ANALYTICSOLUTION FOR HYPERSONIC FLOW PAST A SLENDER ELLIPTIC CONE USING SECOND-ORDER PERTURBATION APPROXIMATIONS

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Abstract An approximate analytical solution is obtained for hypersonic flow past a slender elliptic cone using second-order perturbation techniques in spherical coordinate systems. The analysis is based on perturbations of hypersonic flow past a circular cone aligned with the free stream, the perturbations stemming from the small cross-section eccentricity. By means of hypersonic approximations for the basic cone problem, closed-form second-order approximate solutions for the perturbation equations are obtained within the framework of hypersonic small-disturbance theory. Results for the shock shape, shock-layer structure, and surface pressure are presented for all ranges of Mach numbers, together with comparisons with experimental data. Also a complete vortical layer analysis is presented to prove the suitability of the surface boundary conditions.

Key Words Hypersonic, Elliptic Cone, Second-Order Perturbation, Surface Pressure, Vortical Layer

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INTRODUCTION

A curious singularity appears at the surface of a circular cone inclined at a small angle to an inviscid supersonic stream. This phenomenon was unknown to Stone [1] who expanded the flow quantities formally in ascending powers of the angle of inclination and found the first and second-order perturbations. His results served as the basis for extensive numerical computation by Kopal [2]. The singularity was discovered by Ferri [3] who gave a physical description of the flow near the surface. He deduced that streamlines crossing the shock wave at any circumferential location eventually curve around the body toward the leeward plane of symmetry. Though the entire flow field is rotational, a thin layer of intense vorticity lies near the surface. Ferri called this the vortical layer. All streamlines approach the top ray, and consequently the entropy in that neighborhood is many-valued. Ferri called this the vortical singularity. Munson [4] solved the problem of flow over a circular cone inclined slightly to a uniform stream using the technique of matched asymptotic expansions. He found that the outer expansion is equivalent to Stone's solution of the problem and the inner expansion, valid in a thin layer near the body, represents Ferri's vortical layer. They also showed that the solution to first order in angle of attack so obtained is uniformly valid everywhere in the flow field, but in the second-order expansion an additional non-uniformity appears near the leeward ray. They showed that this defect can be removed by inspection and that density, radial component of velocity and entropy are only second-order quantities which are singular on the surface of the cone.

The problem of supersonic flow over circular cone is only one of the occurrences of vortical singularities. Indeed, they are present in any conical flow without axial symmetry. For supersonic flows past bodies without axial symmetry, the elliptic cone is a basic body shape. Although numerous papers have been directed towards supersonic flows past elliptic cones, their goals have been specific, and no general or comprehensive flow field calculations have been set forth. Work of Doty & Rasmussen [5] was to partially remedy this situation and to present an approximate analytical solution that illustrates the general flow field features of hypersonic flow past a slender elliptic cone with small eccentricity. The analyses were cast in the form of hypersonic similarity theory and the first-order results were presented in appropriate similarity form. There is also the study by Jischke [6], but it does not deal with the vortical layer and the study by Lin et al.[7] is only an inner solution for circular cone case.

In this undertaking, we start with the small-perturbation equations for perturbed flow past an elliptic cone with eccentricity ε and examine the quantities expansions for second-order approximation. Using hypersonic similarity theory, an analytic solution up to second-order approximations are presented. Here the shape and location of shock and coefficient of pressure on the surface of the cone for all Mach numbers are calculated. These results are compared with the existing numerical and experimental results of Zakkay and Visich [8] and Martellucci [9]. Also it is shown that the outer expansion of pressure for any order of ε is uniformly valid and is not singular on the surface of the cone.

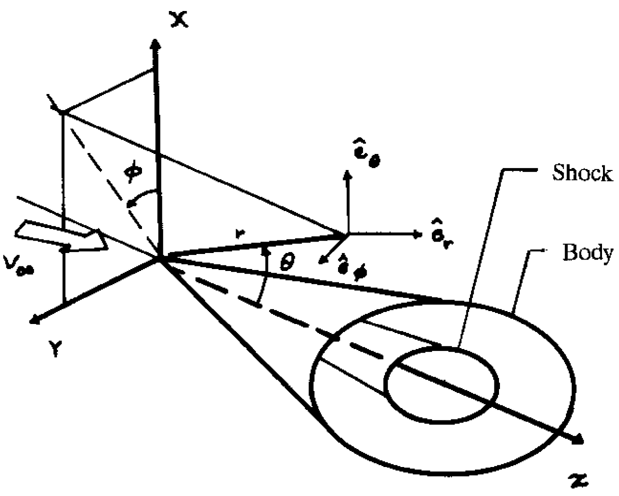

Figure 1. Spherical coordinate system.

FORMULATION OF THE PROBLEM

Governing Equations A spherical coordinate system is chosen as shown in Figure 1. The velocity components u^*, v^*, and w^* are in the directions of increasing r, θ, and ϕ, respectively. The pressure and density are P^* and ρ^*. The cone half-angle is δ. Dimensionless variables are defined as

$$u=\frac{u^*}{V_\infty},\qquad v=\frac{v^*}{V_\infty},\qquad w=\frac{w^*}{V_\infty}$$

$$p=\frac{p^*}{\rho_\infty V_\infty^2},\quad \rho=\frac{\rho^*}{\rho_\infty},\quad s=\frac{(s^*-s_\infty)}{C_v} \tag{1}$$

where V_∞ is the velocity at infinity.

The pressure, density, and velocity are governed by the equations of change for mass, momentum, and energy, plus appropriate equations of state. Here we assume the flow is inviscid, nonconducting, steady, and behaves as a thermally and calorically perfect gas.

Substituting the dimensionless variables into these equations yields:

$$2\,r\,u + \frac{\partial}{\partial q}(rv) + \cot q\; rv + \frac{1}{\sin q}\frac{\partial}{\partial f}(rw) = 0 \quad (2)$$

$$v\frac{\partial u}{\partial q} + \frac{w}{\sin q}\frac{\partial u}{\partial f} - v^2 - w^2 = 0 \quad (3)$$

$$v\frac{\partial v}{\partial q} + \frac{w}{\sin q}\frac{\partial v}{\partial f} + uv - w^2 \cot q + \frac{1}{r}\frac{\partial p}{\partial q} = 0 \quad (4)$$

$$v\frac{\partial w}{\partial q} + \frac{w}{\sin q}\frac{\partial w}{\partial f} + uw + vw\cot q + \frac{1}{r\sin q}\frac{\partial p}{\partial f} = 0 \quad (5)$$

$$v\frac{\partial s}{\partial q} + \frac{w}{\sin q}\frac{\partial s}{\partial f} = 0 \quad (6)$$

$$s = \ln(g M_E^2 p) - g \ln r \quad (7)$$

$$\frac{1}{2}(u^2 + v^2 + w^2) + \frac{g}{g-1}\frac{p}{r} = \frac{1}{2} + \frac{g}{g-1}\frac{1}{M_E^2} \quad (8)$$

These equations are not all independent. Indeed, any one except (2) can be eliminated. It is convenient, however, in the subsequent analysis, to consider all of them, and at certain points to particularize to a given set of five.

Body And Shock Geometry Considering Figure 2, the equation for perturbed cone and its conical shock wave attached to it for no angle of attack is assumed to have the form:

$$q_c = d - e\cos 2f + e^2\left(\frac{1}{2} + \frac{1}{2}\cos 4f\right) \quad (9)$$

$$q_s = b - e g_1 \cos 2f + \frac{e^2}{d}(g_{10} + g_{12}\cos 4f) + O(e^3) \quad (10)$$

in which d specifies the semivertex angle of the basic circular cone about which a perturbation analysis is to be performed, b is the semivertex angle of the basic circular shock corresponding to the basic body with semivertex angle d, and

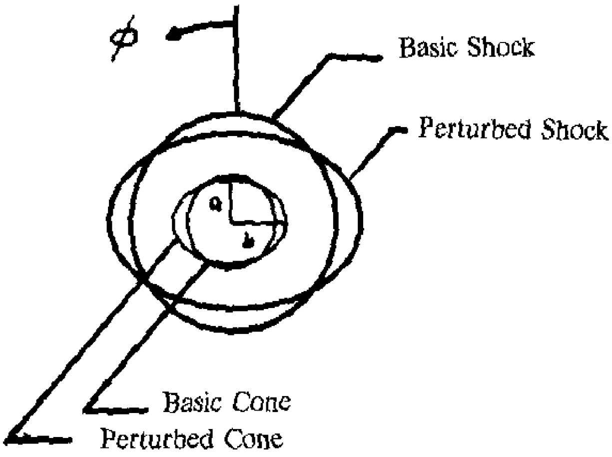

Figure 2. Perturbed cone and shock.

the g factors represent the deviation of the shock eccentricity from the body eccentricity and they are to be determined from the perturbation analysis. The parameter e is a measure of the eccentricity and is the appropriate perturbation parameter to be used in the subsequent analysis.

Expansions For The Flow Variables The velocity vector for conical flow is presented in spherical coordinates by

$$\vec{V} = u(q,f)\,\hat{e}_r + v(q,f)\,\hat{e}_q + w(q,f)\,\hat{e}_f \quad (11)$$

The Fourier representation for the body and shock shapes suggest that any quantities of the velocity components, pressure, and density represented here as $q(q,f,e)$ can be expanded in the following forms, valid outside the vortical layer adjacent to the body surface, for no angle of attack:

$$q(q,f,e) = q_0(q) + e q_1(q)\cos 2f + e^2\left[q_{10}(q) + q_{12}(q)\cos 4f\right] + O(e^3) \quad (12)$$

The lowest-order terms, with the subscript naught, pertain to the basic circular-cone solution, which is persumed known.

Perturbation Equations Substituting (12) into (2)-(8) for any related quantity and equating coefficients of like powers of e gives the following systems of ordinary differential equations:

e^0:

$$2\, r_0\, u_0 + (r_0\, v_0)' + r_0\, v_0 \cot q = 0 \qquad (13\text{-a})$$

$$u_0' = v_0 \qquad (13\text{-b})$$

$$r_0 v_0 v_0' + r_0 u_0\, v_0 + P_0' = 0 \qquad (13\text{-c})$$

$$S_0' = 0 \qquad (13\text{-d})$$

$$S_0 = \ln (g\, M_E^2 P_0) - g \ln r_0 \qquad (13\text{-d})$$

$$\frac{1}{2}(u_0^2 + v_0^2) + \frac{g}{g-1}\frac{P_0}{r_0} - \frac{1}{2} - \frac{1}{(g-1)\, M_E^2} = 0 \qquad (13\text{-f})$$

e^1:

$$2(r_0\, u_1 + u_0\, r_1) + (r_0\, v_1 + v_0\, r_1)' + (r_0 v_1 + v_0\, r_1) \cot q + \frac{2 r_0\, w_1}{\sin q} = 0 \qquad (14\text{-a})$$

$$u_1' = v_1 \qquad (14\text{-b})$$

$$r_0 (v_0 v_1)' + r_1 v_0 v_0' + r_0 u_0 v_1 + v_0 (r_0 u_1 + u_0\, r_1) + P_1' = 0 \qquad (14\text{-c})$$

$$w_1' + \frac{u_0}{v_0} w_1 + w_1 \cot q - \frac{2}{\sin q}\frac{p_1}{r_0 v_0} = 0 \qquad (14\text{-d})$$

$$S_1' = 0 \qquad (14\text{-e})$$

$$S_1 = \frac{P_1}{P_0} - g\frac{r_1}{r_0} \qquad (14\text{-f})$$

$$\frac{1}{2}(u_0^2 + v_0^2) + \frac{r_0}{r_1}(u_0 u_1 + v_0 v_1) + \frac{g}{g-1}\frac{P_1}{r_1} - \frac{1}{2}\frac{1}{(g-1) M_E^2} = 0 \qquad (14\text{-g})$$

e^2:

$$2\left(r_0\, u_{1m} + \frac{1}{2} r_1\, u_1 + u_0\, r_{1m}\right) + \left(r_0\, v_{1m} + \frac{1}{2} r_1\, v_1 + v_0\, r_{1m}\right)' + \left(r_0\, v_{1m} + \frac{1}{2} r_1\, v_1 + v_0\, r_{1m}\right) \cot q + \frac{m}{2}\left(\frac{4\, r_0\, w_{1m} + 2\, r_1\, w_1}{\sin q}\right) = 0 \qquad (15\text{-a})$$

$$v_{1m} = u_{1m}' + (m-1)\frac{w_1\,(2u_1 + w_1 \sin q)}{2 v_0 \sin q} \qquad (15\text{-b})$$

$$r_0\, v_0\, v_{1m}' + \frac{1}{2}(r_0 v_1 + v_0\, r_1)\, v_1' + \left(r_0 v_{1m} + \frac{1}{2} r_1 v_1 + v_0\, r_{1m}\right) v_0' + (m-1)\frac{r_0 w_1 v_1}{\sin q} + r_0 u_0 v_{1m} + \frac{1}{2}(r_0 u_1 + u_0\, r_1) v_1 + \left(r_0 u_{1m} + \frac{1}{2} r_1 u_1 + u_0\, r_{1m}\right) v_0 + \frac{(m-1)}{2} r_0 w_1^2 \cot q + p_{1m}' = 0 \qquad (15\text{-c})$$

$$\frac{m}{2}\left[w_{1m}' + \frac{1}{2}\left(\frac{v_1}{v_0} + \frac{r_1}{r_0}\right) w_1' + \frac{w_1^2}{v_0 \sin q} + \frac{u_0}{v_0} w_{1m} + \frac{1}{2}\left(\frac{u_1}{v_0} + \frac{u_0}{v_0}\frac{r_1}{r_0}\right) w_1 + w_{1m} \cot q + \frac{1}{2}\left(\frac{v_1}{v_0} + \frac{r_1}{r_0}\right) w_1 \cot q - \frac{4 P_{1m}}{r_0 v_0 \sin q}\right] = 0 \qquad (15\text{-d})$$

$$s_{1m}' = (1-m)\frac{w_1\, s_1}{v_0 \sin q} \qquad (15\text{-e})$$

$$S_{1m} = \frac{P_{1m}}{P_0} - g\frac{r_{1m}}{r_0} - \frac{1}{4}\left[\left(\frac{P_1}{P_0}\right)^2 - g\left(\frac{r_1}{r_0}\right)^2\right] \qquad (15\text{-f})$$

$$\frac{1}{2}(u_0^2 + v_0^2) + \frac{r_0}{r_{1m}}\left[u_0 u_{1m} + \frac{1}{4}(u_1^2 + v_1^2 + (1-m)\, w_1^2) + v_0 v_{1m}\right] + \frac{r_1}{2 r_{1m}}(u_0 u_1 + v_0 v_1) + \frac{g}{g-1}\frac{P_{1m}}{r_{1m}} - \frac{1}{2} - \frac{1}{(g-1)\, M_E^2} = 0 \qquad (15\text{-g})$$

with m = 0,2 in each case.

In each of the systems of equations above all the variables are calculated in terms of the radial velocity component (u) and then substituted into the continuity equation of the corresponding systems of equations. The following differential equations are obtained:

$$u_0'' + u_0' \cot q + 2u_0 = \frac{v_0^2}{a_0^2}(u_0'' + u_0) \qquad (16)$$

$$u_1'' + \cot q\, u_1' + \left(2 - \frac{4}{\sin^2 q}\right) u_1 = -\frac{4F_1}{g}\frac{H_0(q)}{\sin^2 q} + \bar{H}_1(q) \qquad (17)$$

$$u_{1m}'' + u_{1m}'\cot q + \left(2 - \frac{8m}{\sin^2 q}\right) u_{1m} = H_{1m}(q) + \bar{H}_{1m}(q) - \frac{2m\, D_{12}}{I \sin^2 q} + \frac{2m}{I \sin^2 q} * ß_b^q \left[\frac{2}{v_0} I(q)\, Q_{12}(q)\right] dq \quad (18)$$

where $I(q)$, $H_0(q)$, $A(q)$, $B(q)$, $C(q)$, $Q_{12}(q)$, $H(q)$, $H_{1m}(q)$, are known functions of q and $\underline{D}_{12}$ is a constant. The functions $H_1(q)$ and $H_{1m}(q)$ are in terms of the variable coefficients including u_1 and u_{1m} which cab be neglected with not much of loss of accuracy.

BOUNDARY CONDITIONS

Boundary Conditions At The Shock The shock jump conditions for flow properties across the shock are well-known Rankine-Hugoniot relations. The dimensionless form of these relations are:

$$\frac{1}{r_s} = \frac{(g-1)\, M_n^2 + 2}{(g+1) M_n^2} ,$$

$$g\, M_È^2 P_s = 1 + \frac{2g}{g+1}(M_n^2 - 1)$$

$$S_s = \ln\left(g\, M_È^2\, P_s\right) + g \ln\left(\frac{1}{r_s}\right)$$

$$\frac{\overset{Ø}{V}_s}{V_È} \cdot n_s = \frac{1}{r_s}\left(\frac{\overset{Ø}{V}_È}{V_È} . \overset{Ø}{n}_s\right)$$

$$\frac{\overset{Ø}{V}_s}{V_È} * \overset{Ø}{n}_s = \left(\frac{\overset{Ø}{V}_È}{V_È} * \overset{Ø}{n}_s\right) \quad (19)$$

where $\overset{Ø}{n}_s$ is the unit outward normal on the shock given as:

$$\overset{Ø}{n}_s = \left[1 - e^2 g_1^2 \frac{1 - \cos 4f}{\sin^2 b}\right]\hat{e}_q + \frac{1}{\sin b}\left[-2 e g^* \sin 2f + e^2\left(\frac{4}{d} g_{12} - g_1^2 \frac{\cos b}{\sin b}\right) \sin 4f\right]\hat{e}_f \quad (20)$$

Since the location of the shock is unknown, using boundary conditions at the shock becomes very complicated. To circumvent this situation the shock angle is expanded in terms of powers of e and by using of a Taylor series the flow variables are transferred to the basic unperturbed shock ($e = 0$).

The boundary conditions at the shock are obtained from these relations.

Surface Boundary Conditions At the body surface, $q = q_c$, the normal velocity must vanish:

$$\overset{Ø}{V}_c / \overset{Ø}{n}_c = 0 \quad (21)$$

By substituting the unit outward normal vector on the elliptic cone surface and the velocity expansions into (21) and transferring the boundary conditions to the basic circular cone surface leads to the surface boundary conditions:

$$v_0(d) = v_1(d) = v_{10}(d) = v_{12}(d) = 0 \quad (22)$$

Rigorously, we should have obtained (22) by matching the outer expansion with an inner expansion for the vortical layer adjacent to the cone surface. Later we will show that (22) is indeed proper.

APPROXIMATE SOLUTION

Basic Cone Flow Approximations For hypersonic flow in the limit $M_È Ø È$ and $\sin q Ø 0$ such that the combination $K Û M_È \sin q$ remains finite, the basic cone flow can be approximated accurately by:

$$u_0(q) = \frac{u_0^*(q)}{v_È} = 1 \ , \ v_0(q) = \frac{v_0^*(q)}{v_È} = -q\left(1 - \frac{d^2}{q^2}\right) \ , \ s = \frac{b}{d} ¡ \frac{g+1}{2} + \frac{1}{k_d^2} \quad (23)$$

where $K_d = M_È d$ is the hypersonic similarity parameter. Also from basic cone solution, following relations are obtained:

$$N = s^2/a_0^2(b) \ , \ J = a_0^2(d)/a_0^2(b) \quad (24)$$

First-Order Approximate Solution A solution to Equation 17 is $u_1(q) = x_1(q)/\sin^2 q$

and by substituting this solution into the differential equation and integrating twice yields:

$$\xi_1(\theta) = \xi_1(\beta) + \frac{\xi_1'(\beta)}{\beta^3} \int_\beta^\theta \theta^3 \, d\theta - \frac{4\Phi_1}{\gamma} * \int_\beta^\theta \{ \theta^3 \int_\beta^\theta [\frac{H_0(\theta)}{\theta^3}] \, d\theta\} d\theta$$

with the assumption of θ and β being small, $\xi_1(\theta)$ can be calculated and by the definition of $\zeta = \theta/\delta$, first-order components of velocity are obtained.

The shock eccentricity factor, g_1, can now be determined from condition (22), as follows:

$$g_1 = 6\sigma^3/\{\frac{3\mathrm{Cos}^{-1}(1/\sigma)}{(\sqrt{\sigma^2 - 1})} + \frac{6(\sigma^6 + \sigma^2)}{(\gamma + 1)} + + 3\sigma^4 - \sigma^2 - 5\} \tag{25}$$

Second-Order Approximate Solution The second-order solution not only provides a better approximations with respect to the first-order solution, it also shows the singularity in flow and vortical layer near the body surface. Clearly, from (15-b) and (15-e) the second-order correction to both the radial velocity and entropy is singular at the surface. It can be shown that the singularity in expressions for v_{1m} & P_{1m} are not real and these quantities are finite on the body surface. To show this and to get rid of the singularity in v_{1m} and P_{1m}, Stone suggested the following change of variables

$$U_{1m}(\theta) = u_{1m}(\theta) + (m - 1) \int_\beta^\theta H(\theta) d\theta \tag{26}$$

Substituting this new variable into (18) we obtain:

$$U_{1m}'' + \cot\theta \, U_{1m}' + (2 - \frac{8m}{\sin^2\theta}) U_{1m} = T_{1m}(\theta) \tag{27}$$

where $T_{1m}(\theta)$ is a known function of θ.

Following the same discussion as in the case of the first-order solution, we assume:

$U_{10}(\theta) = \xi_{10}(\theta) \,\mathrm{Cos}\,\theta$ for $m = 0$ and $U_{12}(\theta) = \xi_{12}(\theta) \; (6/5\sin^4\theta - 1/\sin^2\theta)$ for $m = 2$.

Substituting these solutions into (27), yields:

$$\xi_{10}(\theta) = \xi_{10}(\beta) + \sin\beta\cos^2\beta \; \xi_{10}'(\beta) * \int_\beta^\theta (\frac{1}{\sin\theta \cos^2\theta}) d\theta + \int_\beta^\theta \{\frac{1}{\sin\theta\cos^2\theta} * \int_\beta^\theta [T_{10}(\theta)\sin\theta\cos\theta] d\theta\} d\theta$$

$$\xi_{12}(\theta) = \xi_{12}(\beta) + \xi_{12}'(\theta)/\sin^2\beta \int_\beta^\theta \sin^3\theta \, d\theta - \int_\beta^\theta \{\sin^3\theta [T_{12}(\theta)/\sin\theta] d\theta\} d\theta$$

To assume small angles in above relations, the following results yield:

$$U_{10}(z)/\delta^2 = \frac{\xi_{10}(\beta)}{\delta^2} + \frac{\sigma}{\delta}\xi_{10}'(\beta) \mathrm{Ln}(z/\sigma) + \mathrm{Ln}z \int_\sigma^z [zT_{10}(z)] dz - \int_\sigma^z [z \mathrm{Ln}z T_{10}(z)] dz$$

$$U_{12}(z)/\delta^2 = -\xi_{12}(\beta)/(\delta^4 z^2) - \xi_{12}'(\beta)(z^4 - \sigma^4)/(4\beta^3 z^2 2) + 1/(4z^2)\{z^4 \int_\sigma^z [T_{12}(z)/z] dz - \int_\sigma^z [z^3 T_{12}(z)] dz\} \tag{28}$$

and therefore it can be written:

$$v_{10}(z)/\delta = \sigma/z. \frac{\xi_{12}'(\beta)}{\delta} + 1/z \int_\sigma^z [zT_{10}(z)] dz$$

$$v_{12}(z)/\delta = \frac{2\xi_{12}(\beta)}{\delta^4 z^3} - \frac{\xi_{12}'(\beta)}{2\beta^3}(z + \frac{\sigma^4}{z^3}) + \frac{1}{2}\{z \int_\sigma^z [\frac{T_{12}(z)}{z}] dz + \frac{1}{z^3} \int_\sigma^z [z^3 T_{12}(z)] dz\} \tag{29}$$

The quantities $\xi_{10}'(\beta)$ and $\xi_{12}'(\beta)$ are calculated using boundary conditions at the shock and then g_{10} & g_{12} are obtained using (22).

Pressure Calculation On The Body Knowing the extent of pressure on the body surface is necessary for calculation of lift and drag forces for design and manufacture of flying objects. From the perturbation equations, expressions of pressure perturbations are obtained as follows:

$$P_1 = \Phi_1 P_0 - \rho_0(u_0 u_1 + v_0 v_1) \tag{30}$$

$$P_{1m} = \frac{r_0}{4a_0^2}(u_0u_1 + v_0v_1)^2 - \frac{r_0}{2}F_1(u_0u_1 + v_0v_1) - r_0\left[u_0u_{1m} + v_0v_{1m} + \frac{1}{4}(u_1^2 + v_1^2 - w_1^2)\right] + \frac{F_1^2}{4}P_0 - \frac{P_0S_{1m}}{g-1} \qquad (31)$$

The pressure coefficient, C_p , is defined by:

$$Cp = (P^* - P_È)/((1/2)\ r_È v_È^2) \qquad (32)$$

Thus we can write:

$$Cp = Cp_0 + \varepsilon Cp_1 \cos 2F + \varepsilon^2 (Cp_{10} + Cp_{12} \cos 4F) + o(\varepsilon^3) \qquad (33)$$

where

$$Cp_0 = 2\left[P_0(d) - \frac{1}{gM_È^2}\right], \qquad Cp_1 = 2P_1(d),$$

$$Cp_{12} = 2 P_{12}(d) \qquad (34)$$

using the basic cone solution, we have:

$$\frac{C_{p0}}{d^2} = 1 + \frac{s^2 \operatorname{Ln} s^2}{(s^2 - 1)} \qquad (35)$$

$$C_{p1}/d^2 = 2gN\frac{p_0(d)}{d^2}\left[\frac{g_1}{s^3} - \frac{u_1(1)}{Jd^2}\right] \qquad (36)$$

$$C_{p1m}/d^2 = \frac{2gN}{J}\frac{p_0(d)}{d^2}\frac{p_{1m}(d)}{d^2 r_0(d)} \qquad (37)$$

As mentioned earlier since $v_0(d) = 0$ then S_{1m} and U_{1m} are singular on the surface. From (31) it seems that P_{1m} is also singular on the surface. But noting the change of variable (26) it can be shown that the singularity in S_{1m} and U_{1m} cancel each other out in (31) in a way that P_{1m} is finite on the surface. Paying attention to (15-e), the expression $\int_b^q w_1s_1/v_0 \sin q\, dq$ is the cause of singularity in entropy. After simplification it can be shown that the coefficient of this expression in S_{1m} is $4(m-1)^* F_1 a_0^2(q)/gd^2$ and in U_{1m} is $4(1-m)\ F_1J/N\ g$. Calculating these two coefficients by using (24), it is seen that on the surface they are equal with different signs. Therefore, P_{1m} is finite on the body and Cp_{1m} can be calculated from (37).

VORTICAL LAYER ANALYSIS

Here through asymptotic matching, it will be demonstrated that the boundary conditions on the body as were derived in section (3-2) are indeed correct. New independent variables that are of order unity in the vortical layer are defined according to

$$Q = (q - d)^{\varepsilon}, \qquad F = f \qquad (38)$$

To exhibit the zero in the velocity we define:

$$V = v / (q - d) \qquad (39)$$

This makes V of O(1) in the region of interest. The other independent variables are defined according to:

$$U = u,\ W = w,\ P = p,\ R = r \qquad (40)$$

The expansion of the inner variables in ascending powers of e are according to

$$q(Q,F,\varepsilon) = q_0 + \varepsilon q_1(Q,F) + \varepsilon^2 q_2(Q,F) \qquad (41)$$

where $q(Q,F,\varepsilon)$ represents $U(Q,F,\varepsilon)$ and $V(Q,F,\varepsilon)$ and $W(Q,F,\varepsilon)$ and $P(Q,F,\varepsilon)$ and $R(Q,F,\varepsilon)$.

The zeroth-order variables simply express the fact that these quantities are not functions of Q,meaning that they do not change across the layer.

The inner expansion is valid only in a thin layer near the body and cannot be expected to satisfy the shock conditions. Therefore, the boundary conditions are obtained by matching them with the outer expansion. We apply the following matching principle:

m-term inner expansion of (p-term outer expansion) = p-term outer expansion of (m-term inner expansion) (42)

Here the application of the matching principle considers the three-term outer and inner expansions for the normal velocity component.

Three-term outer expansion of v is:

$v(q) = v_0(q) + \varepsilon v_1(q) \cos 2f + \varepsilon^2[v_{10}(q) + v_{12}(q) \cos 4f]$

We write the functions in inner variables:

$v(q)=v_0(d+Q^{(1/\varepsilon)})+ \varepsilon v_1(d+Q^{(1/\varepsilon)})\cos 2F + \varepsilon^2[v_{10}(d+Q^{(1/\varepsilon)})+v_{12}(d+Q^{(1/\varepsilon)})\cos 4F]$

Expanding this function in powers of ε up to three terms gives

$$v(q)=v_0(d)+ \varepsilon v_1(d) \cos 2F + \varepsilon^2[v_{10}(d)+ + v_{12}(d) \cos 4F] \quad (43)$$

Three-term inner expansion of v is :

$v(q) = Q^{(1/\varepsilon)}[V_0 + \varepsilon V_1(Q,F) + + \varepsilon^2 V_2(Q,F)]$

writing the functions in outer variables:

$v(q) = (q - d) \{V_0 + \varepsilon V_1[(q - d)^{\varepsilon},F] + + \varepsilon^2 V_2[(q - d)^{\varepsilon},F]\}$

Expanding this function in powers of ε gives:

$$v(q) = [Q^{(1/\varepsilon)}]V_0 + 0 = 0 \quad (44)$$

Equating (43) and (44) follows

$v_0(d) = v_1(d) = v_{10}(d) = v_{12}(d) = 0$

which is indeed the same as (22).

PRESENTATION OF RESULTS

The shock eccentricity factors, g's, are plotted in Figures 3 - 5 as a function of k_d for g = 7/5. For dØ0, which corresponds to the limit of linearized theory, the eccentricity factor tends to zero, gØ0, that is, the shock tends to a circular Mach cone. For the limiting hypersonic flow, dØÈ, g approaches the asymptote g= 0.955, and the shock tends to embrace the elliptic cone body. When k_dØÈ and gØ1, then the body, in agreement with hypersonic

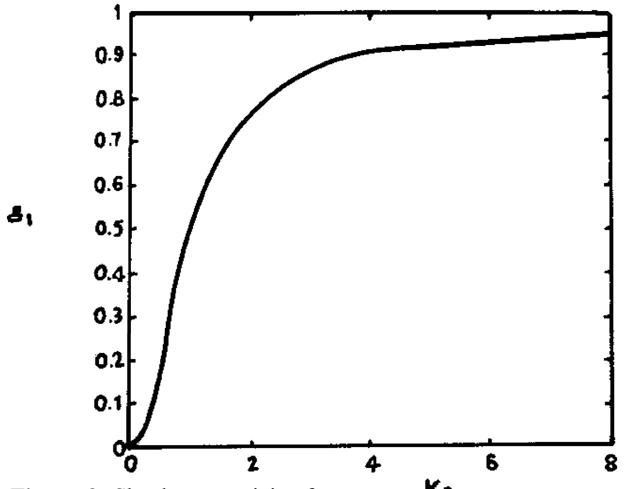

Figure 3. Shock eccentricity factor.

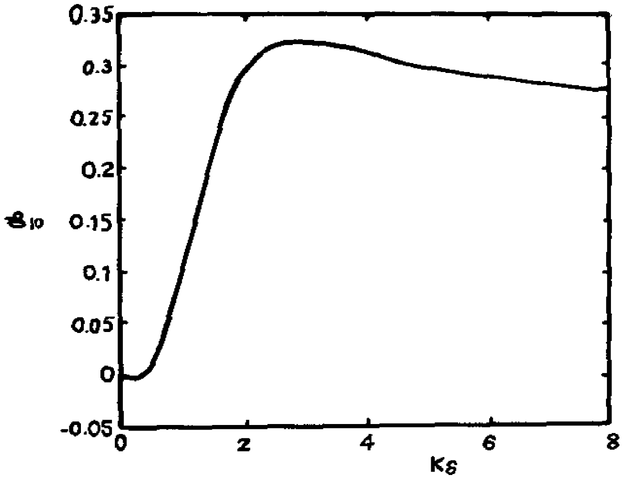

Figure 4. Shock eccentricity factor.

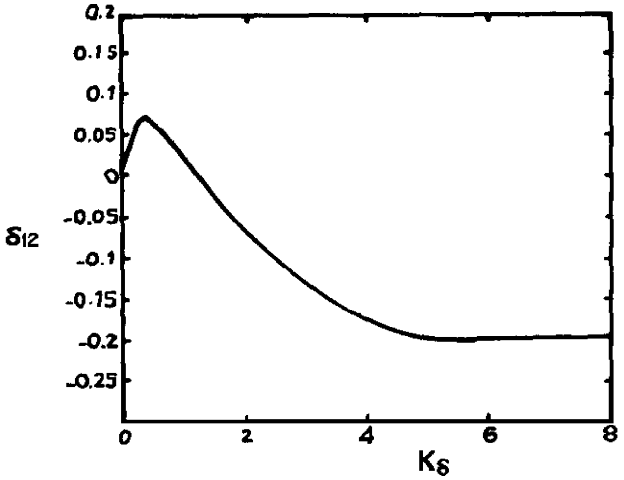

Figure 5. Shock eccentricity factor.

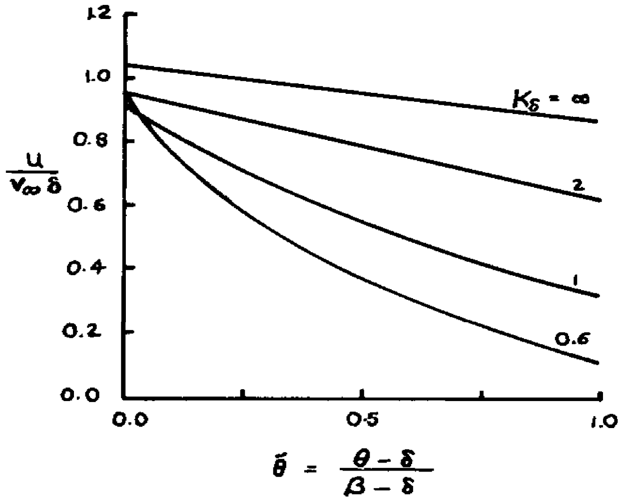

Figure 6. Radial perturbation velocity components.

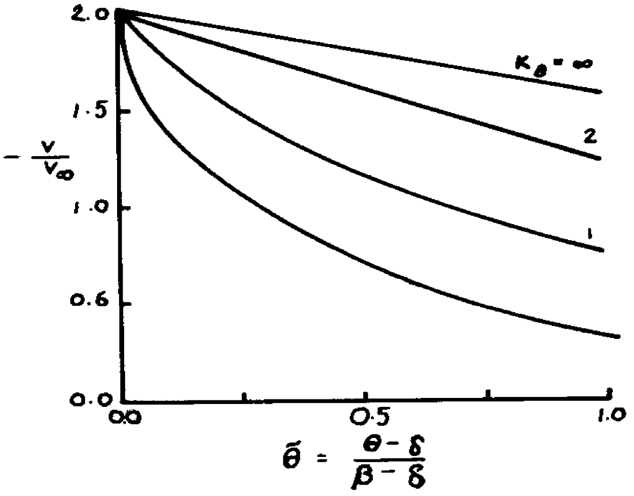

Figure 7. Polar perturbation velocity components.

Newtonian theory [10].

The radial velocity component is plotted in Figure 6. The hypersonic similarity form gives u as a function of z, k_d, and g. Because the thickness of the shock layer varies as a function of k_d, the shock layer is normalized by means of the variable q = (z-1)/(S-1). The body surface corresponds to q = 0 and the shock surface to q = 1. At the body surface u is insensitive to variations in k_d, having approximately the value unity. At the shock surface, it is quite sensitive to variations in k_d. In the hypersonic limit k_d= È, u increases only slightly from the shock to the body.

The polar velocity component v is shown in Figure 7 as a function of q and various values of k_d.The variation of v across the shock layer is analogous to the variation of u.

The azimuthal velocity component w is shown in Figure 8 as a function of q, g=7/5, and various values of k_d. At the shock, w_1 increases as k_d increases. For k_d= 2, variation of w across the shock layer is very slight. At the body surface it decreases as k_d increases. The factor $u^{(c)}(d)/u^{(0)}(d)$ is shown in Figure 9 as a function of k_d and g = 7/5. This is the ratio of the radial component of velocity when the

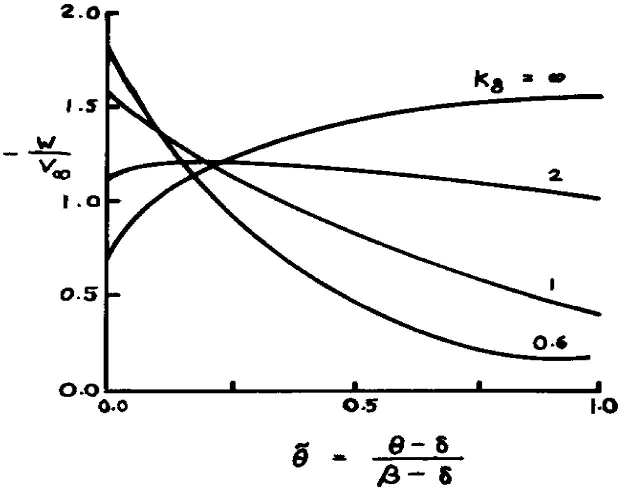

Figure 8. Azimuthal perturbation velocity components.

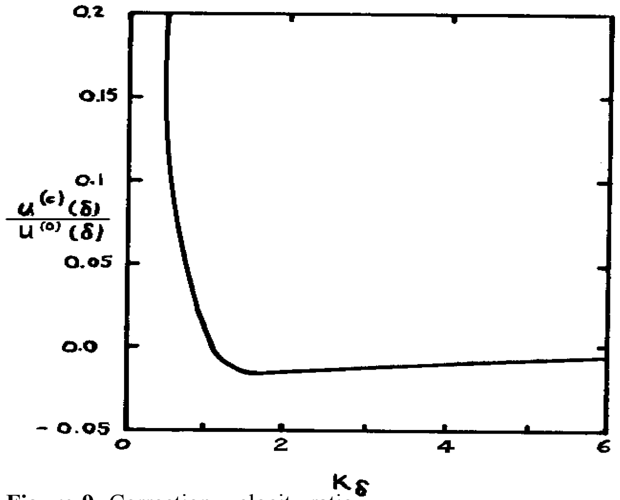

Figure 9. Correction velocity ratio.

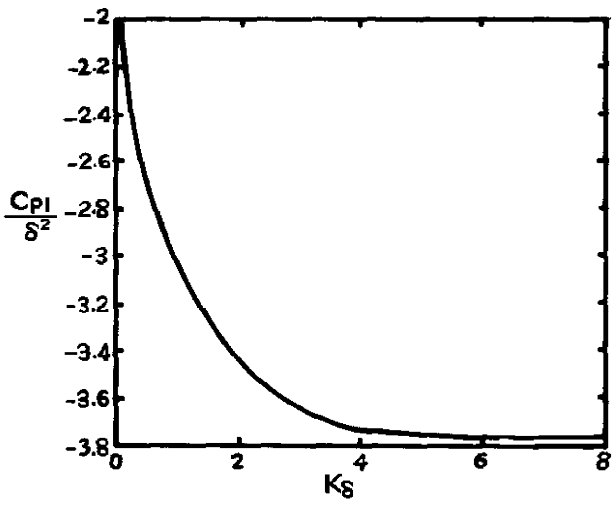

Figure 10. Perturbation pressure coefficient on the body surface.

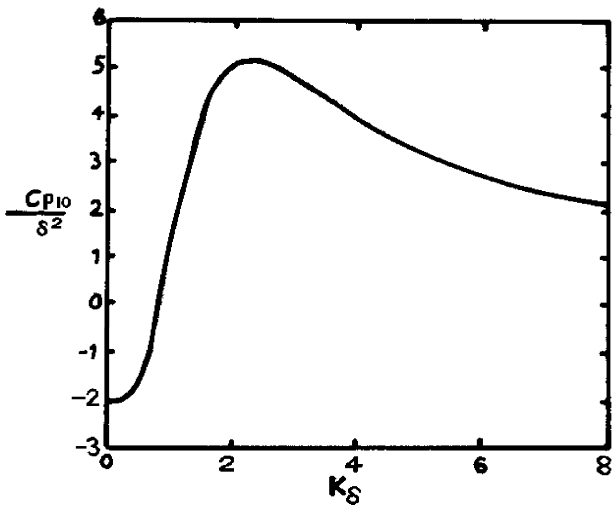

Figure 11. Perturbation pressure coefficient on the body surface.

variable coefficients in Equations 16-18 have not been neglected to when they have been neglected. In the hypersonic flow range, we can expect the approximation obtained by neglecting $u^{(c)}$, to be very accurate. In fact, this is true for k_dÂ1. For k_d=1, the approximation is less accurate. We note,however, that the correct limiting results for k_d= 0, which correspond to the limiting case of linearized theory, are recovered.

Figures 10, 11, and 12 show c_{p1}/d^2, c_{p10}/d^2, and c_{p12}/d^2 plotted as a function of k_d for g=7/5. Surface pressure were measured on two different elliptic cone models, at free stream Mach number 3.09,by Zakkay and Visich [8]. The geometric properties of these models are as follows:

Model I	Model II
e/d =0.155	e/d=0.266
d=16.64 °=0.2904 rad/	d=16.28 °=0.2841 rad

These two models have the same cross sectional areas for the same station along the elliptic-cone axis. The experimental data are compared with the results of the present analysis and also with the analysis of Martellucci [9]. Figure 13 shows the pressure-distribution data on model I for Mach number $M_Ė$=3.09 for which d= 0.900. The present results give good agreement with the data on the semi-major ray, but the data are lower otherwise. The results of Martellucci give a little better agreement with the data between the major and minor rays.

Figure 14 shows the pressure-distribution data on model II for $M_Ė$= 3.09, for which d= 0.888. The agreement with the present analysis is fairly good near the major and minor rays, but poor in between. For this large value ofeccentricity, higher-order perturbation terms are probably required. Martellucci's result shows somewhat better agreement with the data between the major and minor rays.

CONCLUSIONS

General flow field results for the hypersonic flow past an elliptic cone have been obtained. The results are valid for large Mach numbers and small stream deflections such that the hypersonic similarity parameter, K_d= $M_Ė$d, is fixed in the limiting process. The results are more accurate for large k_d, but the proper linearized theory result is recovered when k_dØ0. Second-order perturbation analysis

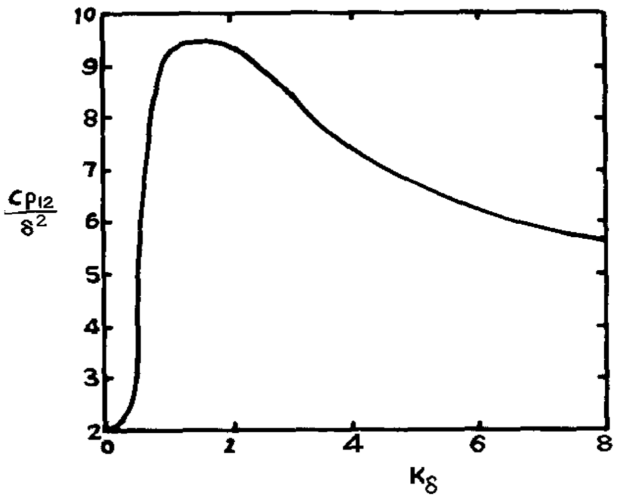

Figure 12. Perturbation pressure coefficient on the body surface.

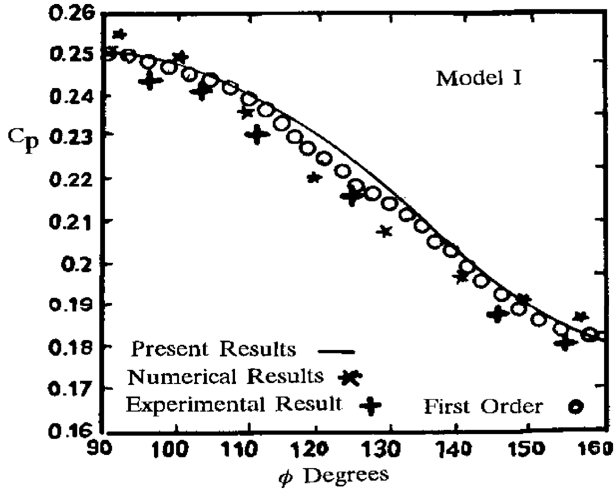

Figure 13. Pressure coefficient on body surface Model I, M = 3.09 and d = 0.900.

proves the existence of vortical layer and singularity of entropy and radial component of velocity and density on the body surface. This layer does not show itself in the zeroth-order and first-order perturbation analysis, since the integration of Equations 13-d and 14-e produces finite and constant values of entropy. It is the second-order entropy that shows the singularity of this quantity on the body and that is because of the term $v_0(d) = 0$ at the denominator of Equation 15-e. It is proved here that in spite of the existence of the vortical layer on the body, the second-order pressure is finite and therefore, the pressure expansion would not be divergent. By proving the validity of the boundary conditions on the surface in the process of vortical layer analysis, our solution procedure here renders the fact that an inner and outer expansions of the quantities are not needed and can be avoided. This is a great simplification and circumvention to the mathematically rigorous inner and outer solutions of the same problem.

LIST OF SYMBOLS

a_0	zeroth-order isentropic speed of sound
A(q), B(q), C(q)	function
c_v	constant density specific heat
c_p	surface pressure coefficient
D_{12}(q)	function
$\hat{e}_r, \hat{e}_q, \hat{e}_f$	unit vectors in spherical coordinate system
F_1	constant
g	shock eccentricity factor associated with elliptic cone
G_{11}, G_{12}, G_{13}	functions
H(q)	function
I(q)	function of integration
K(d)	hypersonic similarity parameter
M	Much number
$\hat{n}$	unit normal vector
p	pressure in outer region
P	pressure in inner region
Q(q)	function
r	spherical coordinate
R	density in inner region
S	entropy
T(q)	function
u, v, w	spherical velocity components in outer region
U, V, W	velocity components in inner region
x(q)	function
z	q/d

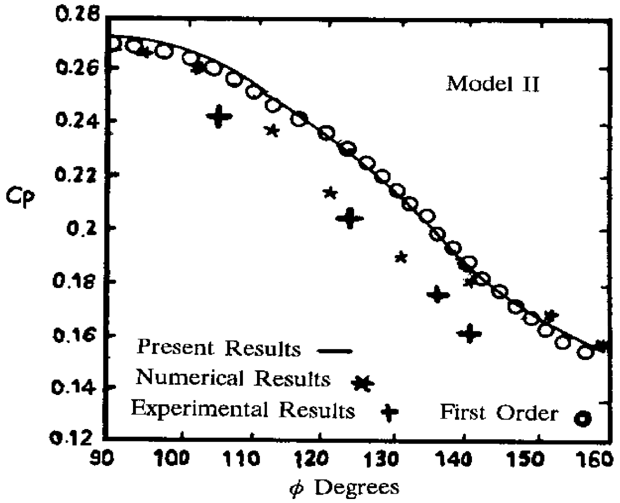

Figure 14. Pressure coefficient on body surface Model II, M = 3.09 and d = 0.888.

GREEKS

b unperturbed shock angle
g ratio of specific heats
d semivertex angle of basic circular cone
e measure of eccentricity
density ratio across the shock
q, f spherical coordinates in outer region
Q, F spherical coordinates in inner region
density
b/d

SUBSCRIPTS

È free stream condition
s condition just downstream of shock
0 zeroth-order
1 first-order
10 second-order
12 second-order

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