of the boundary conditions of Equation 7 are functions of time and assuming steady - state initial conditions for this equation, we have:

$$\tau=0 \rightarrow \frac{\partial f'}{\partial\tau}=0$$

Therefore in this case Equation 7 is reduced to the following form:

$$\eta f'''+f''+Re\left[1-(f')^2+ff''\right]=0 \tag{11}$$

Steady-state solutions are obtained by solving this equation. In all later times $(\tau\neq 0)$ since none of the boundary conditions on f are functions of time and therefore this function does not change with respect to time, the steady-state solution at $(\tau=0)$ is the same and thus $f(\eta,\tau)=f(\eta)$ and consequently Equation 7 can be reduced to Equation 11.

Equation 11 is the same as the one obtained by Wang [5] and its solution is known. Here, Equation 8 is solved for different forms of $V(\tau)$ functions. In what follows, first the self-similar equations and the exact solutions of some particular $V(\tau)$

functions are presented and then, for completeness, the semi-similar equations and their numerical solutions are presented for few examples of these functions.

3.SELF-SIMILAR EQUATIONS

Equation 8 can be reduced to a system of ordinary differential equations if we assume that the function $H(\eta,\tau)$ in Equation 8 is separable as:

$$H(\eta,\tau)=V(\tau).h(\eta) \tag{12}$$

Substituting these separation of variables into Equation 8, gives:

$$\eta\frac{h''}{h}+\frac{h'}{h}+Re(f\frac{h'}{h}-f')=\frac{Re}{V(\tau)}\frac{dV(\tau)}{d\tau} \tag{13}$$

The general solution to the differential equation in (13) with (τ) as an independent variable is as follows:

$$V(\tau)=b.exp\left[(\alpha+i\beta)\tau\right] \tag{14}$$

Here, $i=\sqrt{-1}$ and b, α and β are constants. The boundary conditions are:

$$\eta=1: \quad h=1 \tag{15}$$

$$\eta\to\infty: \quad h=0 \tag{16}$$

Substituting the solution (14) into the differential Equation 13 with η as independent variable results in:

$$\eta h''+h'+\mathrm{Re}\left[f\,h'-f'\,h-\alpha h-iBh\right]=0 \tag{17}$$

Note that in Equation 14 $(b=0)$ corresponds to the case of cylinder with no axial movement, Wang [5]. If $(b\neq0)$ and $(\alpha=\beta=0)$ in Equation 14 gives the case of moving cylinder with constant axial velocity, Gorla [7]. The conditions $(b\neq0)$, $(\beta\neq0)$ and $(\alpha=0)$ corresponds to the case of moving cylinder with a harmonic velocity in its own plane, Gorla [9]. The case of $(b\neq0)$, $(\alpha\neq0)$, $(\beta\neq0)$. is the most general case that is considered in this paper.

To obtain solution of Equation 17, it is assumed that $h(\eta)$ is a complex function as:

$$h(\eta)=h_1(\eta)+ih_2(\eta) \tag{18}$$

Substituting Equation 18 into Equation 17, the following coupled system of differential equations are obtained:

$$\begin{cases}\eta h_1''+h_1'+Re(f\,h_1'-f'h_1-\alpha h_1+\beta h_2)=0\\ \eta h_2''+h_2'+Re(f\,h_2'-f'h_2-\alpha h_2+\beta h_1)=0\end{cases} \tag{19}$$

The boundary conditions for functions *f*, and *h* become:

$$\eta=1: \quad f=0, \quad f'=0, \quad h=1 \tag{20}$$

$$\eta\to\infty: \quad f'=1, \quad h=0 \tag{21}$$

Hence, the boundary conditions on functions h_1 and h_2 are:

$$\eta=1: \quad h_1=1, \quad h_2=0 \tag{22}$$

$$\eta\to\infty: \quad h_1=0, \quad h_2=0 \tag{23}$$

The coupled equations in system of Equations 19 along with Boundary Conditions 22 and 23 have been solved by using the fourth-order Runge-Kutta method of numerical integration along with a shooting method, Press et al. [14]. Using this method, the initial values of $h_1'(1)$, $h_2'(1)$ are guessed and the integration was repeated until convergence was obtained. The value of $h_2(\eta)=0$ was assumed initially and then by repeating the integration of this system of equations, final values of $h_1(\eta)$ *and* $h_2(\eta)$ were obtained.

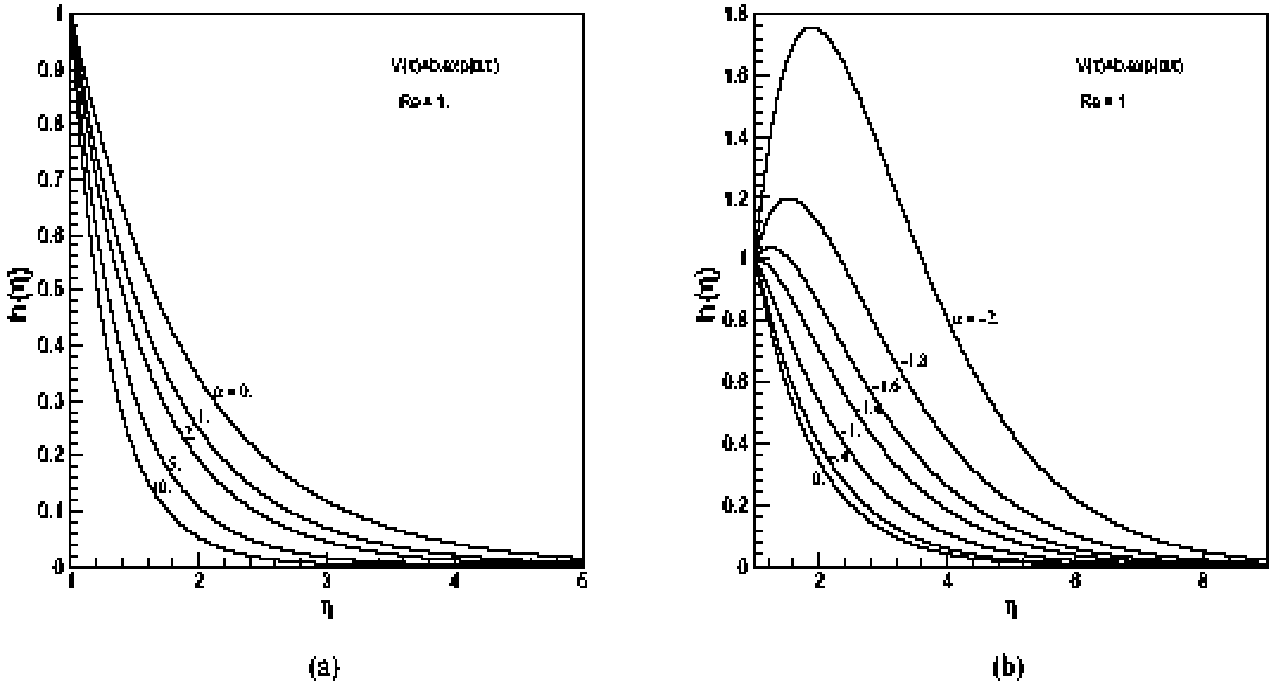

Figure 2. Sample profiles of $h(\eta)$ for cylinder with accelerating and decelerating exponential axial velocity.

The axial velocity of cylinder is:

$$V(\tau) = b\,Exp(\alpha\tau)\left[\cos(\beta\tau) + i\sin(\beta\tau)\right] \quad (24)$$

and thus, the $H(\eta,\tau)$ function from definition (12) becomes as follows:

$$H(\eta,\tau) = b\,Exp(\alpha\tau)\begin{bmatrix}\{h_1 cos(\beta\tau) + h_2 sin(\beta\tau)\} \\ +i\{h_1 sin(\beta\tau) + h_2 cos(\beta\tau)\}\end{bmatrix} \quad (25)$$

Sample axial velocity profiles will be presented in later sections.

4. SEMI-SIMILAR EQUATIONS

Equation 8 can be solved directly for any chosen $V(\tau)$ function. The solutions obtained this way, are called semi-similar solutions. This equation along with Boundary Conditions 10 was solved using a central finite difference method, which lead to a tri- diagonal matrix. Assuming steady state for $\tau \leq 0$, the solution starts from $V(0)$, and marching through time, time-dependent solutions for $\tau \succ 0$ were obtained. The results for same selected velocity functions are presented later, such as step – function, a linear function, and a nonlinear function.

5. SHEAR-STRESS

The shear stress on the cylinder surface is calculated from:

$$\sigma = \left[\mu \frac{\partial w}{\partial r}\right]_{r=a} \quad (26)$$

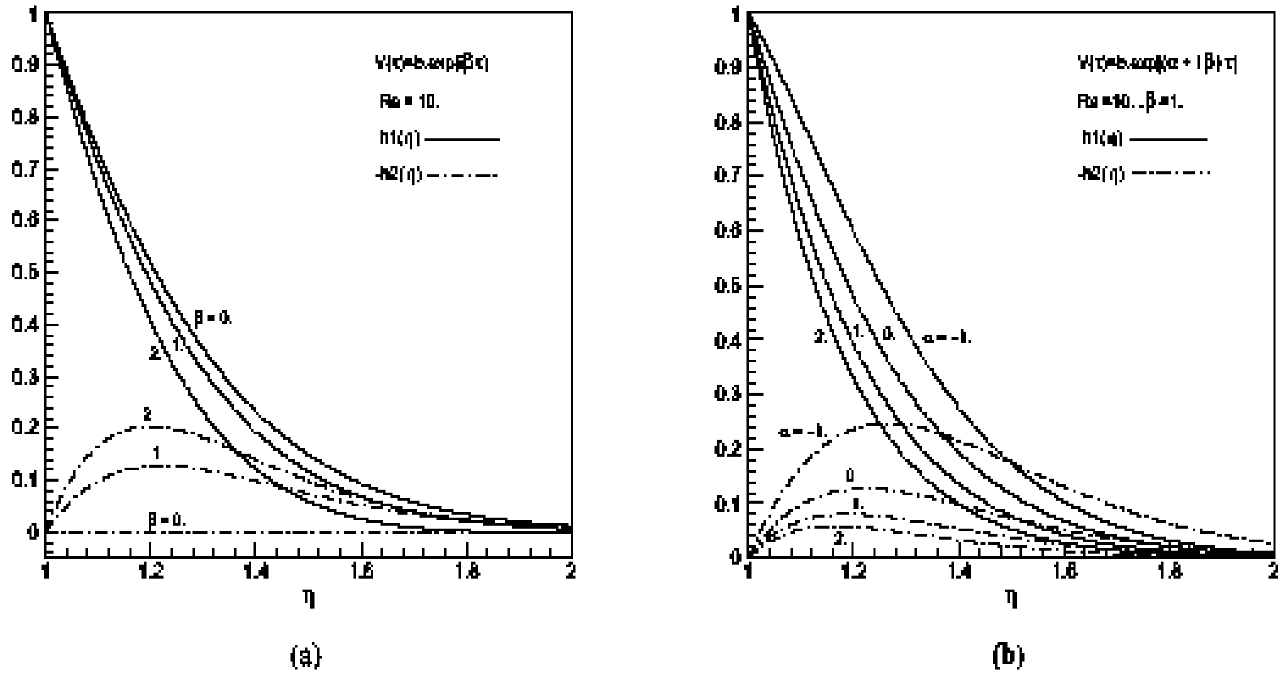

Figure 3. Sample profiles of $h_1(\eta)\,\&\,h_2(\eta)$ for cylinder with (a) Axial harmonic oscillation for $\beta=0,1,2$ (b) accelerating and decelerating oscillatory motion for $\alpha=-1,0,1,2$ and $\beta=1$.

Where, μ is the fluid viscosity. Using Definition 6, the shear stress at the cylinder surface for semi-similar solutions becomes

$$\sigma=\frac{2\mu}{a}\left[2\,k\,f''(1).z+H'(1,\tau)\right] \tag{27}$$

Axial surface shear stress for self-similar solutions is presented by the following form:

$$\sigma=\frac{2\mu}{a}\left\{2k\,f''(1).z+b\,Exp(\alpha\tau)\left[\begin{array}{l}\left\{\begin{array}{l}h_1'(1)cos(\beta\tau)\\-h_2'(1)sin(\beta\tau)\end{array}\right\}+\\ i\left\{\begin{array}{l}h_1'(1)sin(\beta\tau)\\+h_2'(1)cos(\beta\tau)\end{array}\right\}\end{array}\right]\right\} \tag{28}$$

Some numerical values of this component will be presented later for few examples of axial velocities.

6. PRESENTATION OF RESULTS

In this section, the solutions of the self-similar equation (25) and the semi-similar equation (8) along with surface shear stresses for different functions of axial velocity are presented. Also, the axial components of velocity $w(\eta,\tau)$, for some of these cases are given.

Figure 2 presents the sample profiles of $h(\eta)$ for $V(\tau)$ in exponential form for accelerating and decelerating case at Re = 1. It is interesting to note that for $\alpha \succ 0$ as α increases, from Figure 2a, the depth of the diffusion of the fluid velocity field decreases and for $\alpha \prec 0$ as absolute value of α increases, fluid velocity in the vicinity of the cylinder cannot decrease with the same rate as the cylinder axial velocity and therefore in this region, as the Figure 2b shows, the fluid velocity is greater than the cylinder velocity. Also $\alpha=0$ indicates the case of moving cylinder with constant axial velocity [7].

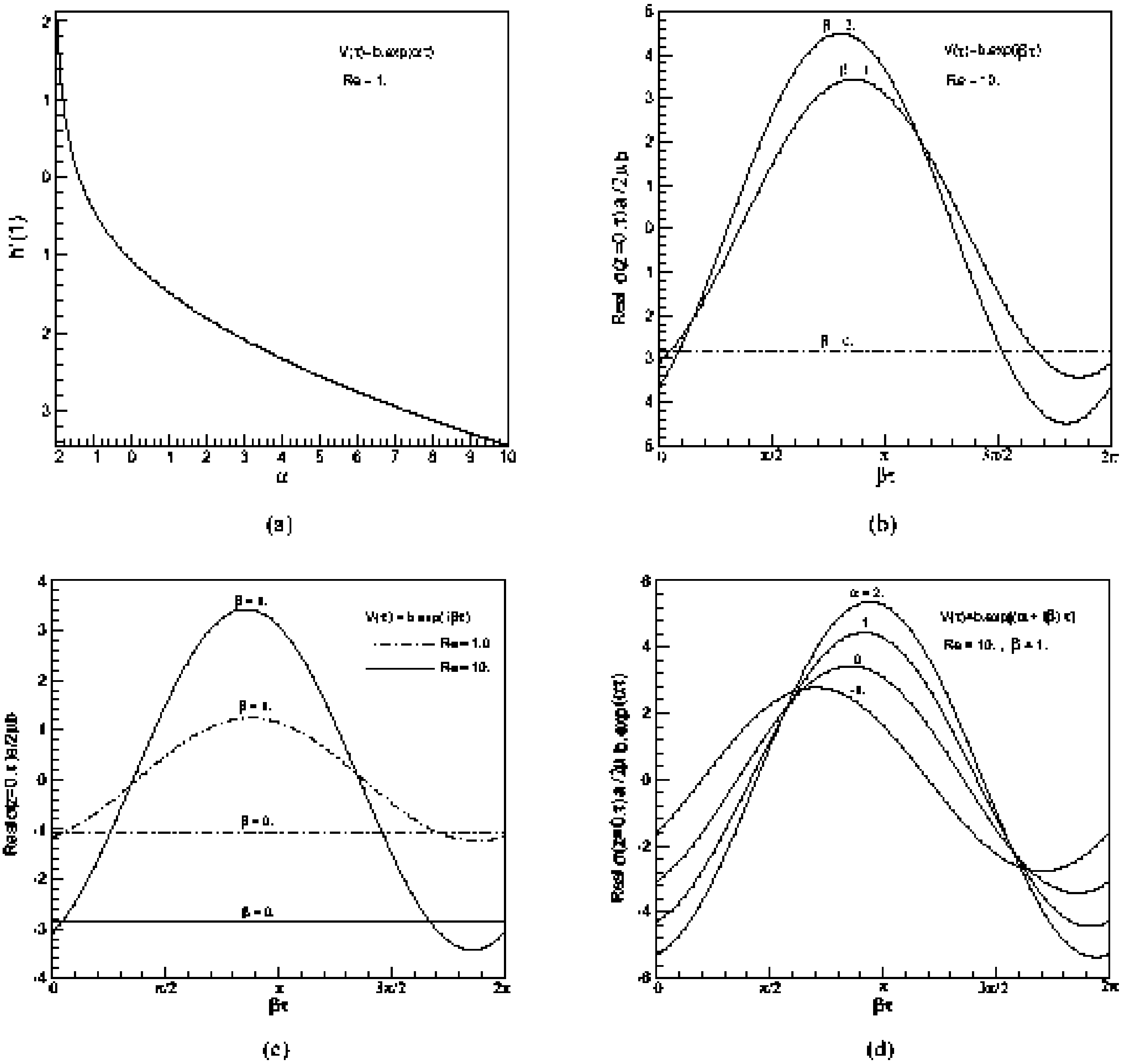

Figure 4. Real Part of Axial Shear Stress at z = 0. For cylinder with (a) Exponential Axial Velocity at Re = 1. (b) and (c) Harmonic Oscillation for $\beta = 0, 1, 2$ at Re = 1, 10. (d) Accelerating and decelerating oscillatory motion for $\alpha = -1, 0, 1, 2$ and $\beta = 1$ at Re = 10.

Figure 3a exhibits the sample $h(\eta)$ solution for pure harmonic motion of the cylinder at different frequencies at Reynolds 10. The case of $\beta = 0$ is the same as in Reference 7 and clearly the imaginary part of $h(\eta)$ is zero. Like the foregoing discussion, as β increases the depth of the diffusion of the fluid velocity field decreases.

Figure 3b presents the sample $h(\eta)$ solution and real part of axial velocity of the cylinder at z = 0 for exponentially oscillating cylinder for different α and β for Re = 10. From this figure it is seen that as α increases the depth of the diffusion of the fluid velocity decreases rapidly.

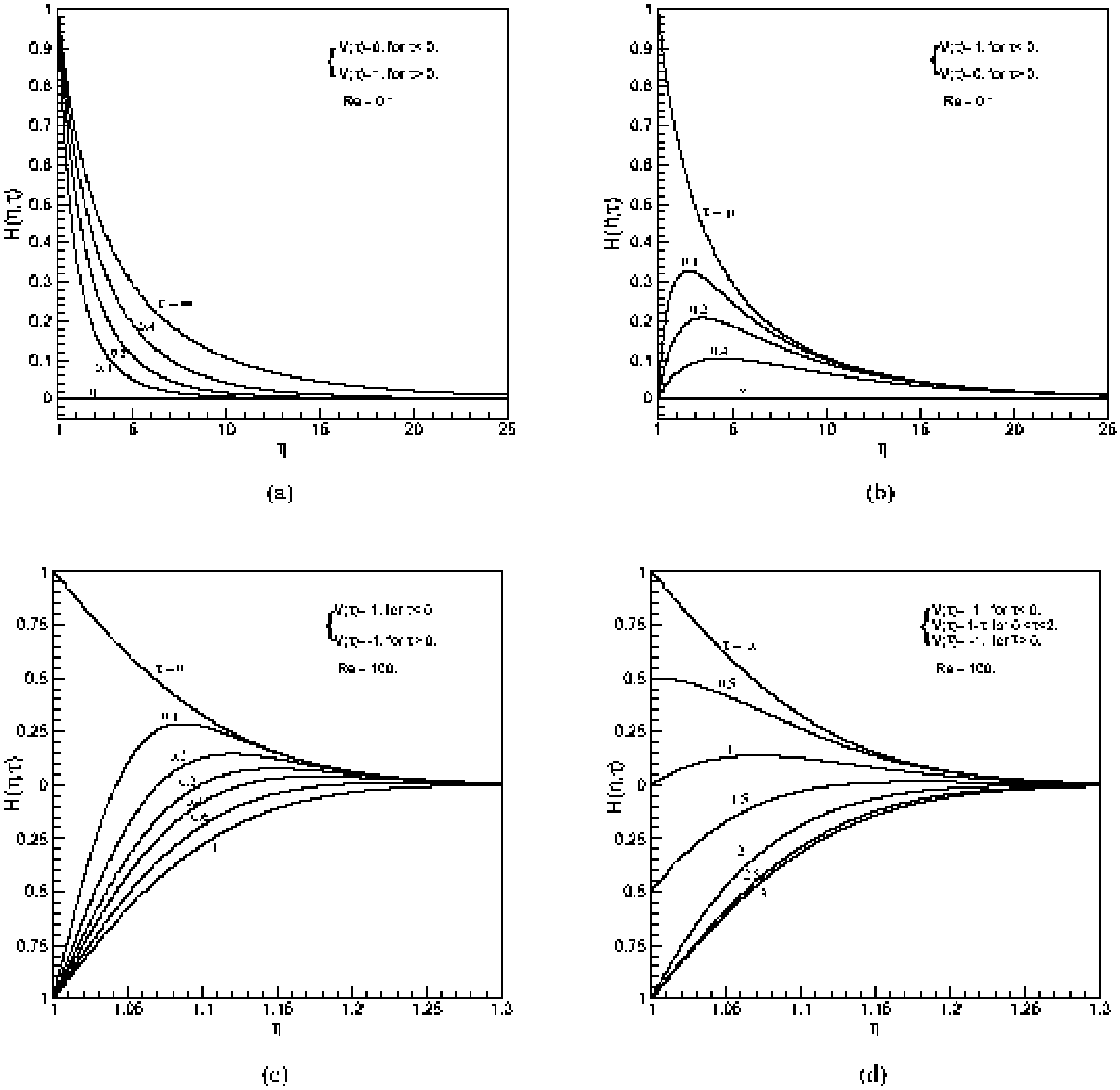

Figure 5. Sample profiles of, $H(\eta,\tau)$ for different axial velocity functions; (a), (b) and (c) for step-functions, (d) for Limited linear function.

Figure (4a) shows the axial shear stress on the surface of the cylinder for exponential axial velocity in terms of accelerating rate at Re=1. It is concluded from this figure that the slope of the curves decreases as α increases, meaning that the sensitivity of shear stress with respect to variation of α decreases as α increases. It is also interesting to note that for particular value of negative α and Reynolds number, the value of shear stress is zero. This interesting result opens the way for an analysis of flows for which a cylinder moving with certain combination of Reynolds number and α is axially stress free.

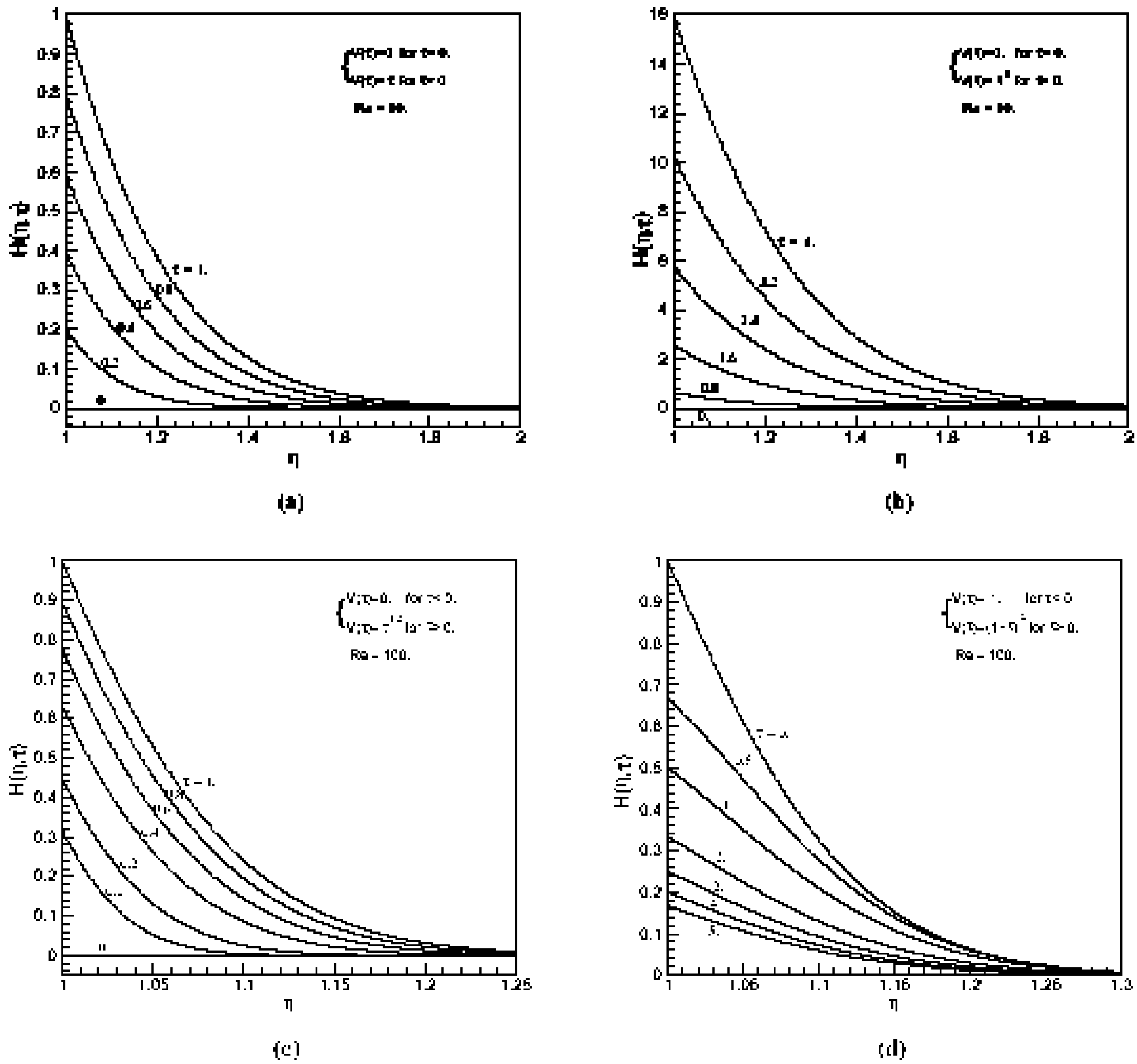

Figure 6. Sample Profiles of, $H(\eta,\tau)$ for Different Axial Velocity Functions: (a) τ *f*, (b), τ^{2} (c), $\tau^{\frac{1}{2}}$ and (d) $\frac{1}{1+\tau}$.

Figure 4 presents the real part of axial shear stress at z = 0 on the surface of the cylinder with pure harmonic and accelerating and decelerating oscillatory motion for Re = 10. This shear stress is for a complete period between 0 and 2π. From Figure 4b as β increases the maximum of the absolute value of shear stress increases. Here, $\beta = 0$ corresponds to the case of constant axial velocity, as in Reference 7. From Figure 4d, when the value of α increases the maximum absolute value of axial shear stress increases. And $\alpha = 0$ corresponds to the shear stress of the pure oscillation case that was obtained by Gorla [9] by using perturbation method and only for very high and very low frequencies. Also from this figure, it can be seen that shear stress and axial velocity have phase differences. Figure 4c shows that the absolute value of surface shear stress increases as Reynolds number increases.

Figure 5 presents the semi-similar solution for different forms of time-dependent axial velocity in which the function $H(\eta,\tau)$ is shown in terms of η and for different non-dimensional time values and Reynolds numbers. The selected functions here are step-function and linear function for axial velocity, which are presented for Reynolds number 0.1 and 100.

Figure 6 shows the semi-similar solution for non-linear forms of time-dependent axial velocity in terms of non-dimensional time values and Reynolds numbers. These selected forms are only for particular non-linear functions and presented for Reynolds numbers 10. And 100. Other velocity functions for particular usage can be selected.

7. CONCLUSIONS

An exact solution of the Navier-Stokes equations is obtained for the problem of stagnation-point flow on a circular cylinder. A general self-similar solution is obtained when the cylinder has different forms of axial motions including: constant axial velocity, exponential axial velocity, and pure harmonic movement, both accelerating and decelerating oscillatory motion. Also, some sample semi-similar solutions for the same problem have been considered when the circular cylinder is moving with different types of time-dependent axial velocity.

Axial component of fluid velocity and surface axial shear stress on the cylinder are obtained in all the above situations, and for different values of Reynolds numbers and transportation rate. Absolute value of axial shear stresses corresponding to all the cases increase with the increase of Reynolds number and suction rate. Also, the maximum value of shear stress increases with increasing oscillation frequency and accelerating and decelerating parameter in the exponential amplitude function. It is also shown that a cylinder moving axially in an exponential manner is axially stress-free for certain combinations of Reynolds number and rate of this exponential function.

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