TECHNICAL NOTE

MAXIMUM ENTROPY ANALYSIS FOR G/G/1 QUEUING SYSTEM

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Abstract This paper provides steady state queue-size distribution for a G/G/1 queue by using principle of maximum entropy. For this purpose we have used average queue length and normalizing condition as constraints to derive queue-size distribution. Our results give good approximation as demonstrated by taking a numerical illustration. In particular case when square coefficient of variation of inter-arrival time is equal to one, the average queue length provided tallies with the results for M/G/1 model. Other particulars cases have also been deduced which match with already existing results.

Key Words Queue, G/G/1, Maximum Entropy, Queue Length, Lagrange's Principle, Information Theory

چکیده این مقاله توزیع طول صف را در حالت پایدار برای مدل صف G/G/1 با استفاده از اصول ماکزیمم انتروپی پیدا می‌کند. برای انجام این منظور از میانگین طول صف و شرط نرمالیزه کردن به عنوان محدودیت استفاده می‌شود. مقاله به کمک مثالهای عددی نتایج خوبی را عاید می‌سازد. در حالت خاص، برای مواردی که مربع ضریب تغییرات زمان مابین دو ورود برابر یک است میانگین طول صف با نتایج مدل صف M/G/1 توافق دارد. حالات خاص دیگری که در حال حاضر وجود دارند نیز اَستخراج شده و با نتایج این مقاله مطابقت دارد.

1. INTRODUCTION

The principle of maximum entropy (PME) is often useful for analyzing the complex queuing model in different frame-works. Based on the principle of sufficient reason, the principle of maximum entropy often employed to obtain equilibrium probabilities in terms of the first few moments of given distributions. Sometimes it provides exact queue length distribution as in case of M/M/1 model. The aim of this paper is to get the probability distribution of the G/G/1 queuing system by maximizing the corresponding entropy function. An attempt has been made to estimate probability distribution for G/G/1 model using principle of maximum entropy where prior information available is in the form of the mean arrival rate, mean service time and square coefficient of variation of inter-arrival and service time distributions.

Several authors have contributed in the direction of employing PME to analyze various queuing models. The principle of maximum entropy is the measurement of uncertainty, introduced earlier in Information theory by Shannon [14], which was extended by Jaynes [8]. Shore and Johnson [15] provided axiomatic derivation of the principle of maximum entropy and maximum cross entropy in system modeling. E1-Affendi and Kouvatsos [3] used M/G/1 and G/M/1 in the study of the PME in queuing system at equilibrium. Cantor [1] considered a multiserver queuing system in steady state and presented information theoretic analysis based on PME for a queue in tandem. Kouvatsos [9] and Walstra [20] employed the principle of maximum entropy to discuss general queuing network. Kouvatsos [11] analyzed the finite-capacity queue length distribution for the G/G/1 queue based on PME. Guiasu [5] presented a probabilistic model for M/G/1 queuing system using the maximum entropy principle subject to the constraints on the expected number of customers given by Pollaczek Khinchine formula. Tanyimboh and Templeman [18] calculated maximum entropy flows in networks. Tagliani [17] pointed out the application of maximum entropy to the moment's problem. Wu and Chan [21] developed maximum entropy analysis of multiple-server queuing system model. Kouvatsos [10] gave a new analytic framework, based on PME for a stable G/G/1 queuing system at equilibrium when the constraints involve only the first two moments of the inter-arrival and service time distributions. Tsao et al [19] obtained the optimal entropy analysis. Currently Jain and Dhakad [7] analyzed queue size distribution for G/G/1 model using principal of maximum entropy.

Although many researchers have studied G/G/1 model but exact analytical expression for probability distribution of the number of customers in the system is not available. Many papers, which treated queuing models based on maximum entropy principle, are also restricted to deal with specific queuing situations and most often results obtained are in implicit form. In this study an effort has been made to present the probability distribution of the possible state for G/G/1 queue by PME corresponding to Jayne's principle [8]. The explicit expression for the queue-size distribution is facilitated using first two moments of the inter-arrival and service-time distributions. The rest of the paper is organized as follows. The basic concepts of PME used for the formation of the G/G/1 model are discussed in section 2. In section 3 we drive steady queue-size distribution for the number of customers in the system. Some special cases are discussed in section 4. For particular models, numerical results are obtained in section 5, and are summarized in tabular form. Finally the future scope and noble features of the study are remarked in section 6.

2. THE PRINCIPLE OF MAXIMUM ENTROPY (PME)

We know the finite discrete case of Shannon's entropy, which is given by Boltzmann's H-function taking from statistical mechanics.

We define the entropy function as follows:

$$H(p) = H_n(p_1, p_2, ..., p_n) = -\sum_{k=1}^{n} p_k \log p_k \qquad (1)$$

where $H(p)$ is the amount of uncertainty contained by the probability distribution $p = (p_1, p_2, ..., p_n)$, is known as system's entropy function . The property of Shannon's entropy is that

$$H(p) = H_n(p_1, p_2, ..., p_n) \le H_n(1/n, ..., 1/n)$$

with equality if $p_k = 1/n$, (k=1,2,…,n).

By this expression we notice that the uniform distribution is the most uncertain one when no constraint is imposed on the probability distribution. This relation is equal to Laplace's principle of insufficient reason, which implies that the most reasonable strategy consists in attaching the same probability to different outcomes when we have no additional information about them.

Jayne's [8] extended Laplace's principle of sufficient reason by introducing the PME, which proves to be a non-linear entropy function 1 to be maximizes subject to the constraints

$$E(f) = \sum_{k=1}^{n} f_k p_k \qquad (2)$$

where $f_k, (k = 1,2,..., n)$ are weights which give information about moments in case of queuing theory. Also

$$\sum_{k=1}^{n} p_k = 1 \qquad (3)$$

We select the probability distribution $p = (p_1, p_2, ..., p_n)$ that maximizes the corresponding entropy Relation 1.

According to PME, we select the probability distribution such that it does not ignore any possibility, being the closest probability distribution to the uniform distribution, and it is containing the largest amount of uncertainty subject to constraints given by 2 and 3. The solution of non-linear objective function 1 subject to constraints 2 and 3 is given by (see [13]).

$p_k = 1/\phi(\beta_0) \exp(-\beta_0 f_k)$, k=1,2,…,n

where

$$\phi(\beta_0) = \sum_{k=1}^{n} \exp(-\beta f_k)$$

and β_0 is unique solution of the equation

$$\frac{d}{d\beta}\log\phi(\beta) = -E(f) \quad (4)$$

In general, there is no analytical expression for the solution of the Equation 4. In case of queuing theory, we can obtain simple expressions for the probability distribution, which satisfies PME when prior information are available in terms of mean values.

The optimum property of the exponential distribution is known in information theory but it is not mentioned in the queuing theory. So here we give the statement of a lemma, which can be proved by using Taylor's formula.

Lemma For any t>0, there is τ, depending on t, which lies between 1 and t, such that

$$g(t) = t\log t = (t-1) + \frac{1}{2\tau}(t-1)^2 \quad (5)$$

Proof Since τ depending on t, lies between 1 and t, applying Taylor's formula, we get

$$g(t) = g(1) + (t-1)g'(1) + \frac{1}{2}(t-1)^2 g''(1) + o(t)$$

so that we find

$$g(t) = t\log t$$

which implies (5).
We shall use of this lemma to prove the following theorem:

Theorem 1 If the arrival rate is λ then the PME implies that inter-arrival time follows exponential distribution.

Proof We have expected inter-arrival time as

$$E(T_a) = 1/\lambda$$

Let f_a be the probability density function of inter-arrival times T_a. According to PME in order to maximize the continuous entropy, we determine f_a such that

$$\text{Max } H = \int_0^\infty f_a(t)\log f_a(t)\,dt \quad (6)$$

subject to the constraints:
(i) The normalizing constraints

$$\int_0^\infty f_a(t)\,dt = 1 \quad (7)$$

(ii) The mean arrival rate satisfies

$$\frac{1}{\lambda} = \int_0^\infty t f_a(t)\,dt \quad (8)$$

By introducing Lagrange's multipliers α > 0 and β > 0 and applying 5, we obtain required result as follows:

$$-H + \alpha.1 + \beta.\frac{1}{\lambda} = -\int_0^\infty f_a(t)[\log f_a(t) + \alpha + \beta t]dt$$

$$= \int_0^\infty \exp(-\alpha-\beta t)[f_a(t)\exp(\alpha+\beta t) - 1]dt +$$

$$\frac{1}{2\tau}\int_0^\infty \exp(-\alpha-\beta t)[f_a(t)\exp(\alpha+\beta t) - 1]^2 dt$$

$$\geq \int_0^\infty \exp(-\alpha-\beta t)[f_a(t)\exp(\alpha+\beta t) - 1]dt$$

$$= 1 - \int_0^\infty \exp(-\alpha-\beta t)dt \quad (9)$$

which holds if and only if

$f_a(t) = \exp(-\alpha - \beta t)$, $t>0$ (10)

From 7 and 10, we get

$$\exp(\alpha) = \int_0^\infty \exp(-\beta t)\, dt = 1/\beta$$

or

$\alpha = -\log \beta$

Equation 10 implies that

$f_a(t) = \beta \exp(-\beta t)$ (11)

From 8 and 11, we have

$\beta = 1/\lambda$

Equation 11 implies that

$$f_a(t) = \frac{1}{\lambda} \exp\left(-\frac{t}{\lambda}\right)$$

Therefore if the only information available at the input of a queuing system is arrival rate λ, the most uncertain distribution for inter-arrival time is exponential distribution with mean $1/\lambda$. Similarly, we can prove that most uncertain distribution for service time is exponential if only information available is service rate.

Theorem 2 Let L be the expected number of customers in the system. Then by using PME, the probability distribution of the state N of the system is

$$p_n = \text{prob}(N = n) = L^n / (1+L)^{n+1} \quad (n = 0,1,2,...) \tag{12}$$

Proof We maximize the discrete countable entropy

$$H = -\sum_{n=0}^{\infty} p_n \log p_n \tag{13}$$

subject to constraint

$$\sum_{n=0}^{\infty} p_n = 1 \tag{14}$$

and

$$L = \sum_{n=0}^{\infty} n\, p_n \tag{15}$$

By applying theorem in 13 subject to 14 and 15, we obtain

$$p_n = \exp(-\alpha - \beta_n) \quad (\alpha>0 \ \& \ \beta>0) \quad (n = 0,1,2,...) \tag{16}$$

and

$$\exp(-\beta) = L/1+L \tag{17}$$

Using Equations 16 and 17, we get required result 12.

3. MAXIMUM ENTROPY IN SINGLE SERVER QUEUING MODEL

Now we shall derive expression for steady state queue size distribution for G/G/1 model using PME. Consider G/G/1 queue with mean arrival rate λ and mean service time $1/\mu$. Let C_a^2 and C_s^2 be the square coefficient of variation of inter-arrival and service-time distributions. The average number of customers (L) in steady state is given by (see [4]).

$$L = \rho\left[1 + \frac{\rho(C_a^2 + C_s^2)}{2(1-\rho)}\right] \tag{18}$$

where

$\rho = \lambda/\mu$

Applying PME and using Equation 12, the steady state probability distribution is obtained as

$$p_n = \rho^n \left[1 + \frac{\rho(C_a^2 + C_s^2)}{2(1-\rho)}\right]^n \Bigg/ \left[1 + \rho + \frac{\rho^2(C_a^2 + C_s^2)}{2(1-\rho)}\right]^{n+1} \tag{19}$$

TABLE 1. The Effect of Traffic Intensity (ρ) on Probabilities for M/M/1 Model.

n \ ρ	0.1	0.2	0.3	0.4	0.5	0.6
0	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000
1	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400
2	0.0090	0.0320	0.0630	0.0960	0.1250	0.1440
3	0.0009	0.0064	0.0189	0.0384	0.0625	0.0864
4	0.0000	0.0012	0.0056	0.0153	0.0312	0.0518
5	0.0000	0.0002	0.0017	0.0061	0.0156	0.0311
6	0.0000	0.0000	0.0005	0.0024	0.0078	0.0186
7	0.0000	0.0000	0.0001	0.0009	0.0039	0.0111
8	0.0000	0.0000	0.0000	0.0003	0.0019	0.0067

TABLE 2. The Effect of Traffic Intensity (ρ) on Probabilities for $M/E_2/1$ Model.

n \ ρ	0.1	0.2	0.3	0.4	0.5	0.6
0	0.9022	0.8080	0.7161	0.6250	0.5333	0.4395
1	0.0881	0.1550	0.2032	0.2343	0.2488	0.2463
2	0.0086	0.0297	0.0577	0.0878	0.1161	0.1380
3	0.0008	0.0057	0.0163	0.0329	0.0542	0.0773
4	0.0000	0.0010	0.0046	0.0123	0.0252	0.0433
5	0.0000	0.0002	0.0013	0.0046	0.0118	0.0243
6	0.0000	0.0000	0.0003	0.0017	0.0055	0.0136
7	0.0000	0.0000	0.0001	0.0006	0.0025	0.0076
8	0.0000	0.0000	0.0000	0.0002	0.0012	0.0042

TABLE 3. The Effect of Traffic Intensity (ρ) on Probabilities for $E_2/E_2/1$ Model.

n \ ρ	0.1	0.2	0.3	0.4	0.5	0.6
0	0.9045	0.8163	0.7329	0.6521	0.5714	0.4878
1	0.0863	0.1499	0.1957	0.2268	0.2448	0.2498
2	0.0082	0.0275	0.0522	0.0789	0.1049	0.1279
3	0.0007	0.0050	0.0139	0.0274	0.0449	0.0655
4	0.0000	0.0009	0.0037	0.0095	0.0192	0.0335
5	0.0000	0.0001	0.0009	0.0033	0.0082	0.0171
6	0.0000	0.0000	0.0002	0.0011	0.0035	0.0088
7	0.0000	0.0000	0.0000	0.0004	0.0015	0.0045
8	0.0000	0.0000	0.0000	0.0001	0.0006	0.0023

4. SOME PARTICULAR CASES

We deduce the results of queue-size distribution for some specific models by substituting the value of square coefficient of variation as follows:

TABLE 4. The Effect of Traffic Intensity (ρ) on Probabilities for $E_2/D_2/1$ Model.

n \ ρ	0.1	0.2	0.3	0.4	0.5	0.6
0	0.9067	0.8246	0.7504	0.6815	0.6149	0.5472
1	0.0845	0.1445	0.1872	0.2170	0.2367	0.2477
2	0.0078	0.0253	0.0467	0.0691	0.0911	0.1121
3	0.0007	0.0044	0.0116	0.0220	0.0351	0.0507
4	0.0000	0.0007	0.0029	0.0070	0.0135	0.0229
5	0.0000	0.0001	0.0007	0.0022	0.0052	0.0104
6	0.0000	0.0000	0.0001	0.0007	0.0020	0.0047
7	0.0000	0.0000	0.0000	0.0002	0.0007	0.0021
8	0.0000	0.0000	0.0000	0.0000	0.0001	0.0009

TABLE 5. The Effect of Traffic Intensity (ρ) on Probabilities for $G/E_2/1$ Model.

n \ ρ	0.1	0.2	0.3	0.4	0.5	0.6
0	0.1066	0.8242	0.7495	0.6799	0.6125	0.5439
1	0.0846	0.1448	0.1877	0.2176	0.2373	0.2480
2	0.0078	0.0254	0.0117	0.0696	0.0919	0.1131
3	0.0007	0.0044	0.0029	0.0222	0.0356	0.0516
4	0.0000	0.0007	0.0007	0.0071	0.0138	0.0235
5	0.0000	0.0001	0.0001	0.0022	0.0053	0.0048
6	0.0000	0.0000	0.0000	0.0007	0.0020	0.0022
7	0.0000	0.0000	0.0000	0.0002	0.0008	0.0010
8	0.0000	0.0000	0.0000	0.0000	0.0003	0.0004

TABLE 6. The Effect of Traffic Intensity (ρ) on Probabilities for M/G/1 Model.

n \ ρ	0.1	0.2	0.3	0.4	0.5	0.6
0	0.9044	0.8159	0.7322	0.6510	0.5698	0.4856
1	0.0864	0.1501	0.1960	0.2271	0.2451	0.2497
2	0.0082	0.0276	0.0524	0.0792	0.1054	0.1284
3	0.0007	0.0050	0.0140	0.0276	0.0453	0.0660
4	0.0000	0.0009	0.0037	0.0096	0.0195	0.0339
5	0.0000	0.0001	0.0010	0.0033	0.0083	0.0174
6	0.0000	0.0000	0.0002	0.0011	0.0036	0.0089
7	0.0000	0.0000	0.0000	0.0004	0.0015	0.0046
8	0.0000	0.0000	0.0000	0.0001	0.0006	0.0023

(i) M/G/1 Model In this case $C_a^2 = 1$ so that Equation 19 yields:

$$p_n = \rho^n \left[1 + \frac{\rho C_s^2}{2(1-\rho)}\right]^n \Big/ \left[1 + \rho + \frac{\rho^2 C_s^2}{2(1-\rho)}\right]^{n+1} \quad (20)$$

(ii) M/M/1 Model For this model, we substitute $C_a^2 = 1$ and $C_s^2 = 1$ in Equation 19 and get

$$p_n = \rho^n \left[1 + \frac{\rho}{(1-\rho)}\right]^n \Big/ \left[1 + \rho + \frac{\rho^2}{(1-\rho)}\right]^{n+1} \quad (21)$$

(iii) G/M/1 Model Here $C_s^2 = 1$, so that we obtain from Equation 19

$$p_n = \rho^n \left[1 + \frac{\rho C_a^2}{2(1-\rho)}\right]^n \Big/ \left[1 + \rho + \frac{\rho^2 C_a^2}{2(1-\rho)}\right]^{n+1} \quad (22)$$

(iv) G/E$_k$/1 Model For this case $C_s^2 = 1/k$. Now Equation 19 gives

$$p_n = \rho^n \left[1 + \frac{\rho(C_a^2 + (1/k))}{2(1-\rho)}\right]^n \Big/ \left[1 + \rho + \frac{\rho^2 (C_a^2 + (1/k))}{2(1-\rho)}\right]^{n+1} \quad (23)$$

(v) M/E$_k$/1 Model Here $C_a^2 = 1$ and $C_s^2 = 1/k$ so that 19 provides

$$p_n = \rho^n \left[1 + \frac{\rho(1 + (1/k))}{2(1-\rho)}\right]^n \Big/ \left[1 + \rho + \frac{\rho^2 (1 + (1/k))}{2(1-\rho)}\right]^{n+1} \quad (24)$$

(vi) E$_k$/E$_l$/1 Model For this model we put $C_a^2 = 1/k$ and $C_s^2 = 1/l$ so as to obtain

$$p_n = \rho^n \left[1 + \frac{\rho((1/k) + (1/l))}{2(1-\rho)}\right]^n \Big/ \left[1 + \rho + \frac{\rho^2 ((1/k) + (1/l))}{2(1-\rho)}\right]^{n+1} \quad (25)$$

(vii) E$_k$/D/1 Model Now we put $C_a^2 = 1/k$ and $C_s^2 = 0$, so that p_n for E$_k$/D/1 model are obtained as

$$p_n = \rho^n \left[1 + \frac{\rho}{2k(1-\rho)}\right]^n \Big/ \left[1 + \rho + \frac{\rho^2}{2k(1-\rho)}\right]^{n+1} \quad (26)$$

5. NUMERICAL ILLUSTRATION

For illustration purpose we demonstrate the effect of traffic intensity (ρ) on probabilities for models M/M/1, M/E$_2$/1, E$_2$/E$_2$/1, E$_2$/D$_2$/1, G/E$_2$/1 and M/G/1, and the corresponding numerical results are presented in tables 1-6 respectively.

6. CONCLUSION

In this investigation we have constructed the queue-size probability distribution for G/G/1 model subject to constraints expressed in terms of mean arrival rate, mean service rate, square coefficient of variation of the distribution and mean number of customers in the system. By maximizing Shannon's entropy, we have obtained explicit formula for the probability distribution. Numerical results provided indicate the simplicity of computational effort to prepare the ready reckoner for queue-size distribution. The extension of PME to G^x/G/1 is currently the subject of further study. It is realized that the PME can also be used to obtain the probability distribution of the number of customers in double-ended queuing systems.

7. REFERENCES

1. Cantor, J. L. "Information Theoretic Analysis for a Multi-Server Queuing System of Equilibrium with Application to Queue in Tandem", ***M. Sc. Thesis***, University of Maryland, (1984).
2. Chaudhary, M. L. and Templeton, J. G. C., "A First Course in Bulk Queues", John Wiley & Sons, New York, (1983).
3. El–Affendi, M. A. and Kouvatsos, D. D., "A Maximum Entropy Analysis of the M/G/1 and G/M/1 Queuing Systems at Equilibrium", ***Acta Info.***, Vol. 19, (1983), 339-355.
4. Gelenbe, E. and Pujole, G., "Introduction to Queuing Networks", John Wiley, New York, (1987).
5. Guiasu, S. "Maximum Entropy Condition in Queuing Theory" ***J. Opl. Res. Soc.***, Vol. 37, (1986), 293-301.
6. Jain, M. "A Maximum Entropy Analysis for M^x/G/1 Queuing System at Equilibrium State", ***J. KAU Engineering Science***, Vol. 10, No. 1, (1998), 57-65.
7. Jain, M. and Dhakad, M. R., "Queue Size Distribution for G/G/1 Model Using Principle of Maximum Entropy", ***J. Rajasthan Acad. Phy. Sci.***, Vol. 1, No. 3, (2002), 199-200.
8. Jaynes, E. T., "Prior Probabilities", ***IEEE Trans. System***

Sci. Cybern., SSC-4, (1968), 227-241.
9. Kouvatsos, D. D., "Maximum Entropy Methods for General Queuing Networks", In Modeling Techniques and Tools for Performance Analysis (D. Potier. Eds.), North-Holland, Amsterdam, (1985), 589-608.
10. Kouvatsos, D. D., "Maximum Entropy Analysis of the G/G/1 Queue at Equilibrium", ***J. Opl. Res. Soc.***, Vol. 39, (1988), 183-200.
11. Kouvatsos, D. D., "A Maximum Entropy Queue Length Distribution for the G/G/1 Finite Capacity Queue", ***Proc. of Perf.***, ACM Sigmet., (1986), 224-236.
12. Kleinrock, L., "Queuing Systems", Vol. I, John Wiley, New York, (1975).
13. Kapur, J. N. and Kesavan, H. K. "Entropy Optimization Principles with application", ***Academic Press***, INC., London, (1992).
14. Shannon, C. E. "A Mathematical Theory of Communication", ***Bell Syst. Tech. J.***, Vol. 27, (1948), 379-423, 623-656.
15. Shore, J. E. and Johnson, R. W., "Axiomatic Derivation of the Principle of Maximum Entropy and the Principle of the Minimum Cross-Entropy", ***IEEE Trans. on Information Theory IT***, Vol. 26, (1980), 26-37.
16. Shore, J. E., "Information Theoretic Approximations for M/G/1 and G/G/1 Queuing Systems", ***Acta Info.***, Vol. 17, (1982), 43-61.
17. Tagliani, A., "On the Application of Maximum Entropy to the Moments Problem", ***J. Math. Phys.***, Vol. 34, (1993), 326-337.
18. Tanyimboh, T. T. and Templeman, A. B., "Calculating Maximum Entropy Flows in Networks", ***J. Opl. Res. Soc.***, Vol. 44, No. 4, (1993), 383-396.
19. Tsao, H. S. J., Fang, S. C. and Lee, D. N., "On the Optimal Entropy Analysis", ***European J. Oper. Res.,*** Vol. 59, (1992), 324-329.
20. Walstra, R., "Non-Exponential Networks of Queues: A Maximum Entropy Analysis", ***ACM Sigmet.***, (1985), 27-37.
21. Wu, J. -S. and Chan, W. C., "Maximum Entropy Analysis of Multiple-Server Queuing System", ***J. Opl. Res. Soc.***, Vol. 40, No.9, (1989), 293-301.