SEEPAGE WITH NONLINEAR PERMEABILITY BY LEAST SQUARE FEM

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Abstract In seepage problems, the coefficients of permeability in Laplace equation are usually assumed to be constant vs. both space and time; but in reality these coefficients are variable. In this study, the effect of material deformation due to external loads (consolidation) and variation of head in the consolidation process are considered. For the first case, formulation of k_x and k_y can be defined by a second order binominal equation in order to take into account the material changes due to volume changes. For the second case, k_x and k_y can be defined as a function of unknown total head. The solution of the resulting non-linear differential equation is found using the Least Square Finite Element formulation. In order to increase the accuracy of the solution, eight nodal (isoperimetric) elements were obtained. This method was used satisfactorily to solve several seepage problems and to examine the accuracy and convergence of the results. The effect of a variable coefficient of permeability may not be significant on small dams, but as the height of the dam increases, the effect becomes more considerable. It is believed that a variable permeability analysis such as the one described in this paper should be taken into account.

Key Words Dams, Numerical Modeling and Analysis, Permeability, Seepage, Consolidation, Anisotropy

چکیده برای سادگی در مسائل تراوش، ضرایب نفوذ در معادله لاپلاس معمولا نسبت به زمان و مکان ثابت فرض می‌شوند. اما در واقع این ضرایب متغیرند. در این مقاله اثرات بارهای خارجی و تغییر بار آبی در شکل تغییر مصالح، بمانند تحکیم مورد توجه قرار می‌گیرد. در حالت اول برای تعریف تغییر حجم مصالح در سیستم، می‌توان تاثیر آن را بوسیله یک معادله درجه دوم معرفی نمود. در حالت دوم معادله ضریب نفوذ پذیری می‌تواند به عنوان تابع نامعلومی از بار کلی معرفی شود. معادله دیفرانسیل غیرخطی بدست آمده با استفاده از روش حداقل مربعات المانهای محدود قابل حل می‌باشد. به منظور افزایش دقت حل مساله، از المان‌های هشت گرهی استفاده می‌شود. از این روش برای حل چندین مساله تراوش به صورت رضایت بخشی استفاده گردیده و دقت و همگرایی نیز مورد بررسی قرار گرفته است. اثر تغییر ضریب نفوذ پذیری در سدهای کوتاه قابل چشم پوشی است؛ ولی در سدهای بلند این اثر اهمیت یافته و بایستی مورد توجه واقع شود. بر این باوریم که تغییر نفوذ پذیری همانطور که در این مقاله به‌طور کامل مورد بررسی قرار گرفته می‌تواند در مسائل تراوش در نظر گرفته شود.

1. INTRODUCTION

In most geotechnical analyses, soil properties are assumed to be spatially and temporally invariant and thus, average property values are used. In reality, however, these soil parameters usually vary from point to point (heterogeneous) and even at one point they may have different values in various measured directions (anisotropy). Moreover, these parameters may vary in time while a geotechnical process is occurring due to an external influence such as surface pressure or due to the change of chemical compositions. For computations in flow problems using numerical techniques usually homogeneous conditions are assumed for the coefficient of permeabilities and anisotropic conditions are assumed throughout. In this research, the coefficients of permeability are assumed to vary in term of geometry, external load influences such as those causing consolidation effects, and the effect of head variation in the system where seepage is taking place. In order to define these variations two conditions are presented in this paper. The first condition can be explained for example by applying an embankment load over a confined saturated fine grain soil layer. This load would begin to consolidate underlying materials. At the end of consolidation process the permeability of the materials are changed and can be described by a governing differential equation, which can then be solved. In addition to the first case, a second case can be defined in which variation of the head can also have an effect on the consolidation process resulting in permeability variations. This effect simply can be seen in Terzaghi's effective stress equation. The influence of head variation is introduced by a defined function, which can be solved numerically. This changes the governing differential equation to a non-linear one, where one of the parameters (head), which define the coefficients of the governing differential equation, is unknown. A numerical solution is required in such cases.

The finite element (FE) method is a very powerful tool to solve many sophisticated engineering problems. FE analysis has been implemented in a number of areas in engineering such as solid mechanics, heat transfer and hydrodynamics as well as geotechnical interests such as Desai and Christian [1] for general geotechnical uses, Beacher and Ingra [2] and Righetti and Harrop-Williams [3] for stress analysis and Finn [4] and Smith and Freeze [5,6] and Griffiths and Fenton [7] for seepage analysis.

General finite element formulations, such as the Variational or Weighted residual processes methods used by Zienkiewicz [8] cannot be employed to solve a non-linear equations such as Navier Stokes, Burgers or Laplace equations. The Galerkin and least square methods are an extension of the Weighted Residual method. Using the Galerkin method for the solution of the Navier-Stokes equation has many associated difficulties such as (a) the coefficient matrix is not symmetric and in the pressure variation direction in the continuity equation would perform as ill-conditioned, and (b) convergence of this system in non-linear problems is very slow and sometimes may come up with some difficulties with iterations.

In the present study in order to solve the non-linear governing differential equation the Least Square Finite Element Formulation (LSFEF) was utilized. This method has recently been used by researchers such as Zienkiewicz et al. [9], Lynn and Arya [10], and Winterscheidt and Surana [11,12] in many areas such as solution of partial and hyperbolic differential equations or boundary layer flow, gas dynamics, and compressible fluid and gas problems. LSFEF method was used based on the minimizing of the error function in differential equations with non-linear partial differentiation.

2. VARIABILITY OF THE COEFFICIENT OF PERMEABILITY

In flow problems, both the magnitude and direction of governing fluid flows are highly sensitive to the coefficient of the permeability. For simplicity, this parameter is usually assumed to be a constant in space and time. In this study, the coefficient of permeability is assumed to be spatially variable. The variation of coefficient of permeability was defined for different cases, and then the resulted governing differential equation was solved. In order to define a function for the variation of the permeability two conditions were proposed.

First Condition This is a simple condition where the coefficient of the permeability is a function of material properties and geometrical conditions. From most classical soil mechanics literature it is well known that coefficient of permeability is directly proportional to the void ratio of the soil. As the void ratio increases or decreases, so does the coefficient of permeability, Lambe and Whitman, [13]. Only confined flow was considered here. As an example, one can consider construction of an embankment dam over a saturated fine grain soil. As the construction starts, the consolidation of the material beneath the embankment will begin. Due to non-uniformity of the applied load, the consolidation of the materials under the embankments will vary, which will result in void ratios that vary in space and time. This would therefore introduce variation of coefficient of permeability at different locations and directions under the embankments. Generally, these coefficients of permeability are a minimum at the centerline of the embankments and increase as the distance from the centerline increases. These coefficients of permeability would also be time dependent as long as the consolidation process is occurring. In order to define good estimates for coefficient of permeability in flow problems for any given point, mainly dependent upon soil type, fabric and structure, and consolidation stage one should undertake laboratory testing to define the equations for k_x and k_y, the coefficients of permeability in x and y direction, respectively. The variation of k_x in horizontal direction can be simply expressed by any order binominal equation, which in this study was considered to be second order.

$$k_x = a_x x^2 + b_x x + c_x \tag{1}$$

Where a_x, b_x, and c_x are the coefficients that can be determined from a curve fitting procedure based on the results from laboratory and field-testing.

Similarly k_y the coefficient of permeability in vertical direction can be expressed by a similar second order binominal equation of the form:

$$k_y = a_y y^2 + b_y y + c_y \tag{2}$$

where a_y, b_y, and c_y are the coefficient which can be determined from curve fitting procedure based on the results from laboratory testing. Generally k_y in vertical direction can vary by either the effect of overburden pressure of the natural soils or the influence of excess stresses due to an embankment load. For the first case, as the overburden pressure increases with depth, there would be a tendency for the material to become more compacted, therefore reducing k_y with depth. For the second case, as the depth increases the effect of embankment load decreases i.e. less consolidation, and thus k_y increases. The effect of the second imposed condition is opposite to the first case, and these physical effects with depth should be superimposed in order to define Equation 2 for every starting point at interface of embankment and natural soil in the vertical direction.

Second Condition: In this condition, the coefficient of permeability, in addition to the first case, can be affected by the variation of heads in the upstream, downstream, or in the soil. In the next section the relationship between hydraulic head and the coefficient of permeability is described.

Relationship Between Effective Stress and Soil Void Ratio In the soil consolidation process, the relationship between effective stress and void ratio can be demonstrated in e vs. $\log p$ space, as an example Figure 1, Leroueil et al. [14]. The first portion of the curve with lower slope, which is due to unloading of the sample, is not considered here. Only the second portion with slope of c_c, which is mainly due to loading, is considered. The void ratio "e" of the material at any stage of the consolidation can be determined by:

$$e = c_c \log \frac{\sigma'}{\sigma'_1} + e_1 \tag{3}$$

where σ' is the applied effective stresses (head) corresponding to e and σ'_1 is the known effective stress corresponding to e_1. Equation 3 can be written as:

$$e = c_c \log \sigma' - c_c \log \sigma'_1 + e_1 \tag{4}$$

or

$$e = a \log \sigma' + b \tag{5}$$

where $a = c_c$ and $b = -c_c \log \sigma'_1 + e_1$

Relationship Between Void Ratio and Coefficient of Permeability It can be observed from previous research of Lambe and Whitman [13] and Cedergren [15] and Leroueil [14] that the

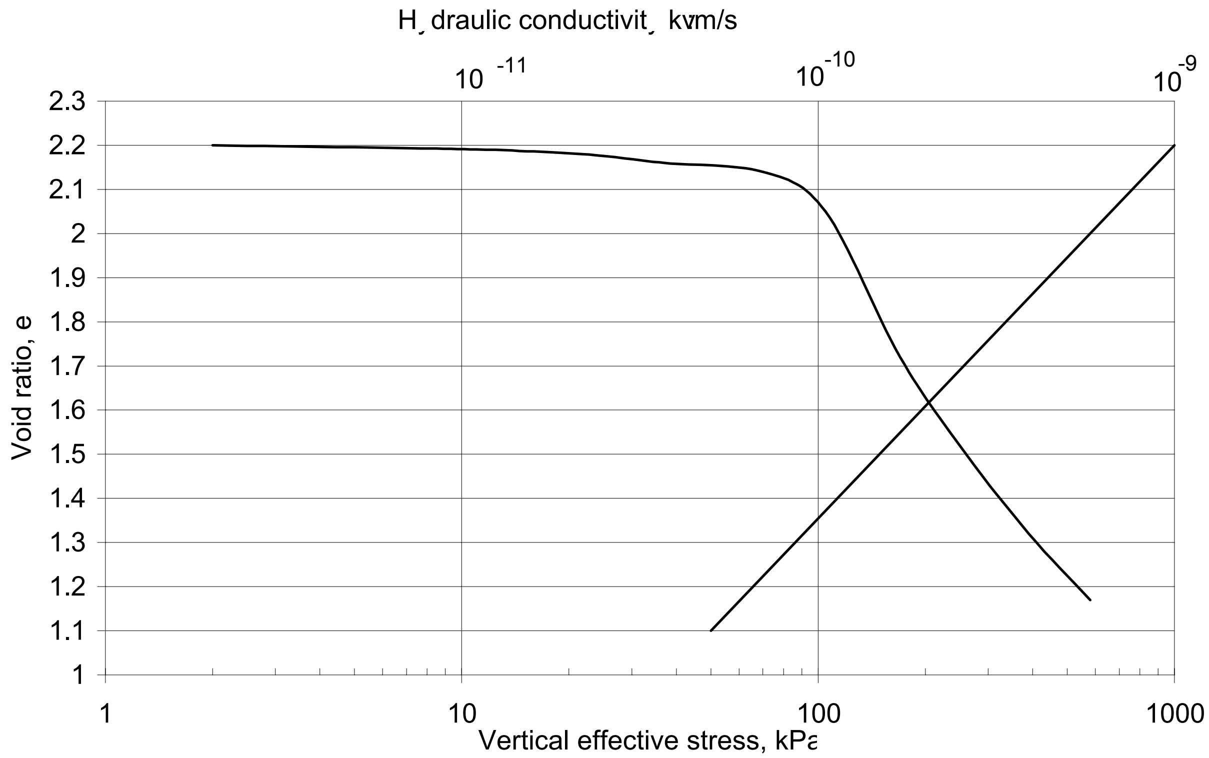

Figure 1. Typical *e* against *log* σ′ and *e* against *log k* curve (from Leroueil et al. [14]).

relationship between void ratio and logarithm of coefficient of permeability is linear, Figure 1. Similar to previous case, "*e*" void ratio of the material at any stage can be determine by:

$$e = c_k \log \frac{k}{k_1} + e_1 \qquad (6)$$

where c_k is the slope of the curve, k is the unknown coefficient of permeability corresponding to e, and k_1 is the known coefficient of permeability corresponding to e_1. By rearranging the Equation 6, the coefficient of permeability can be found as follows:

$$\log k = \frac{e}{c_k} - \frac{e_1}{c_k} + \log k_1 \qquad (7)$$

Since $- e_1/c_k + \log k_1$ is a constant value assumed to be equal to d, and $c = 1/c_k$, therefore:

$$\log k = c\, e + d \qquad (8)$$

and finally k can be written as:

$$k = 10^{(c\, e + d)} \qquad (9)$$

and with substitution of Equation 5 into Equation 9, it can be written as:

$$k = 10^{(\alpha \log \sigma' + \beta)} \qquad (10)$$

where α is equal to ca and β is equal to $cb + d$, which all of the parameters *a, b, c,* and *d* are constant and can be determined from laboratory or in-situ testing.

Relationship Between Heads (Total or Pressure) and Coefficient of Permeability

From the effective stress Terzaghi's Equation and from the information in Figure 2, the effective stress at any point can be written as:

$$\sigma' = (-y\, \gamma_{sat} + \gamma_w H) - (h + y)\, \gamma_w \qquad (11)$$

where γ_{sat} is saturated density of the soil, h is the

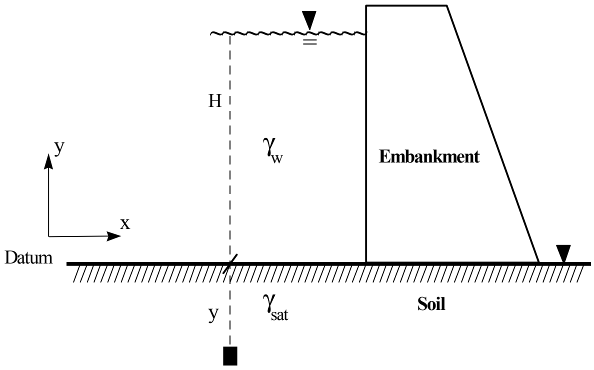

Figure 2. Schematic diagram of an embankment dam.

total head, $h + y$ is the pressure head, H is the upstream water height and γ_w is water density.

By substituting Equation 11 into Equation 10, it can be written as:

$$k = 10^{\alpha \log [-y\gamma sat + (H - h - y)\gamma w] + \beta} \quad (12)$$

Equation 12 can be simplified to

$$k = 10^{\beta}[-y\ \gamma_{sat} + (H - h - y)\gamma_w]^{\alpha} \quad (13)$$

In the above equation α, β, γ_{sat}, and γ_w are constants that depend on material properties and can be determined from laboratory or in-situ testing. The value of total head h depends on the geometry of the considered point and is an unknown value, H is the height of water at upstream and y is the depth of the considered point from datum. It can be concluded from the above equation that at any point within the confined flow the coefficient of permeability can be defined as a function of total head "h" which will directly influence the solution of the governing differential equation.

3. WATER FLOW NON- LINEAR GOVERNING DIFFERENTIAL EQUATION

The 2-D governing equation of water flow in porous media under laminar conditions, where Darcy's law is applicable is given by:

$$\frac{\partial}{\partial x}\left(\gamma_w k_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\gamma_w k_y \frac{\partial h}{\partial y}\right) = 0 \quad (14)$$

The above equation can be simplified by assuming γ_w, water density, to stay constant at all times, and therefore:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right) = 0 \quad (15)$$

Under conditions of homogeneity, k_x and k_y are assumed to be constants which do not vary in space. In addition, applying anisotropy conditions requires $k_x \neq k_y$. Generally for simplicity k_x and k_y are assumed to be constant and for more simplicity, they are assumed to be equal and constant. However, in this research these coefficients are assumed to be variable which would change the differential equation to a non-linear one. Equation 15 can be expressed as follows:

$$\frac{dk_x}{dx}\frac{dh}{dx} + k_x \frac{d^2 h}{dx^2} + \frac{dk_y}{dy}\frac{dh}{dy} + k_y \frac{d^2 h}{dy^2} = 0 \quad (16)$$

k_x and k_y can now be expressed by Equations 1 and 2 and Equation 13 and can be substituted in Equation 16. Formulation of the least square finite element method requires first order differential equations. This can be adopted by assigning hydraulic gradients in the x and y directions as follow:

$$P_x = \frac{dh}{dx} = I_x \qquad P_y = \frac{dh}{dy} = I_y \quad (17)$$

Equation 17 was used in least square finite element formulation.

Secondary Solutions In seepage problems, in addition to evaluation and calculation of heads at various locations in the system, three other parameters are important to be evaluated. These are total discharge rate, exit hydraulic gradient, and uplift pressure. These parameters are known as the secondary solutions. Total discharge rate can be calculated on the bases of discharge for each element at any section, and the summation of these discharge rates would be the total discharge rate of the system, which will be

$$Q = -\sum_{i=1}^{N} d_i [k_{xi}][I_{xi}] \tag{18}$$

where d_i is the width of the element I, with the value I_x as the average of the eight node hydraulic gradient for each element. The exit hydraulic gradient would be known at the downstream section of the system. Uplift pressure can be calculated on the bases of Bernoulli's equation by knowing total head (h) from analysis and evaluation of the concerning point from geometry assuming, that $v^2/2g = 0$.

4. NUMERICAL EXAMPLES

In this section two examples are provided for the proposed types of variations on k_x and k_y.

Example 1 To illustrate the proposed methods, consider Example 18.2 of Lambe and Whitman [13]. A schematic diagram of a concrete dam is given in Figure 3. This system consists of two sheet piles of 21 meters height at upstream and downstream of the dam. In order to analyses the problem, the permeable section of the system was divided into 18 elements with 77 nodes. Sheet piles were considered as impermeable boundaries, where $P_x = \partial h / \partial x = 0$ and other impermeable boundaries where $P_y = \partial h / \partial y = 0$ are at the bottom of the 64 meter thick permeable layer and dam itself. A thirty-meter distance away from the system (sheet piles) was chosen as a limit for numerical analysis where it assumed there is no flow-taking place away from these limits in the permeable layer. Variable heads at

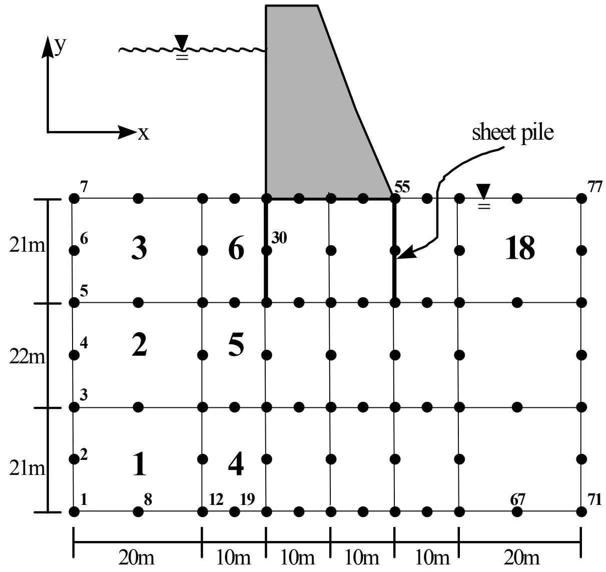

Figure 3. Schematic diagram of the system and FE mesh.

upstream and downstream locations were chosen in order to examine the effect of the proposed solution. In order to apply Equation 13, the following values were used based on Effati [16].

$\alpha = -0.034049$
$\beta = -1.0$
$\gamma_{sat} = 22\ kN/m^3$
$\gamma_w = 10\ kN/m^3$

Results for head, the coefficient of permeability k and discharge rate were obtained based on the above values employing computer program which was prepared based on the least square finite element formulation. Comparing the results of the heads obtained here with flow nets in Lambe and Whitman [13] shows only very small differences. Any conclusions based on head results and flow nets alone may not be justified due to the accuracy of the results. A flow net drawing is based on a trial and error procedure and is not affected by upstream or downstream heads. In Figure 4 variation of coefficient of permeability k is shown against head (water height) in the upstream. It is clear from this figure that, (i) when the head at any node varies, it would influence the permeability

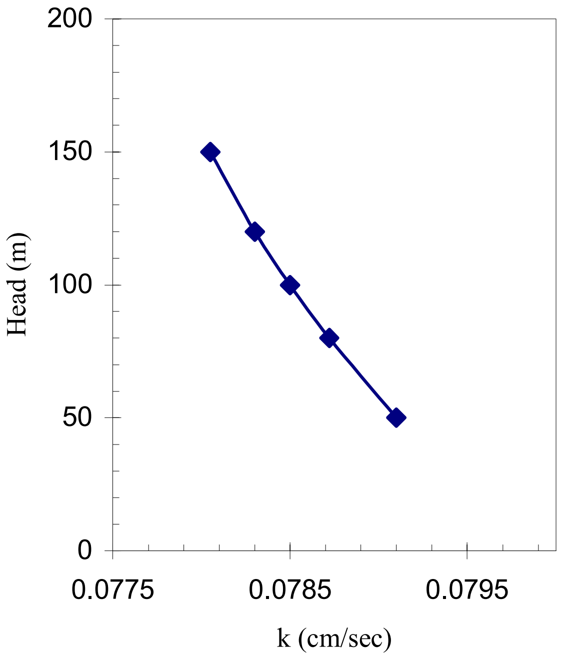

Figure 4. Variation of coefficient of permeability against head.

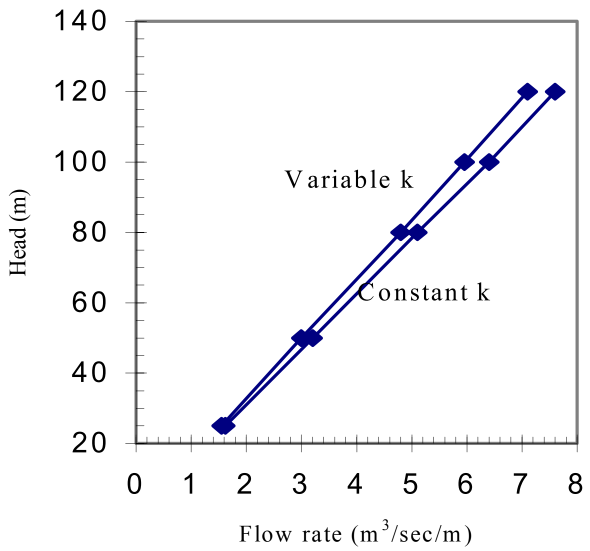

Figure 5. Comparison of discharge rate variation under dam against head for constant and variable coefficient of permeability.

of that node, (ii) as the head increases the values of permeability decrease, and (iii) when the head at any point increases, consolidation of the material occurs resulting in reduced the permeability. The variation of k against head is linear because the proposed function for k in Equation 13 is non-linear. Figure 5 shows the variation of the discharge rate under the dam against upstream head. Two types of curves are shown in Figure 5, one with constant permeability and the other with variable permeability (proposed method). In the one with constant permeability, similar to most classical seepage problems, permeability is assumed constant throughout the analysis and the system, and if it varies it is not due to the effect of upstream head. In this case k was assumed to be 0.07782 *cm/sec*. But in the other one variable k refers to the influence of head on discharge rate. It is can be seen from Figure 4 that for both cases when upstream head increases, the discharge rate also increases. It should be clarified, however, that in the actual case, head effects influenced permeability. The discharge rate is different from that of the constant permeability case. The difference would be higher for longer values of upstream head, i.e. for h = 120 m the effect of the head on discharge rate is about 5.4%. This is mainly due to the effect of upstream head on permeability.

Example 2 In these example variations of k_x and k_y are not effected by a direct influence of head but they are based on other effects using Equation 1 and Equation 2. A schematic diagram of a concrete dam is given in Figure 6.

The permeable section of the system was divided into 20 elements with 85 nodes. The top and bottom portion of the permeable section with thickness of 40 meters were considered as impermeable boundaries where $P_y = \partial h / \partial y = 0$. Sixty meters from the toe and heel of the dam were chosen as a limit for numerical analysis where no flow was assumed to take place away from these limits in the permeable layer. The proposed variations for k_x and k_y are based on Equations 1 and 2 and Effati [16].

$$k_x = 0.375E - 3x^2 - 0.375E - 2x + 10$$

at $x = 0.0$ $k_x = 10$ *m/day*

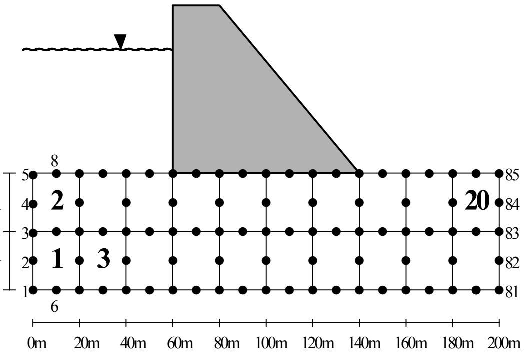

Figure 6. Schematic diagram of the system with FE mesh.

$$k_y = 0.255E - 2y^2 + 0.375E - 2y + 2.5$$

at $y = 0.0$ $\quad k_y = 2.5\ m/day$

In this example results for exit gradient and uplift pressure are presented based on above values for k_x and k_y using a computer program which was prepared based on the least square finite element formulation.

Figure 7 shows the exit gradient in vertical direction against upstream head for constant and variable permeability based on data in this example. For low upstream head the difference between constant and variable permeability conditions is sometimes negligible, but as the upstream head increases, in large dams the difference becomes more significant which might influence the design of the whole system. For the upstream height of 180 meters, the exit gradient difference is about 32%, which would reduce the factor of safety against piping to a low point, which, in turn, may result in changing the geometry of the dam. It should be noted that constant values of permeability considered in the computation would result in higher values for exit gradient, which would be on the safer side.

Figure 8 shows the uplift pressure at the bottom of the dam against upstream head for constant and variable permeability based on data in this example. As the head in the upstream of the dam will increase, obviously the uplift pressure under dam increases, but as it can be seen, there is a difference between constant or variable permeability conditions. This difference would be around 7% for an upstream head of 150 meters. Again it should be noted that with constant permeability the value of uplift pressure is higher than that with the variable permeability, which would also be on the safe side.

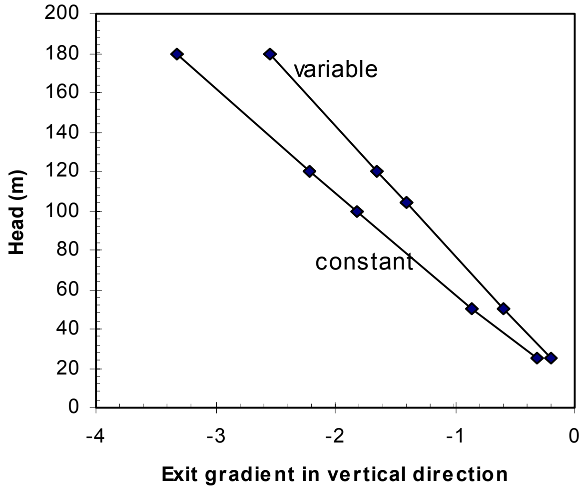

Figure 7. Variation of exit hydraulic gradient against head for constant and variable permeability.

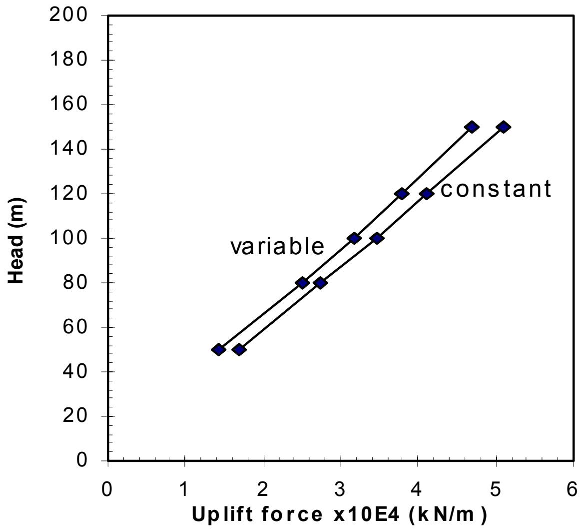

Figure 8. Comparison of the uplift pressure against head for constant and variable permeability.

5. CONCLUSIONS

This paper presents a non-linear governing differential equation for a confined seepage problem under non-homogeneous and anisotropic conditions. This non-linear performance is introduced by the governing equation based on actual material behavior and solving the resulting non-linear differential equation numerically using the least square finite element formulation. This method was used to solve several seepage problems to examine the accuracy of the results. The solutions show good accuracy and convergence. The advantage of this method is its capability to solve non-linear problems compared to routine methods with constant coefficients in order to increase the accuracy of the solution; eight nodal (isoperimetric) elements were used. Some clear conclusions can be drawn from this study.

a) Generally, results of head changes (i.e., flow net analyses) either by first or second conditions for variable permeability conditions compare favorably to the case when the permeability is assumed to be constant and very little difference is observed.

b) Comparison of the results for discharge rate between constant and variable permeability conditions shows little effect of low head on discharge rate results. However, as upstream head increases, the effect of variable permeabilities becomes more significant. Usually the difference in discharge rate between variable and constant permeability for a typical head is not more than 8%.

c) Results of exit gradient for a critical condition shows that the values are less affected for low head, but the effect increases for higher head. The results assuming variable permeability conditions would give a lower safety factor regarding piping, etc.

d) In terms of uplift pressure, as the head increases the uplift pressure also increases, but there is only a slight difference between uplift pressure under constant or variable permeability conditions for any given head. This result is consistent with part (a).

e) In general, the effect of variable coefficient of permeability may not be significant on small dams, but as the height of the dam increases, the effect becomes more considerable. It is believed that this would influence the geometry and design of the dam and that variable permeability analysis such as the one described in this paper should be conducted.

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