EXPLICIT STIFFNESS OF TAPERED AND MONOSYMMETRIC I BEAM-COLUMNS

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Abstract A formulation for finite element analysis of tapered and monosymmetric I shaped beam-columns is presented. This is a general way to analyze these types of complex elements. Based upon the formulation, member stiffness matrix is obtained explicitly. The element considered has seven nodal degrees of freedom. In addition, the related stability matrix is found. Numerical studies of the aforementioned members show the validity and accuracy of the formulation in comparison to the other ones.

Key Words Explicit Stiffness, Tapered Member, Monosymmetric Section, Finite Element, Stability Matrix, Subspace Iteration

»Ç°n§¼¥y±{»«³À̖A¬nB̄kB°n±z@̊B¥ñ{ ̗° B˭̄ṿo¼¼° ÅNy˥kd«°Åj A° p̊w³wⅯ 4k½a ¥B̃c̶̀eeRn±~Ṁsî»Ṫwvu ¼d½Bk,n±̄d¼°p̊w³w̄M° B̊̋doṼS wA²k¼¼⑫2 Åj A³˭±¢@A¥¼d̶U° ÅM¦!-@ ,§ÇAoⅯ4°qS̄Ak{B¾vk»µo£°jA̗C]rj S %µ° Å⅓ ,³̒S£nÅv»wⅿ½Jh±«B̗ @Årk´—»Äj̖ /S wA»̃̃B¾kⅰy ³wⅯ»¼ÆB³°S'j¶k°̊j ─Bz̄²k{jB¾¼d̖ oṼjkî R B̊§̋w« /S wA²k«CS wkⅯd¼⁴x±̄d½k° nAk1̄½u ¼d½Bk /S wAo¢½⁄° B{°nB¾̄v̄⅔«rj²k{³ÀA̖° p̊w

INTRODUCTION

Progress in fabrication techniques of steel members has resulted in optimized use of the material. To gain the lowest weight in steel members, tapered and monosymmetric sections have been widely used. However, the complexity of the section results in difficulty of stability analysis of the members. There is no classical way to determine the critical actions of tapered elements. In the case of monosymmetric members, the critical actions can be found only for some simple loading conditions by solving the differential equations of equilibrium. Numerical methods are generally used for stability analysis of tapered, monosymmetric or tapered and monosymmetric members.

Finite element method is widely used in stability analysis of structures. In the finite element modelling of beam-columns, any complex section, loading or boundary conditions can be considered. In steel frames, members with I sections are used more widely than other shapes. In this article, a finite element formulation is presented for determining elastic critical actions of beam-columns having tapered and monosymmetric I sections. The element used has seven nodal degrees of freedom.

HISTORICAL OVERVIEW

The finite element method has been utilized recently to investigate the stability of I beam-columns. Rajeskaran [1] and Yang and Mcguire [2] have presented the stiffness and stability matrices of such members. Conci [3] has found the stiffness and stability matrices of monosymmetric I beam-columns for finite element analysis. Wekezer [4] used the finite

element method to investigate the instability of thin walled taperd bars. He used total lagrangian formulation where the volume should be divided to triangular subdomains. Yang and Yau [5] presented explicit stiffness and stability matrices of tapered I beam-columns. Flexural torsional buckling of variable and open section columns have been analyzed, using exact shape functions for establishing stiffness and stability matrices, by Eisenberger and Cohen [6].

Utilizing the finite integral method, Kitipornchai and Trahair [7] have offered solutions for buckling loads of tapered and monosymmetric simple I beams. Elastic buckling of tapered and monosymmetric beams have been investigated by Bradford and Cuk [8], using finite elements. They have presented the stiffness and stability matrices of an element with eight degrees of freedom. These matrices are computed using four point Gaussian quadrature. Recently, Rajeskaran [9,10] has found critical actions of tapered thin walled beams of generic open section using the finite element method. The related matrices are also established by numerical integration.

THEORETICAL BASES

The assumptions used for the analysis of a thin walled tapered and monosymmetric I beam-column are [11]:

1- The material is elastic and homogeneous.
2- The section is thin walled.
3- Every cross section is rigid in its own plane.
4- Shearing deformation of the middle surface of the section is negligible.
5- Longitudinal displacements are neglected compared to transverse displacements.

For a tapered and monosymmetric I beam, shear center axis is not parallel to the centroid axis. Therefore, the minor axis bending and
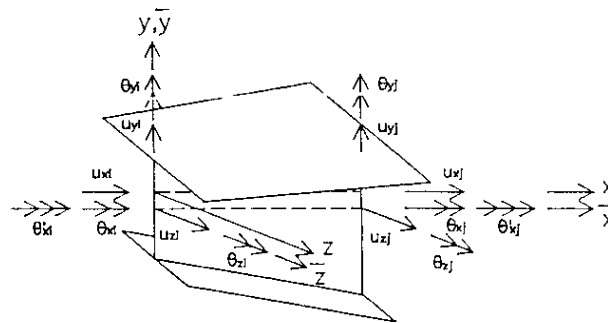

Figure 1. Tapered monosymmetric element.

torsion can not be seperated [7]. The finite element analysis of tapered and monosymmetric I shape element, uses two systems of coordinates, passing through centroid (c) and midheight of the web (o). It has been shown that this leads to the same elastic buckling predictions using the classical systems of coordinates passing through centroid and shear center [8]. As shown in Figure 1, a right handed coordinates system is chosen, where x is the centroidal axis and y and z are principal axes of the cross section. The position vector of any arbitrary point that is located on the midsurface can be expressed as [5]:

$$r = xi + yj + zk \tag{1}$$

Because the element is nonprismatic, this position vector will have derivatives with respect to x, y and z as follows:

$$
\begin{aligned}
r_{,x} &= i + y\mathring{j} + z\mathring{k} \\
r_{,y} &= j \\
r_{,z} &= k
\end{aligned}
\tag{2}
$$

Linear and nonlinear components of strain tensor, neglecting large displacement and rotations, can be generally written as:

$$
\begin{aligned}
e_{mn} &= \frac{1}{2}\,[r_{,m}.u_{,n} + r_{,n}.u_{,m}] &\quad m,n = x,y,z \\
h_{mn} &= \frac{1}{2}\,[u_{,m}.u_{,n}] &\quad m,n = x,y,z
\end{aligned}
$$

By inserting position vector derivatives in strain tensors, strain components can be found as:

$$e_{xx} = u\mathring{_x}_c - yu\mathring{\mathring{_y}}_o - zu\mathring{\mathring{_z}}_o - wq\mathring{\mathring{_x}}_o - y\,q\mathring{_x}_o \tag{1}$$

$$e_{xy} = -\frac{1}{2}\left(z + \frac{\tilde{A}w}{\tilde{A}y}\right) q\mathring{A}_o \qquad (2)$$

$$e_{xz} = -\frac{1}{2}\left[\left(y - \frac{\tilde{A}w}{\tilde{A}z}\right) - y_o\right] q\mathring{A}_o \qquad (3)$$

$$h_{xx} = \frac{1}{2}[u\mathring{A}_{yo}^2 + u\mathring{A}_{zo}^2 - 2zu\mathring{A}_{yo}q\mathring{A}_o + 2yu\mathring{A}_{zo}q\mathring{A}_o$$

$$+ (y^2 + z^2)q\mathring{A}_{xo}^2 + y_o^2 q\mathring{A}_{xo}^2 - 2y_o u\mathring{A}_{zo}q\mathring{A}_{xo}] \qquad (4)$$

$$h_{xy} = \frac{1}{2}[-(u\mathring{A}_{xc} - zu\mathring{A}\mathring{A}_{zo})u\mathring{A}_{yo} + u\mathring{A}_{zo}q\mathring{A}_o$$

$$- y_o q\mathring{A}_{xo}q\mathring{A}_{xo}] \qquad (5)$$

$$h_{xz} = \frac{1}{2}[-(u\mathring{A}_{xc} - yu\mathring{A}\mathring{A}_{yo})u\mathring{A}_{zo} - u\mathring{A}_{yo}q\mathring{A}_{xo}] \qquad (6)$$

in which:

$$w = y\mathring{A}z + z\mathring{A}y$$

$$y = w\mathring{A} - (y - y_o)z\mathring{A} + y\mathring{A}z$$

SECTION PROPERTIES

The areas of the cross section, torsion constant and moments of inertia with reference to Figure 2 are expressed as:

$$A = b_T t_T + b_B t_B + h t_w$$

$$J = \frac{1}{3}(b_T t_T^3 + b_B t_B^3 + h t_w^3)$$

$$I_y = \frac{1}{12}(b_T^3 t_T + b_B^3 t_B)$$
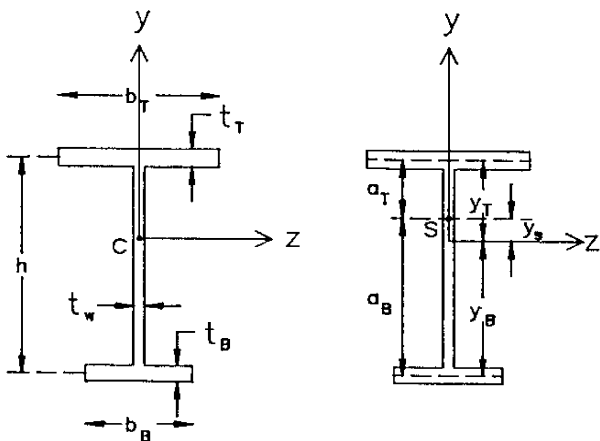
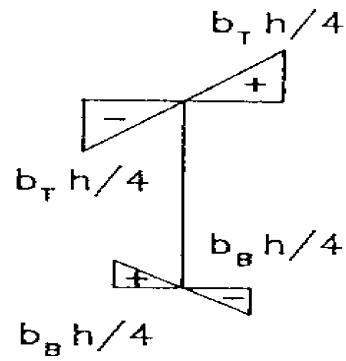$$I_z = b_T t_T y_T^2 + b_B t_B y_B^2$$

Figure 2. Monosymmetric section.

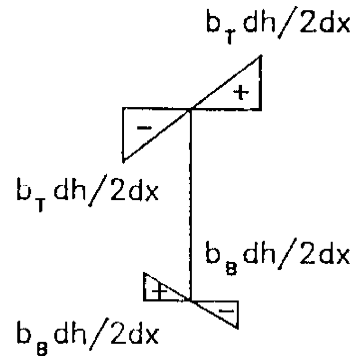**Figure 3.** Diagram of w.

Figure 4. Diagram of y.

The diagrams of w and y using midheight of the web as the pole for sectorial area, are shown in Figures 3 and 4. With reference to these figures, the section properties dependent on w and y can be found as:

$$I_w = \int w^2\, dA = I_y \frac{h^2}{4}$$

$$I_y = \int y^2\, dA = I_y h\mathring{A}^2$$

$$I_{wy} = \int wy\, dA = I_y \frac{hh\mathring{A}}{2}$$

$$I_{yz} = \int yz\, dA = (I_{yT} - I_{yB}) h\mathring{A}$$

$$I_{wz} = \int wz\, dA = (I_{yT} - I_{yB}) \frac{h}{2}$$

Other section properties that will be used in the formulation are y_o, r_{po}^2 and b_E. These are defined by:

$$y_o = \frac{h}{2A}(b_T t_T - b_B t_B)$$

$$r_{po}^2 = \frac{I_y + I_z}{A} + y_o^2$$

$$b_E = \frac{1}{I_z}\{y_T(I_{yT} + b_T t_T y_T^2 + \frac{1}{4}t_w y_T^3) - y_B(I_{yB} +$$

$b_B t_B y_B{}^2 + \frac{1}{4} t_B y_B{}^3)\} - 2y_o$

To find relations between the section properties of the first element node (i), and these properties at any arbitrary point, taper coefficients are used. Showing the properties of the top flange, bottom flange and web, by T, B and w, respectively. These coefficients are defined by:

$$g_T = \frac{b_{Tj} - b_{Ti}}{b_{Ti}}$$

$$g_B = \frac{b_{Bj} - b_{Bi}}{b_{Bi}}$$

$$g_h = \frac{h_j - h_i}{h_i}$$

The thicknesses of the flanges and the web are assumed to be constant along the length of an element. Using taper coefficients, width of flanges and height of the web (distance between flange centroids) at any point along longitudinal axis are found as:

$b_T = (1 + g_T x) b_{T0}$
$b_B = (1 + g_B x) b_{B0}$
$h = (1 + g_h x) h_0$

In these relations, x is nondimensional longitudinal coordinate and is equal to x/L. It should be noted that $h\mathring{=} g_h h_0 / L$, because the taper of the web and also the flanges are assumed to be linear.

With the aid of taper constants, section properties at any point can be written as a function of the properties at x= 0. These are given in Appendix 1.

STRESS RESULTANTS

The cross section stress resultants of a taperd monosymmetric I beam column are defined by [2]:

$F_x = \beta\, s_x\, dA$ (7)

$M_z = - \beta\, s_x\, y\, dA$ (8)

$M_y = \beta\, s_x\, z\, dA$ (9)

$F_y = \beta\, t_{xy}\, dA$ (10)

$F_z = \beta\, t_{xz}\, dA$ (11)

$M_x = \beta\, [t_{xz}(y - y_s) - t_{xy}(z - z_s)]\, dA$ (12)

$M_w = - \beta\, s_x\, w\, dA$ (13)

where s_x is the normal stress and t_{xy} and t_{xz} are shear stresses. The normal stress may be found from Hook's law as follows:

$s_x = E\, e_x$

Stress resultants can be written in terms of the strains by substituting s_x in eqs (7) to (9) and Equation 13 as:

$EAu\mathring{A}_{xc} = F_x$

$- EI_y u\mathring{A}\mathring{A}_{zs} = M_y$

$EI_z u\mathring{A}\mathring{A}_{ys} = M_z$

$EI_w q\mathring{A}\mathring{A}_{xs} = M_w$

VIRTUAL WORK EQUATION

Internal linear virtual work can be generally written as:

$$W_{is} = \frac{1}{2} \beta \{ E\, d(e_{xx}{}^2) + 2G\, d(e_{xy}{}^2 + e_{xz}{}^2) \}\, dv \tag{14}$$

Using linear strain expressions in Equation 58 will lead to:

$$W_{is} = \frac{1}{2} \beta_0{}^L E \{ A\, d(u\mathring{A}_{xc}{}^2) + I_y\, d(u\mathring{A}\mathring{A}_{zo}{}^2) + I_z$$

$d(u\mathring{A}\mathring{A}_{yo}{}^2) + I_w\, d(q\mathring{A}\mathring{A}_{xo}{}^2)$

$+ I_y\, d(q\mathring{A}_{xo}{}^2) + 2I_{wy}\, d(q\mathring{A}_{xo} q\mathring{A}\mathring{A}_{xo}) + 2I_{yz}$

$$d(q\mathring{A}_{xo} u\mathring{A}\mathring{A}_{zo}) + \frac{G}{E} J\, d(q\mathring{A}_{xo}{}^2) \}\, dx \tag{15}$$

It should be noted that I_{wz} will be equal to zero, if shear center is used as the pole of sectorial area. Internal geometric virtual work can be divided into two parts. The first part is common between symmetric and monosymmetric sections and is shown by superscript s. The second part is found only in monosymmetric sections and is denoted by superscript m. Internal geometric virtual work is generally expressed as:

$$W_{ig} = \beta \{ s_x\, dh_{xx} + 2\, t_{xy}\, dh_{xy} + 2\, t_{xz}\, dh_{xz} \}\, dv \tag{16}$$

Substituting Equations 4 to 6 in Equation 16

will lead to:

$$W_{ig}^{s} = \int_0^L \{ \frac{1}{2} F_x d(u'^2_{yo}+u'^2_{zo}+r^2_{po}q'^2_{xo}) -$$

$$M_z d(u'_{zo}q'_{xo}) - M_y d(u'_{yo}q'_{xo}) - F_y d(u'_{xc}u'_{yo}$$

$$- u'_{zo}q_{xo}) - F_z d(u'_{xc}u'_{zo}+ u_{yo}oq_{xo}) \} dx \quad (17)$$

$$W_{ig}^{m} = \int_0^L \{ - F_x d(y_o u'_{zo}q'_{xo}) - F_y d(y_o q_{xo}q'_{xo})$$

$$- \frac{1}{2} M_z d(b_E q'^2_{xo}) \} dx \qquad (18)$$

The principle of virtual work in an updated Lagrangian formulation is written as:

$$W_{is} + W_{ig}^{s} + W_{ig}^{m} = W_{e2} - W_{e1} \qquad (19)$$

where W_{e2} and W_{e1} are external works in the current and neighboring deformed equilibrium state. It should be reminded that in an updated Lagrangian formulation the local reference state for each element is the current configuration.

FINITE ELEMENT FORMULATION

To use finite elements for the stability analysis of a beam-column, stiffness and stability matrices are needed. They are found by using shape functions which relate the displacements of any arbitrary point to the nodal degrees of freedom. Linear shape function is used for axial displacement and a cubic one for other displacements. These shape functions are defined by:

$$[N_1] = [(1- x) \quad (x)]$$

$$[N_3] = [(1- 3x^2+2 x^3) \quad L(x- 2x^2+ x^3) \quad (3x^2- 2x^3)$$

$$L(- x^2+ x^3)]$$

Using these functions, the incremental displacements can be expressed as follows:

$$u_{xc} = [N_1] \{D_x\} \qquad (20)$$

$$u_{yo} = [N_3] \{D_y\} \qquad (21)$$

$$u_{zo} = [N_3] \{D_z\} \qquad (22)$$

$$q_{xo} = [N_3] \{D_q\} \qquad (23)$$

According to Figure 1, the nodal degrees of freedom are defined by:

$$\{D_x\} = \{ u^c_{xi} \quad u^c_{xj} \}$$

$$\{D_y\} = \{ u^o_{yi} \quad q^o_{zi} \quad u^o_{yj} \quad q^o_{zj} \}$$

$$\{D_z\} = \{ u^o_{zi} - q^o_{yi} \quad u^o_{zj} - q^o_{yj} \}$$

$$\{D_q\} = \{ q^o_{xi} \quad q'^o_{xi} \quad q^o_{xj} \quad q'^o_{xj} \}$$

Subscripts c and o are used to show that the generalized displacements are referred to centroid or midheight of the web. By substituting Equations 20 to 23, internal parts of virtual work can be written as:

$$W_{is} = \{dD_x\}^T \int (EA/L)[N'_1]^T[N'_1] dx \{D_x\} + \{dD_y\}^T \int (EI_z/L^3)[N''_3]^T[N''_3] dx \{D_y\} + \{dD_z\}^T [\int (EI_y/L^3)[N''_3]^T[N''_3] dx \{D_z\} + \int (EI_{yz}/L^3)[N''_3]^T[N'_3] dx \{D_q\} + \int EI_{wz}/L^3) [N''_3]^T[N''_3] dx \{D_q\}] + \{dD_y\}^T [\int (EI_w/L^3) [N''_3]^T[N''_3] dx \{D_q\} + \int (EI_y/L)[N'_3]^T[N'_3] dx \{D_q\} + \int (EI_{wy}/L^2) ([N'_3]^T [N''_3] + [N''_3]^T [N'_3]) dx \{D_q\} + \int (EI_{yz}/L^3) [N'_3]^T [N''_3] dx \{D_z\} + \int (EI_{wz}/L^3) [N''_3]^T [N''_3] dx \{D_z\} + \int (GJ/L) [N'_3]^T [N'_3] dx \{D_q\}] \qquad (24)$$

$$W_{ig}^{s} = \{dD_x\}^T [\int ((M_{zi}+M_{zj})/L^2) [N'_1]^T[N'_3] dx \{D_y\} - \int ((M_{yi}+M_{yj})/L^2) [N'_1]^T[N'_3] dx \{D_z\}] + \{dD_y\}^T [\int ((M_{zi}+M_{zj})/L^2)[N'_3]^T[N'_1] dx \{D_x\} + \int (F_{xj}/L) [N'_3]^T[N'_3] dx \{D_y\} + \int ((M_{yi}/L) (1-x)- (M_{yj}/L)x) [N'_3]^T[N'_3] dx \{D_q\} - \int ((M_{yi}+M_{yj})/L) [N'_3]^T[N'_3] dx \{D_q\}] + \{dD_z\}^T [\int (- (M_{yi}+M_{yj})/L^2) [N'_3]^T [N'_1] dx \{D_x\} + \int (F_{xj}/L) [N'_3]^T[N'_3] dx \{D_z\} + \int ((M_{zi}/L) (1-x)- (M_{zj}/L)x) [N'_3]^T[N'_3] dx \{D_q\} - \int ((M_{zi}+M_{zj})/L) [N'_3]^T[N'_3] dx \{D_x\}] + \{dD_q\}^T [\int (- (M_{yi}+M_{yj})/L) [N_3]^T[N'_3] dx \{D_y\} + \int ((M_{yi}/L) (1-x)- (M_{yj}/L)x) [N'_3]^T[N'_3] dx \{D_y\} - \int ((M_{zi}+M_{zj})/L) [N_3]^T[N'_3] dx \{D_z\} + \int ((M_{zi}/L)(1-x)- (M_{zj}/L)x) [N'_3]^T[N'_3] dx \{D_z\} + \int (F_{xj}/L)r^2_{po} [N'_3]^T[N'_3] dx \{D_q\}] \qquad (25)$$

$W_{ig}^m = \{dD_z\}^T \int (- (F_{xj}/L) \, y_0 \, [N\mathring{A}]^T[N\mathring{A}] \, dx$
$\{D_q\} + \{dD_q\}^T \, [\int (- (F_{xj}/L) \, y_0[N\mathring{A}]^T[N\mathring{A}] \, dx$
$\{D_z\} - \int ((M_{zi}+M_{zj}/L) \, y_0 \, ([N\mathring{A}]^T[N_3] + [N_3]^T$
$[N\mathring{A}]) \, dx \, \{D_q\} + \int ((M_{zi}/L) \, (1-x) + (M_{zj}/L)x)$
$b_E \, [N\mathring{A}]^T[N\mathring{A}]) \, dx \, \{D_q\} \,]$ (26)

It should be noted that in these equations, the stress resultants at section x of the element have been expressed in terms of those at the element nodes as [2]:

$F_x = F_{xj})$
$F_y = - (M_{zi} + M_{zj})/L$
$F_z = (M_{yi} + M_{yj})/L$
$M_x = M_{xj}$
$M_y = - M_{yi} (1-x/L) + M_{yj} (x/L)$
$M_z = - M_{zi} (1-x/L) + M_{zj} (x/L)$

Substituting for section properties in eqs (24) to (26), and performing the integration, the stiffness matrix ([s]) and stability matrix ([g]) are derived. The stability matrix is divided into two parts. The first part is found from Equation 75 whose notation is $[g_s]$. The second part, which is derived from Equation 76, is shown by $[g_m]$. The matrices [s] and $[g_s]$ are found completely in explicit form. In $[g_m]$, the terms having b_E can not be derived explicitly. If b_E is written in terms of x, it will have a sixth degree polynomial as denominator, therefore the terms containing it can not be integrated explicitly. For these terms, the integration with respect to x is carried out by three point Gaussian quadrature. The stiffness and stability matrices are given in appendices (2), (3), and (4). It should be noted that these matrices are symmetric, so only the nonzero entries of the upper triangle will be given in the appendices.

SUBSPACE ITERATION

The global stiffness and stability matrices of a beam-column ([S], $[G_s]$ and $[G_m]$) are found by assembling element matrices ([s], $[g_s]$ and $[g_m]$). The stiffness matrix is considered to be constant and the stability matrix as a multiple of its base value obtained from a base load. The equation of equilibrium can be written as:

$([S] + l \, [G]) \, \{f\} = \{0\}$ (27)

where $[G]=[G_s]+[G_m]$ is the total base stability matrix and l is a load factor that will cause buckling. It should be noted that {f} is the buckling mode shape. To find the critical eigenvalue l, a subspace iteration with shift is used. It should be mentioned that in a monosymmetric section, the stability matrix is not positive definite. It could have zero, negative or positive diagonal terms, depending on the value of b_E. To solve the eigenproblem, first stiffness and stability matrices are interchanged, leading to:

$[G] \, \{f\} = \frac{1}{l} \, [S] \, \{f\}$ (28)

Then a shift is applied causing the stability matrix to become positive definite, finding a new eigenproblem written as:

$([G] - r \, [S]) \, \{f\} = m[S] \, \{f\}$

where r is the shift and m is a new eigenvalue. Using the subspace iteration method, m is found. After adding r and m, the biggest and smallest eigenvalues are symmetric with respect to zero. Therefore, the biggest negative eigenvalue of Equation 28 is $-(r+m)$. To find the smallest eigenvalue of Equation 27, $-(r+m)$ should be inversed, so that l can be derived as:

$l_{min} = - \frac{1}{m+r}$

The shift is proposed to be initialized to -0.0001. If after shifting, the stability matrix does not become positive definite, the shift should be increased.

NUMERICAL STUDIES

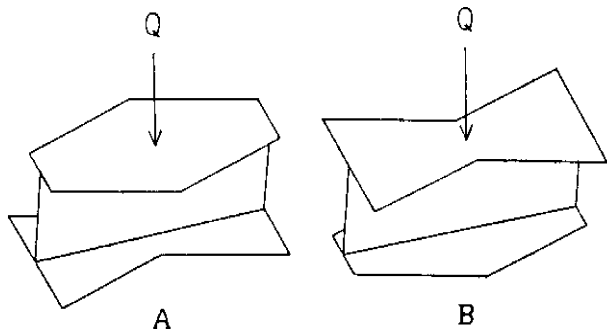A computer program has been designed to perform geometrical nonlinear analysis using

Figure 5. Beam section.

the derived stiffness and stability matrices. Critical eigenvalue is found by subspace iteration with a shift. To show the accuracy of the method, one example of tapered and monosymmetric simple beam is solved and compared to the existing solutions. Two new problems are also solved.

Example 1- A simple monosymmetric beam that has linearly varying flange width and is under central concentrated load (Q), as shown in Figure 5, is considered. The same beam has been investigated using the finite integral method by Kitipornchai and Trahair [7], and also by Bradford and Cuk [8], and Rajeskaran [10] by finite elements. The web height and thickness are 72.76mm (2.865in) and 2.13mm (0.084in), respectively. The flange thickness is 3.11mm (0.1225in) and the maximum width of the flange is equal to 31.55mm (1.242in). The ratio of the smallest to the largest width of the flange is shown by a. The material is aluminum with E= 65.16GPa (9450Ksi) and G= 25.65GPa (3740Ksi). In type A, the top flange is wider at midspan, while in type B, it is wider at the supports. The results of using ten elements for beam modelling are shown in Figure 6. Good agreement exists between the results of this paper and those of Bradford and Cuk and also Rajeskaran. It should be added that using finite integral method, results differ appreciably in
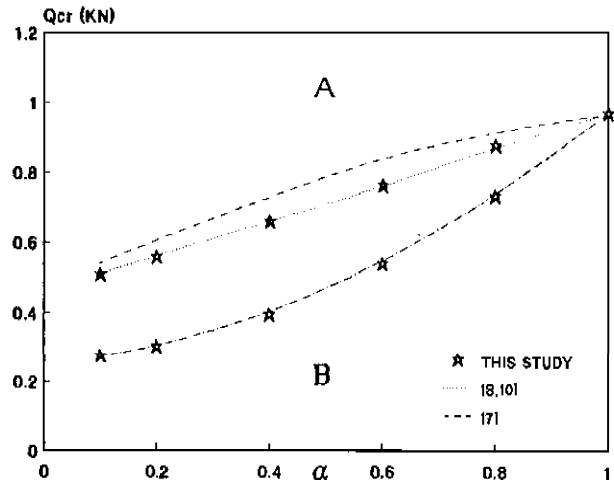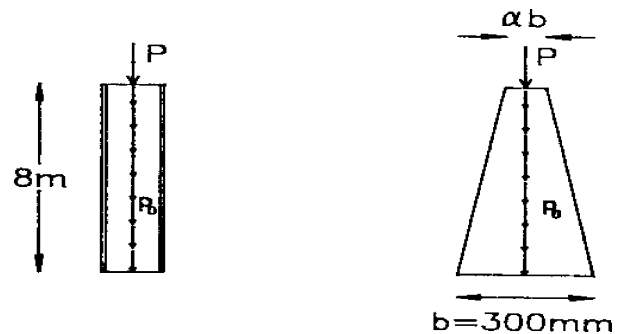

Figure 6. Critical load.
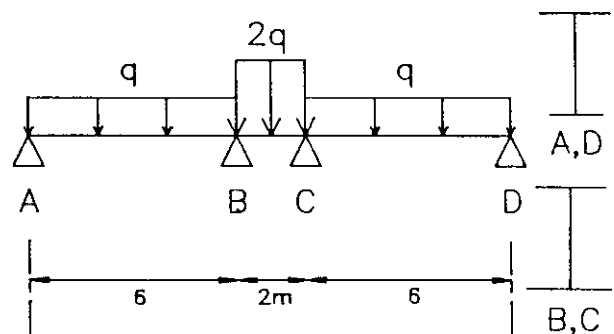

Figure 7. Tapered column.

Figure 8. Continuous beam.

type A.

Example 2- An I section with linearly tapered flanges is shown in Figure 7. It is under a concentrated axial load (P) at its top and uniformly distributed axial load (p_0) along its length, so that $p_0L = P/4$. The section dimensions at bottom are: b = 300, h = 289, tf

= 19 and tw=11mm. The length of the column is 8m and the modulus of elasticity and shear modulus are equal to 200GPa and 80GPa, respectively. The width of the flange at the top end equals ah, where a is the taper ratio. The critical load (P_{cr}) has been found for four types of boundary conditions and presented in Table 1. It can be seen that in tapered columns, the critical load does not change along prismatic columns by differing boundary conditions.

Example 3- A statically indeterminate beam has three spans and is under distributed load on its top flange (q and $2q$), as shown in Figure 8. The top flange width and the web height are constant and equal to 200 and 300 millimeters, respectively. The thickness of the web and the flanges are also constant and are tw=8mm and tf= 15mm. The width of the bottom flange varies linearly from 150mm at the outer supports to 250mm at the middle supports, so that the beam is prismatic in the middle span. In this situation, when the beam is under positive moment (top fibers in compression), the top flange is larger and vice versa. It should benoted

that this structure is statically indeterminate, so it should be analyzed based on its nonprismatic section properties. In order to consider the variations of the moments, sixteen elements are used for modelling the beam. Assuming E=200GPa and G=80GPa, the obtained critical load is q_{cr}=63.72KN/m.

CONCLUSION

After a brief review of the developments in the area of stability analysis of thin walled I beam-columns, a formulation is presented. The finite element method uses this formulation along with an element which has fourteen degrees of freedom. All entries of the stiffness matrix for tapered and monosymmetric elements are calculated explicitly. Most of the entries of the related stability matrix are explicitly found as well. Finally, this formulation is used to solve some examples. Numerical results show the validity and also the accuracy of the derived matrices, compared to other techniques.

TABLE 1. Critical Load P_{cr}(KN).

a	0.2	0.4	0.6	0.8	1
cantilever	152	275	389	498	604
hinged-hinged	268	646	1116	1673	2310
hinged-fixed	542	1346	2352	3537	4887
fixed-fixed	1026	2551	4457	6697	9246

APPENDIX 1. SECTION PROPERTIES AND TAPER CONSTANTS

$A = (1+C_a x)A_0$, $J = (1+C_j x)J_0$, $I_{yT} = (1+C_{it1}x + C_{it2}x^2 + C_{it3}x^3)I_{yT0}$,

$I_{yB} = (1+C_{ib1}x+C_{ib2}x^2+C_{ib3}x^3)I_{yB0}$, $I_y = (1+C_{iy1}x+C_{iy2}x^2+C_{iy3}x^3)I_{y0}$,

$I_z = \dfrac{1+C_{iz1}x+C_{iz2}x^2+C_{iz3}x^3+C_{iz4}x^4+C_{iz5}x^5}{(1+C_a x)^2} I_{z0}$, $I_W = (1+C_{iW1}x +C_{iW2}x^2+ C_{iW3}x^3 + C_{iW4}x^4 + C_{iW5}x^5)I_{W0}$,

$I_y = (1 + C_{iy1}x + C_{iy2}x^2 + C_{iy3}x^3)I_{y0}$, $I_{Wy} = (1 + C_{Wy1}x + C_{Wy2}x^2 + C_{Wy3}x^3 + C_{Wy4}x^4)I_{Wy0}$, $I_{yz} = I_{yz0}$

$+ C_{yz1}x + C_{yz2}x^2 + C_{yz3}x^3$, $I_{wz} = I_{wz0} + C_{wz1}x + C_{wz2}x^2 + C_{wz3}x^3$, $y_o = \dfrac{y_{o0} + C_{yo1}x + C_{yo2}x^2}{1 + C_a x}$,

$r_{po}^2 = \dfrac{r_{po0}^2 + C_{rp1}x + C_{rp2}x^2 + C_{rp3}x^3 + C_{rp4}x^4 + C_{rp5}x^5}{(1 + C_a x)^3}$

In the presented taper constants, AT0, AB0 and AW0 are the areas of the top flange, bottom flange and web, respectively.

$C_a = \dfrac{1}{A_0}(A_{T0}g_T + A_{B0}g_B + A_{W0}g_h)$, $C_j = \dfrac{1}{3J_0}(b^3_{T0}t_T g_T + b^3_{B0}t_B g_B + h^3_0 t_w g_h)$, $C_{it1} = 3g_T$, $C_{it2} = 3g^2_T$, $C_{it3} = g^3_T$

$C_{ib1} = 3g_B$, $C_{ib2} = 3g^2_B$, $C_{ib3} = g^3_B$, $C_{iy1} = \dfrac{1}{I_{y0}}(C_{ib1}I_{yB0} + C_{it1}I_{yT0})$, $C_{iy2} = \dfrac{1}{I_{y0}}(C_{ib2}I_{yB0} + C_{it2}I_{yT0})$, $C_{iy3} = \dfrac{1}{I_{y0}}$

$(C_{ib3}I_{yB0} + C_{it3}I_{yT0})$, $C_{izn} = (A_{B0} + A_{T0})(4A_{B0}A_{T0} + A^2_{W0}) + 8A_{B0}A_{T0}A_{W0}$, $C_{iz1} = \dfrac{1}{C_{izn}}[4A^2_{T0}A_{B0}(g_B + 2g_h +$

$2g_T) + 4A_{T0}A^2_{B0}(2g_B + 2g_h + g_T) + 8A_{T0}A_{B0}A_{W0}(g_B + 3g_h + g_T) + A_{T0}A^2_{W0}(4g_h + g_T) + A_{B0}A^2_{W0}(g_B + 4g_h)]$,

$C_{iz2} = \dfrac{1}{C_{izn}}[A^2_{T0}A_{B0}(8g_B g_h + 8g_B g_T + 4g^2_h + 16 g_h g_T + 4g^2_T) + A_{T0}A^2_{B0}(4g^2_B + 16 g_B g_h + 8 g_B g_T + 4g^2_h + 8 g_h g_T)$

$+ A_{T0}A_{B0}A_{W0}(24 g_B g_h + 8 g_B g_T + 24 g^2_h + 24 g_h g_T) + A_{T0}A^2_{W0}(6g^2_h + 4 g_h g_T) + A_{B0}A^2_{W0}(4 g_B g_h + 6 g^2_h)]$,

$C_{iz3} = \dfrac{1}{C_{izn}}[4A^2_{T0}A_{B0}(g_B g^2_h + 4 g_B g_h g_T + g_B g^2_T + 2 g^2_h g_T + 2 g_h g^2_T) + 4A_{T0}A^2_{B0}(2g^2_B g_h + g^2_B g_T + 2 g_B g^2_h +$

$4 g_B g_h g_T + g^2_h g_T) + 8g_h A_{T0}A_{B0}A_{W0}(3 g_B g_h + 3 g_B g_T + g^2_h + 3 g_h g_T) + 2g^2_h A_{T0}A^2_{W0}(2g_h +$

$3 g_T) + 2 g^2_h A_{B0}A^2_{W0}(3g_B + 2g_h)]$, $C_{iz4} = \dfrac{1}{C_{izn}}[4g_h g_T A^2_{T0}A_{B0}(2g_B g_h + 2 g_B g_T + g_h g_T) + 4 g_B g_h A_{T0}A^2_{B0}(g_B g_h$

$+ 2 g_B g_T + 2 g_h g_T) + 8g^2_h A_{T0}A_{B0}A_{W0}(g_B g_h + 3 g_B g_T + g_h g_T) + g^3_h A_{T0}A^2_{W0}(g_h + 4g_T) + g^3_h A^2_{B0}A^2_{W0}(4g_B$

$+ g_h)]$, $C_{iz5} = \dfrac{1}{C_{izn}}[4g_B g^2_h g^2_T A^2_{T0}A_{B0} + 4g^2_B g^2_h g_T A_{T0}A^2_{B0} + 8 g_B g^3_h g_T A_{T0}A_{B0}A_{W0} + g^4_h g_T A_{T0}A^2_{W0} +$

$g_B g^4_h A_{B0}A^2_{W0}]$, $C_{iw1} = \dfrac{1}{I_{y0}}(C_{ib1}I_{yB0} + C_{it1}I_{yT0} + 2g_h I_{y0})$, $C_{iw2} = \dfrac{1}{I_{y0}}[(2C_{ib1}g_h + C_{ib2})I_{yB0} + (2C_{it1}g_h + C_{it2})$

$I_{yT0} + g^2_h I_{y0}]$, $C_{iw3} = \dfrac{1}{I_{y0}}[(C_{ib1}g^2_h + 2C_{ib2}g_h + C_{ib3})I_{yB0} + (C_{it1}g^2_h + 2C_{it2}g_h + C_{it3})I_{yT0}]$, $C_{iw4} = \dfrac{1}{I_{y0}}$

$[(C_{ib2}g^2_h + 2C_{ib3}g_h)I_{yB0} + (C_{it2}g^2_h + 2C_{it3}g_h)I_{yT0}]$, $C_{iw5} = \dfrac{1}{I_{y0}}[C_{ib3}g^2_h I_{yB0} + C_{it3}g^2_h I_{yT0}]$, $C_{wy1} = \dfrac{1}{I_{y0}}$

$[C_{ib1}I_{yB0} + C_{it1}I_{yT0} + g_h I_{y0})$, $C_{wy2} = \dfrac{1}{I_{y0}}[(C_{ib1}g_h + C_{ib2})I_{yB0} + (C_{it1}g_h + C_{it2})I_{yT0}]$, $C_{wy3} = \dfrac{1}{I_{y0}}[(C_{ib2}g_h + C_{ib3})I_{yB0}$

$+ (C_{it2}g_h + C_{it3})I_{yT0}]$, $C_{wy4} = \dfrac{1}{I_{y0}}[C_{ib3}I_{yB0} + C_{it3}I_{yT0}]$, $C_{yz1} = \dfrac{g_h h_0}{L}[C_{it1}I_{yT0} - C_{ib1}I_{yB0}]$, $C_{yz2} = \dfrac{g_h h_0}{L}$

$[C_{it2}I_{yT0} - C_{ib2}I_{yB0}]$, $C_{yz3} = \dfrac{g_h h_0}{L}[C_{it3}I_{yT0} - C_{ib3}I_{yB0}]$, $C_{wz1} = \dfrac{h_0}{2}[C_{it1}I_{yT0} - C_{ib1}I_{yB0} + g_h(I_{yT0} - I_{yB0})]$, $C_{wz2} = \dfrac{h_0}{2}$

$[(C_{it1}g_h + C_{it2})I_{yT0} - (C_{ib1}g_h + C_{ib2})I_{yB0}]$, $C_{wz3} = \dfrac{h_0}{2}[(C_{it2}g_h + C_{it3})I_{yT0} - (C_{ib2}g_h + C_{ib3})I_{yB0}]$, $C_{wz4} = \dfrac{h_0}{2}$

$[C_{it3}g_h I_{yT0} - C_{ib3}g_h I_{yB0}]$, $C_{ya0} = \dfrac{I_{y0}}{A_0} = \dfrac{1}{12A_0}(b^3_{B0}t_B + b^3_{T0}t_T)$, $C_{ya1} = \dfrac{1}{4A_0}(b^3_{B0}t_B g_B + b^3_{T0}t_T g_T)$, $C_{ya2} = \dfrac{1}{4A_0}$

$(b^3_{B0}t_B g^2_B + b^3_{T0}t_T g^2_T)$, $C_{ya3} = \dfrac{1}{12A_0}(b^3_{B0}t_B g^3_B + b^3_{T0}t_T g^3_T)$, $C_{za0} = \dfrac{I_{z0}}{A_0} = \dfrac{h^2_0}{4A^3_0}[4A^2_{B0}A_{T0} + A_{B0}$

$(4A^2_{T0}+8A_{T0}A_{W0}+A^2_{W0}) + A_{T0}A^2_{W0}]$, $C_{za1}= \dfrac{h^2_0}{4A^3_0}\{4A^2_{B0}A_{T0}(2g_B+2\,g_h+\,g_\Gamma)+A_{B0}[4A^2_{T0}\,(g_B+2\,g_h+2\,g_\Gamma)$

$+\ 8A_{T0}A_{W0}\,(g_B+3\,g_h+g_\Gamma)+A^2_{W0}(g_B+4\,g_h)\ ]+\ A_{T0}A^2_{W0}\ ]\ (4g_h+g_\Gamma)\}$, $C_{za2}= \dfrac{h^2_0}{2A^3_0}\{2A^2_{B0}A_{T0}$

$(g^2_B+\,4g_Bg_h+2\,g_Bg_\Gamma+g^2_h+2\,g_hg_\Gamma)\ +\ A_{B0}\,[2A^2_{T0}\,(\,2g_Bg_h+2\,g_Bg_\Gamma+g^2_h+4\,g_hg_\Gamma+g^2_T)\ +\ 4\,A_{T0}A_{W0}$

$(3g_Bg_h+g_Bg_\Gamma+3\,g^2_h+3\,g_hg_\Gamma)\ +\ A^2_{W0}\,(2g_Bg_h+3\,g^2_h)\ ]\ +\ A_{T0}A^2_{W0}\ ]\,(3g^2_h+2\,g_hg_\Gamma)\,\}$, $C_{za3}= \dfrac{h^2_0}{2A^3_0}\{2A^2_{B0}A_{T0}$

$(2g^2_Bg_h+\,g^2_Bg_\Gamma+2g_Bg^2_h+4g_Bg_hg_\Gamma+g^2_hg_\Gamma)\ +\ A_{B0}\,[2A_{T0}\,(g_Bg^2_h\ +\ 4g_Bg_hg_\Gamma\ +\ g_Bg^2_T\ +\ 2g^2_hg_\Gamma\ +\ 2g_hg^2_T)\ +$

$4A_{T0}A_{W0}(3g_Bg^2_h+3\,g_Bg_hg_\Gamma+g^3_h+3\,g^2_hg_\Gamma)\ +A^2_{W0}\,(3g_Bg^2_h+2\,g^3_h)]+A_{T0}A^2_{W0}]\,(2g^3_h+3\,g^2_hg_\Gamma)\,\}$, $C_{za4}= \dfrac{g_hh^2_0}{4A^3_0}$

$\{4A^2_{B0}A_{T0}(g^2_Bg_h+2g^2_Bg_\Gamma+2g_Bg_\Gamma g_h)+A_{B0}[4A^2_{T0}(2g_Bg_\Gamma g_h+2g_Bg^2_T+g_hg^2_T)+8A_{T0}A_{W0}(g_Bg^2_h+\,3g_Bg_hg_\Gamma+$

$g^2_hg_\Gamma)\ +\ 2A^2_{W0}\,(4g_Bg_h+g^2_h)\ ]\ +\ A_{T0}A^2_{W0}]\,(g^3_h+4\,g^2_hg_\Gamma)\,\}$, $C_{za5}= \dfrac{g^2_hh^2_0}{4A^3_0}\{4A^2_{B0}A_{T0}g^2_B\ +\ A_{B0}\,[4A^2_{T0}$

$g_Bg^2_T+8A_{T0}A_{W0}g_Bg_hg_\Gamma+g^2_hg_\Gamma+A^2_{W0}g_Bg^2_h]\ +\ A_{T0}A^2_{W0}g^2_hg_\Gamma)\,\}$, $C_{yo0}=y_{o0}=\dfrac{h_0}{2A_0}[A_{B0}-A_{T0}]$, $C_{yo1}=\dfrac{h_0}{2A_0}[$

$(g_B+g_h)\,A_{B0}-\,(g_\Gamma+g_h)\,A_{T0}\,]$, $C_{yo2}=\dfrac{g_hh_0}{2A_0}[g_BA_{B0}-g_\Gamma A_{T0}]$, $C_{rp0}=r^2_{po0}=C_{ya0}+C^2_{yo0}+C_{za0}$, $C_{rp1}=$

$C_a(2C_{ya0}+C^2_{yo0})+C_{ya1}+2C_{yo0}C_{yo1}+C_{za1}$, $C_{rp2}=C^2_aC_{ya0}+2C_a\,(C_{ya1}+C_{yo0}C_{yo1})+C_{ya2}+2C_{yo0}C_{yo2}+$

$C^2_{yo1}+C_{za2}$, $C_{rp3}=C^2_aC_{ya1}+C_a(2C_{ya2}+2C_{yo0}C_{yo2}+C^2_{yo1})+C_{ya3}+2C_{yo1}C_{yo2}+C_{za3}$, $C_{rp4}=C^2_aC_{ya2}+$

$2C_a(C_{ya3}+C_{yo1}C_{yo2})+C^2_{yo2}+C_{za4}$, $C_{rp5}=C^2_aC_{ya3}+C_aC^2_{yo2}+C_{za5}$

APPENDIX 2. STIFFNESS MATRIX

Only the elements that are related to I_z, I_{yz} and I_{wz} will be given here. The other elements are identical to those of Yang and Yau [5], replacing C_a, C_j, C_y, C_w and C_i in that paper with C_a, C_j, C_{iy}, C_{iw} and C_{wy}, given in appendix (1), respectively. First, the elements which are used when $C_a \cancel{=} 0$, related to I_z are given:

$s(2,2)=-s(2,9)=s(9,9)=\dfrac{36ELn(1+C_a)I_{z0}}{C^8_aL^3}[C^6_a(C_{iz1}-4)+2C^5_a(4C_{iz1}-C_{iz2}-4)+3C^4_a(4C_{iz1}-4C_{iz2}+C_{iz3})$

$-4C^3_a(4C_{iz2}-4C_{iz3}+C_{iz4})+5C^2_a(4C_{iz3}-4C_{iz4}+C_{iz5})-24C_a(C_{iz4}-C_{iz5})+28C_{iz5}]+\dfrac{3EI_{z0}}{5C^7_a(1+C_a)L^3}[60C^7_a-C^6_a$

$(180C_{iz1}-20C_{iz2}-10C_{iz3}-8C_{iz4}-7C_{iz5}-480)-C^5_a(840C_{iz1}-320C_{iz2}+30C_{iz3}+12C_{iz4}+9C_{iz5}-480)-2C^4_a$

$(360C_{iz1}-600C_{iz2}+230C_{iz3}-20C_{iz4}-7C_{iz5})+10C^3_a(96C_{iz2}-156C_{iz3}+60C_{iz4}-5C_{iz5})-20C^2_a(60C_{iz3}-96C_{iz4}$

$+37C_{iz5})+120C_a(12C_{iz4}-19C_{iz5})-1680C_{iz5}]$, $s(2,6)=-s(6,9)=\dfrac{12ELn(1+C_a)I_{z0}}{C^8_aL^2}[C^6_a(2C_{iz1}-7)+2C^5_a(7C_{iz1}$

$-2C_{iz2}-6)+3C^4_a(6C_{iz1}-7C_{iz2}+2C_{iz3})-4C^3_a(6C_{iz2}-7C_{iz3}+2C_{iz4})+5C^2_a(6C_{iz3}-7C_{iz4}+2C_{iz5})-6C_a(6C_{iz4}$

$-7C_{iz5})+42C_{iz5}]+ \dfrac{EI_{z0}}{5C_a^7(1+C_a)L^2}[120C_a^7- C_a^6(360C_{iz1}- 30C_{iz2}- 10C_{iz3}- 7C_{iz4}- 6C_{iz5}- 780)- C_a^5(1380C_{iz1}$

$- 630C_{iz2}+ 50C_{iz3}+ 13C_{iz4}+ 8C_{iz5}- 720)- 2C_a^4(540C_{iz1}- 990C_{iz2}+ 450C_{iz3}- 35C_{iz4}- 8C_{iz5})+ 30C_a^3(48C_{iz2}$

$- 86C_{iz3}+ 39C_{iz4}- 3C_{iz5})- 60C_a^2(30C_{iz3}- 53C_{iz4}+ 24C_{iz5})+ 540C_a(4C_{iz4}- 7C_{iz5})- 2520C_{iz5}], \ s(2,13)=$

$-s(9,13)= \dfrac{12ELn(1+C_a)I_{z0}}{C_a^8L^2}[C_a^6(C_{iz1}- 5)+2C_a^5(5C_{iz1}- C_{iz2}- 6)+3C_a^4(6C_{iz1}- 5C_{iz2}+C_{iz3})- 4C_a^3(6C_{iz2}$

$- 5C_{iz3}+C_{iz4})+5C_a^2(6C_{iz3}- 5C_{iz4}+C_{iz5})- 6C_a(6C_{iz4}- 5C_{iz5})+42C_{iz5}]+ \dfrac{EI_{z0}}{5C_a^7(1+C_a)L^2}[60C_a^7- C_a^6(180C_{iz1}$

$- 30C_{iz2}- 20C_{iz3}- 17C_{iz4}- 15C_{iz5}- 660) - C_a^5(1140C_{iz1} - 330C_{iz2} + 40C_{iz3}+ 23C_{iz4}+ 19C_{iz5}- 720)- 2C_a^4$

$(540C_{iz1}- 810C_{iz2}+ 240C_{iz3}- 25C_{iz4}- 13C_{iz5})+ 30C_a^3(48C_{iz2}- 70C_{iz3}+ 21C_{iz4}- 2C_{iz5})- 60C_a^2(30C_{iz3}- 43C_{iz4}$

$+13C_{iz5})+180C_a(12C_{iz4}- 17C_{iz5})- 2520C_{iz5}], \ s(6,6)= \dfrac{4ELn(1+C_a)I_{z0}}{C_a^8L}[4C_a^6(C_{iz1}- 3)+2C_a^5(12C_{iz1}- 4C_{iz2}$

$- 9)+ 3C_a^4(9C_{iz1}- 12C_{iz2}+ 4C_{iz3})- 4C_a^3(9C_{iz2}- 12C_{iz3}+ 4C_{iz4})+ 5C_a^2(9C_{iz3}- 12C_{iz4}+ 4C_{iz5})- 18C_a(3C_{iz4}$

$- 4C_{iz5})+63C_{iz5}]+ \dfrac{EI_{z0}}{15C_a^7(1+C_a)L}[240C_a^7- C_a^6(690C_{iz1}- 60C_{iz2}- 15C_{iz3}- 8C_{iz4}- 6C_{iz5}- 1260)- C_a^5(2250C_{iz1}$

$- 1200C_{iz2}+ 105C_{iz3}+ 22C_{iz4}+ 10C_{iz5}- 1080)- C_a^4(1620C_{iz1}- 3240C_{iz2}+ 1710C_{iz3}- 150C_{iz4}- 29C_{iz5})+ 15C_a^3$

$(144C_{iz2}- 282C_{iz3}+ 148C_{iz4}- 13C_{iz5})- 30C_a^2(90C_{iz3}- 174C_{iz4}+ 91C_{iz5})+ 270C_a(12C_{iz4}- 23C_{iz5})- 3780C_{iz5}],$

$s(6,13) = \dfrac{4ELn(1+C_a)I_{z0}}{C_a^8L}[C_a^6(2C_{iz1}- 9) + 2C_a^5(9C_{iz1}- 2C_{iz2}- 9)+3C_a^4(9C_{iz1}- 9C_{iz2}+2C_{iz3})- 4C_a^3(9C_{iz2}- 9$

$C_{iz3} + 2C_{iz4})+5C_a^2(9C_{iz3}- 9C_{iz4}+2C_{iz5})- 54C_a(C_{iz4}- C_{iz5})+63C_{iz5}]+ \dfrac{EI_{z0}}{15C_a^7(1+C_a)L}[120C_a^7- C_a^6(390C_{iz1}$

$- 30C_{iz2}- 15C_{iz3}- 13C_{iz4}- 12C_{iz5}- 1080)- C_a^5(1890C_{iz1}- 690C_{iz2}+ 45C_{iz3}+ 17C_{iz4}+ 14C_{iz5}- 1080) - C_a^4$

$(1620C_{iz1}- 2700C_{iz2}+ 990C_{iz3}- 60C_{iz4}- 19C_{iz5})+ 15C_a^3(144C_{iz2}- 234C_{iz3}+ 86C_{iz4}- 5C_{iz5})- 30C_a^2(90C_{iz3}-$

$144C_{iz4} + 53C_{iz5}) + 270C_a(12C_{iz4}- 19C_{iz5})- 3780C_{iz5}], \ s(13,13)= \dfrac{4ELn(1+C_a)I_{z0}}{C_a^8L}[C_a^6(C_{iz1}- 6) + 2C_a^5$

$(6C_{iz1}- C_{iz2}- 9)+ 3C_a^4(9C_{iz1}- 6C_{iz2}+ C_{iz3})- 4C_a^3(9C_{iz2}- 6C_{iz3}+ C_{iz4})+ 5C_a^2(9C_{iz3}- 6C_{iz4}+ C_{iz5})- 18C_a(3C_{iz4}$

$- 2C_{iz5}) + 63C_{iz5}]+ \dfrac{EI_{z0}}{15C_a^7(1+C_a)L}[60C_a^7- C_a^6(150C_{iz1}- 60C_{iz2}- 45C_{iz3}- 38C_{iz4}- 33C_{iz5}- 900)- C_a^5(130C_{iz1}$

$- 300C_{iz2}+ 75C_{iz3}+ 52C_{iz4}+ 43C_{iz5}- 1080)- C_a^4(1620C_{iz1}- 2160C_{iz2}+ 450C_{iz3}- 90C_{iz4}- 59C_{iz5})$

$+15C_a^3(144C_{iz2}- 186C_{iz3}+40C_{iz4}- 7C_{iz5})- 30C_a^2(90C_{iz3}- 114C_{iz4}+25C_{iz5})+810C_a(4C_{iz4}- 5C_{iz5})- 3780C_{iz5}]$

If $C_a=0$, then the presented elements are replaced with:

$s(2,2)= -s(2,9)= s(9,9)= \dfrac{EI_{z0}}{L^3}[12+6C_{iz1}+ \dfrac{24}{5}C_{iz2}+ \dfrac{21}{5}C_{iz3}+ \dfrac{132}{35}C_{iz4}+ \dfrac{24}{7}C_{iz5}], s(2,6)= -s(6,9)= \dfrac{EI_{z0}}{L^2}$

$[6+2C_{iz1}+ \dfrac{7}{5}C_{iz2}+ \dfrac{6}{5}C_{iz3}+ \dfrac{38}{35}C_{iz4}+C_{iz5}], s(2,13)= -s(9,13)= \dfrac{EI_{z0}}{L^2}[6+4C_{iz1}+ \dfrac{17}{5}C_{iz2}+3C_{iz3}+ \dfrac{94}{35}C_{iz4}$

$$+ \frac{17}{7}C_{iz5}], s(6,6) = \frac{EI_{z0}}{L}[4 + C_{iz1} + \frac{8}{15}C_{iz2} + \frac{2}{5}C_{iz3} + \frac{12}{35}C_{iz4} + \frac{13}{42}C_{iz5}], s(6,13) = \frac{EI_{z0}}{L}[2 + C_{iz1} + \frac{13}{15}C_{iz2} +$$

$$\frac{4}{5}C_{iz3} + \frac{26}{35}C_{iz4} + \frac{29}{42}C_{iz5}], s(13,13) = \frac{EI_{z0}}{L}[4 + 3C_{iz1} + \frac{38}{15}C_{iz2} + \frac{11}{5}C_{iz3} + \frac{68}{35}C_{iz4} + \frac{73}{42}C_{iz5}]$$

The entries related to I_{yz} and I_{wz} are given below:

$$s(3,4) = - s(3,11) = - s(4,10) = s(10,11) = \frac{3E}{35L^3}(7C_{yz1} + 7C_{yz2} + 6C_{yz3}) + \frac{3E}{35L^3}(140I_{wz0} + 70C_{wz1} + 56C_{wz2}$$

$$+49C_{wz3} + 44C_{wz4}), s(3,7) = - s(7,10) = \frac{E}{35L^2}(35I_{yz0} + 7C_{yz1} - 2C_{yz3}) + \frac{E}{35L^2}(210I_{wz0} + 70C_{wz1} + 49C_{wz2} +$$

$$42C_{wz3} + 38C_{wz4}), s(3,14) = - s(10,14) = - \frac{E}{35L^2}(35I_{yz0} + 28C_{yz1} + 21C_{yz2} + 16C_{yz3}) + \frac{E}{35L^2}(210I_{wz0} +$$

$$140C_{wz1} + 119C_{wz2} + 105C_{wz3} + 94C_{wz4}), s(4,5) = \frac{E}{70L^2}(70I_{yz0} + 21C_{yz1} + 14C_{yz2} + 11C_{yz3}) - \frac{E}{35L^2}(210I_{wz0}$$

$$+70C_{wz1} + 49C_{wz2} + 42C_{wz3} + 38C_{wz4}), s(4,12) = - s(11,12) = - \frac{E}{70L^2}(70I_{yz0} + 49C_{yz1} + 42C_{yz2} + 38C_{yz3})$$

$$- \frac{E}{35L^2}(210I_{wz0} + 140C_{wz1} + 119C_{wz2} + 105C_{wz3} + 94C_{wz4}), s(5,7) = \frac{E}{210L}(105I_{yz0} + 14C_{yz1} + 7C_{yz2} +$$

$$6C_{yz3}) - \frac{E}{105L}(420I_{wz0} + 105C_{wz1} + 56C_{wz2} + 42C_{wz3} + 36C_{wz4}), s(5,11) = - \frac{E}{70L^2}(70I_{yz0} + 21C_{yz1} +$$

$$14C_{yz2} + 11C_{yz3}) + \frac{E}{35L^2}(210I_{wz0} + 70C_{wz1} + 49C_{wz2} + 42C_{wz3} + 38C_{wz4}), s(5,14) = \frac{E}{210L}(105I_{yz0} +$$

$$49C_{yz1} + 35C_{yz2} + 27C_{yz3}) - \frac{E}{105L}(210I_{wz0} + 105C_{wz1} + 91C_{wz2} + 84C_{wz3} + 78C_{wz4}), s(7,12) = - \frac{E}{210L}$$

$$(105I_{yz0} + 56C_{yz1} + 42C_{yz2} + 36C_{yz3}) - \frac{E}{105L}(210I_{wz0} + 105C_{wz1} + 91C_{wz2} + 84C_{wz3} + 78C_{wz4}), s(12,14) =$$

$$- \frac{E}{210L}(105I_{yz0} + 91C_{yz1} + 84C_{yz2} + 78C_{yz3}) - \frac{E}{105L}(420I_{wz0} + 315C_{wz1} + 266C_{wz2} + 231C_{wz3} + 204C_{wz4})$$

APPENDIX 3. STABILITY MATRIX COMMON BETWEEN SYMMETRIC AND MONOSYMMETRIC SECTION

In this appendix the elements related to wagner effect are presented. The other entries are identical with those given by Yang and Yau [5], and, Yang and McGuire [2]. First, the aforementioned elements are presented, assuming that $C_a \acute{U} 0$:

$$g_s(4,4) = - g_s(4,11) = g_s(11,11) = \frac{36F_{xj}Ln(1+C_a)}{C_a^{10}L}[C_a^7C_{rp0} + 3C_a^6(2C_{rp0} - C_{rp1}) + 6C_a^5(C_{rp0} - 2C_{rp1} + C_{rp2}) -$$

$$10C_a^4(C_{rp1} - 2C_{rp2} + C_{rp3}) + 15C_a^3(C_{rp2} - 2C_{rp3} + C_{rp4}) - 21C_a^2(C_{rp3} - 2C_{rp4} + C_{rp5}) + 28C_a(C_{rp4} - 2C_{rp5}) - 36C_{rp5}]$$

$$- \frac{3F_{xj}}{35C_a^9L}[C_a^6(1260C_{rp0} - 140C_{rp1} - 35C_{rp2} - 14C_{rp3} - 7C_{rp4} - 4C_{rp5}) + 21C_a^5(120C_{rp0} - 140C_{rp1} + 20C_{rp2} + 5C_{rp3}$$

$$+ 2C_{rp4} + C_{rp5}) - 42C_a^4(100C_{rp1} - 125C_{rp2} + 20C_{rp3} + 5C_{rp4} + 2C_{rp5}) + 70C_a^3(90C_{rp2} - 117C_{rp3} + 20C_{rp4} + 5C_{rp5})$$

$$- 420C_a^2 (21C_{rp3} - 28C_{rp4} + 5C_{rp5}) + 840C_a(14C_{rp4} - 19C_{rp5}) - 15120C_{rp5}]$$

$$g_s(4,7) = - g_s(7,11) = \frac{6F_{xj}Ln(1+C_a)}{C_a^{10}} [C_a^7(5C_{rp0} - C_{rp1}) + 3C_a^6(7C_{rp0} - 5C_{rp1} + C_{rp2}) + 6C_a^5(3C_{rp0} - 7C_{rp1} + 5C_{rp2}$$

$$- C_{rp3}) - 10C_a^4(3C_{rp1} - 7C_{rp2} + 5C_{rp3} - C_{rp4}) + 15C_a^3(3C_{rp2} - 7C_{rp3} + 5C_{rp4} - C_{rp5}) - 21C_a^2(3C_{rp3} - 7C_{rp4} + 5C_{rp5})$$

$$+ 28C_a (3C_{rp4} - 7C_{rp5}) - 108C_{rp5}] - \frac{F_{xj}}{70C_a^9 L} [210C_a^7 C_{rp0} + C_a^6(5040C_{rp0} - 1680C_{rp1} + 35C_{rp2} - 7C_{rp3} - 7C_{rp4} -$$

$$5_{rp5}) + 21C_a^5(360C_{rp0} - 540C_{rp1} + 200C_{rp2} - 5C_{rp3} + C_{rp4} + C_{rp5}) - 42C_a^4(300C_{rp1} - 475C_{rp2} + 185C_{rp3} - 5C_{rp4} +$$

$$C_{rp5}) + 70C_a^3(270C_{rp2} - 441C_{rp3} + 177C_{rp4} - 5C_{rp5}) - 420C_a^2(63C_{rp3} - 105C_{rp4} + 43C_{rp5}) + 840C_a(42C_{rp4} - 71$$

$$C_{rp5}) - 45360C_{rp5}], \quad g_s(4,14) = - g_s(11,14) = \frac{6F_{xj}Ln(1+C_a)}{C_a^{10}} [2C_a^7 C_{rp0} + 3C_a^6(5C_{rp0} - 2C_{rp1}) + 6C_a^5(3C_{rp0}$$

$$- 5C_{rp1} + 2C_{rp2}) - 10C_a^4(3C_{rp1} - 5C_{rp2} + 2C_{rp3}) + 15C_a^3(3C_{rp2} - 5C_{rp3} + 2C_{rp4}) - 21C_a^2(3C_{rp3} - 5C_{rp4} + 2C_{rp5}) +$$

$$28C_a (3C_{rp4} - 5C_{rp5}) - 108C_{rp5}] - \frac{F_{xj}}{70C_a^9(1+C_a)} [C_a^7(2730C_{rp0} - 210C_{rp1} - 35C_{rp2} - 7C_{rp3} + 2_{rp5}) + C_a^6 (10080$$

$$C_{rp0} - 6720C_{rp1} + 595C_{rp2} + 98C_{rp3} + 21C_{rp4} + 2_{rp5}) + 21C_a^5 (360C_{rp0} - 900C_{rp1} + 590C_{rp2} - 55C_{rp3} -$$

$$9C_{rp4} - 2C_{rp5}) - 14C_a^4(900C_{rp1} - 2175C_{rp2} + 1410C_{rp3} - 135C_{rp4} - 22C_{rp5}) + 70C_a^3 (270C_{rp2} - 639C_{rp3} + 411C_{rp4}$$

$$- 40C_{rp5}) - 420 C_a^2 (63 C_{rp3} - 147 C_{rp4} + 94 C_{rp5}) + 840C_a (42 C_{rp4} - 97 C_{rp5}) - 45360 C_{rp5}], \quad g_s(7,7) =$$

$$\frac{F_{xj}Ln(1+C_a)L}{C_a^{10}} [C_a^7 (22C_{rp0} - 8C_{rp1} + C_{rp2}) + 3C_a^6(24C_{rp0} - 22C_{rp1} + 8C_{rp2} - C_{rp3}) + 6C_a^5 (9C_{rp0} - 24C_{rp1} +$$

$$22C_{rp2} - 8C_{rp3} + C_{rp4}) - 10C_a^4 (9C_{rp1} - 24C_{rp2} + 22C_{rp3} - 8C_{rp4} + C_{rp5}) + 15C_a^3 (9C_{rp2} - 24C_{rp3} + 22C_{rp4}$$

$$- 8C_{rp5}) - 21C_a^2(9C_{rp3} - 24C_{rp4} + 22C_{rp5}) + 84C_a(3C_{rp4} - 8C_{rp5}) - 324C_{rp5}] + \frac{F_{xj}L}{420C_a^9} [210C_a^8 C_{rp0} - 210C_a^7$$

$$(8C_{rp0} - C_{rp1}) - C_a^6 (18900C_{rp0} - 10080C_{rp1} + 1785C_{rp2} - 56C_{rp3} - 14C_{rp4} - 8_{rp5}) - 21C_a^5 (1080C_{rp0} - 1980C_{rp1} +$$

$$1140 C_{rp2} - 215C_{rp3} + 8C_{rp4} + 2C_{rp5}) + 42C_a^4(900C_{rp1} - 1725C_{rp2} + 1030C_{rp3} - 200C_{rp4} + 8C_{rp5}) - 210C_a^3 (270$$

$$C_{rp2} - 531C_{rp3} + 324C_{rp4} - 64C_{rp5}) + 3780C_a^2(21C_{rp3} - 42C_{rp4} + 26C_{rp5}) - 2520C_a(42C_{rp4} - 85C_{rp5}) - 136080$$

$$C_{rp5}] \quad g_s(7,14) = \frac{F_{xj}Ln(1+C_a)L}{C_a^{10}} [C_a^7(11C_{rp0} - 2C_{rp1}) + 3C_a^6(18C_{rp0} - 11C_{rp1} + 2C_{rp2}) + 6C_a^5 (9C_{rp0} - 18C_{rp1}$$

$$+ 11C_{rp2} - 2C_{rp3}) - 10C_a^4(9C_{rp1} - 18C_{rp2} + 11C_{rp3} - 2C_{rp4}) + 15C_a^3(9C_{rp2} - 18C_{rp3} + 11C_{rp4} - 2C_{rp5}) - 21C_a^2(9C_{rp3}$$

$$- 18C_{rp4} + 11C_{rp5}) + 252C_a(C_{rp4} - 2C_{rp5}) - 324C_{rp5}] - \frac{F_{xj}L}{420C_a^9(1+C_a)} [420C_a^8 C_{rp0} + C_a^7(12180C_{rp0} - 3360C_{rp1} +$$

$$105C_{rp2} + 14C_{rp3} + 7C_{rp4} + 6_{rp5}) + C_a^6(34020C_{rp0} - 30240C_{rp1} + 8505C_{rp2} - 301C_{rp3} - 35C_{rp4} - 15_{rp5}) + 21C_a^5$$

$$(1080C_{rp0} - 3060C_{rp1} + 2670C_{rp2} - 755C_{rp3} + 28C_{rp4} + 3C_{rp5}) - 42C_a^4(900C_{rp1} - 2475C_{rp2} + 2135C_{rp3} - 605C_{rp4}$$

$$+ 23C_{rp5}) + 630C_a^3(90C_{rp2} - 243C_{rp3} + 208C_{rp4} - 59C_{rp5}) - 1260C_a^2(63C_{rp3} - 168C_{rp4} + 143C_{rp5}) + 7560C_a$$

$$(14C_{rp4} - 37C_{rp5}) - 136080C_{rp5}] , \quad g_s(14,14) = \frac{F_{xj}Ln(1+C_a)L}{C_a^{10}} [4C_a^7 C_{rp0} + 12C_a^6(3C_{rp0} - C_{rp1}) + 6C_a^5(9C_{rp0} -$$

$12C_{rp1} + 4C_{rp2}) - 10C_a^4(9C_{rp1} - 12C_{rp2} + 4C_{rp3}) + 15C_a^3(9C_{rp2} - 12C_{rp3} + 4C_{rp4}) - 21C_a^2(9C_{rp3} - 12C_{rp4} + 4C_{rp5})$

$+ \ 84C_a(3C_{rp4} - 4C_{rp5}) - 324C_{rp5}] - \dfrac{F_{xj}L}{140C_a^9(1+C_a)^2} [C_a^8(1890C_{rp0} + 280C_{rp1} - 595C_{rp2} + 168C_{rp3} \ - \ 14C_{rp4} \ - $

$12_{rp5}) + C_a^7(10640C_{rp0} - 3990C_{rp1} - 2030C_{rp2} + 2121C_{rp3} - 532C_{rp4} + 18C_{rp5}) + C_a^6(16380C_{rp0} - 20580C_{rp1} + $

$5985C_{rp2} + 5418C_{rp3} - 4592C_{rp4} + 1080C_{rp5}) + 7C_a^5(1080C_{rp0} - 4140C_{rp1} + 4800C_{rp2} - 1125C_{rp3} - 1492C_{rp4} + $

$1144C_{rp5}) \ - \ 14C_a^4(900C_{rp1} - 3225C_{rp2} + 3550C_{rp3} - 690C_{rp4} + 1222C_{rp5}) + 210C_a^3(90C_{rp2} - 309C_{rp3} + 328C_{rp4}$

$- \ 54C_{rp5}) - 2940C_a^2(9C_{rp3} - 30C_{rp4} + 31C_{rp5}) + 840C_a(42C_{rp4} - 137C_{rp5}) - 45360C_{rp5}]$

If $C_a = 0$, then the elements given will be written as:

$g_s(4,4) = -g_s(4,11) = g_s(11,11) = \dfrac{F_{xj}}{70L}(84C_{rp0} + 42C_{rp1} + 24C_{rp2} + 15C_{rp3} + 10C_{rp4} + 7C_{rp5})$

$g_s(4,7) = -g_s(7,11) = \dfrac{F_{xj}}{420}(42C_{rp0} + 42C_{rp1} + 30C_{rp2} + 21C_{rp3} + 15C_{rp4} + 11C_{rp5})$, $g_s(4,14) = -g_s(11,14) = \dfrac{F_{xj}}{420}$

$(42C_{rp0} \ - \ 12C_{rp2} - 15C_{rp3} - 14C_{rp5})$, $\quad g_s(7,7) = \dfrac{F_{xj}L}{840}(112C_{rp0} + 28C_{rp1} + 16C_{rp2} + 11C_{rp3} + 8C_{rp4} + 6C_{rp5})$

$g_s(7,14) = -\dfrac{F_{xj}L}{840}(28C_{rp0} + 14C_{rp1} + 12C_{rp2} + 11C_{rp3} + 10C_{rp4} + 9C_{rp5})$, $g_s(14,14) = \dfrac{F_{xj}L}{840}(112C_{rp0} + 84C_{rp1}$

$+72C_{rp2} + 65C_{rp3} + 60C_{rp4} + 56C_{rp5})$

It should be noted that the term $-F_{yi}a_{Li}$ and $-F_{yi}a_{Lj}$ are added to $g_s(4,4)$ and $g_s(11,11)$, respectively, if the transverse load is not applied at the web midheight. F_y and a_L are the nodal transverse load (positive acting downward), and the distance between its point of application with the web midheight (positive upward), respectively.

APPENDIX 4. STABILITY MATRIX RELATED TO MONOSYMMETRIC SECTION PROPERTIES

When $C_a \acute{U} 0$, the entries of this matrix are given as:

$g_m(3,4) = -g_m(3,11) = -g_m(4,10) = g_m(10,11) = \quad - \dfrac{36F_{xj}Ln(1+C_a)}{C_a^7L} [C_a^4C_{yo0} + C_a^3(2C_{yo0} - C_{yo1}) + C_a^2(C_{yo0}$

$- 2C_{yo1} + C_{yo2}) + C_a(2C_{yo2} - C_{yo1}) + C_{yo2}] - \dfrac{3F_{xj}}{5C_a^6L} [C_a^5(5C_{yo0} + 2C_{yo1} + C_{yo2}) - C_a^4(20C_{yo0} + 5C_{yo1} + 2C_{yo2}) \ - 5C_a^3$

$(18C_{yo0} \ - \ 4C_{yo1} \ - \ C_{yo2}) - 10C_a^2(6C_{yo0} - 9C_{yo1} + 2C_{yo2}) + 30C_a(2C_{yo1} - 3C_{yo2}) - 60C_{yo2}]$, $g_m(3,7) = -g_m(4,5) = $

$g_m(5,11) = -g_m(7,10) = -\dfrac{6F_{xj}Ln(1+C_a)}{C_a^7}[C_a^5C_{yo0} + C_a^4(5C_{yo0} - C_{yo1}) + C_a^3(7C_{yo0} - 5C_{yo1} + C_{yo2}) + C_a^2(3C_{yo0} - $

$7C_{yo1} \ + \ 5C_{yo2}) \ + \ C_a(7C_{yo2} - 3C_{yo1}) + 3C_{yo2}] + \dfrac{F_{xj}}{10C_a^6}[C_a^5(5C_{yo0} - C_{yo1} - C_{yo2}) + C_a^4(150C_{yo0} - 5C_{yo1} + C_{yo2}) + $

$5C_a^3(66C_{yo0} - 30C_{yo1} + C_{yo2}) + 30C_a^2(6C_{yo0} - 11C_{yo1} + 5C_{yo2}) - 30C_a(6C_{yo1} - 11C_{yo2}) + 180C_{yo2}]$ $g_m(3,14) = -$

$g_m(4,12) = g_m(10,14) = -g_m(11,12) = \quad - \dfrac{6F_{xj}Ln(1+C_a)}{C_a^7}[2C_a^4C_{yo0} + C_a^3(5C_{yo0} - 2C_{yo1}) + C_a^2(3C_{yo0} - 5C_{yo1} + $

$2C_{yo2}) \ + \ C_a(5C_{yo2} - 3C_{yo1}) + 3C_{yo2}] - \dfrac{F_{xj}}{10C_a^6}[C_a^5(5C_{yo0} + C_{yo1}) - C_a^4(30C_{yo0} + 5C_{yo1} + C_{yo2}) - 5C_a^3(42C_{yo0}$

$- \ 6C_{yo1} \ - \ C_{yo2}) \ - \ 30C_a^2(6C_{yo0} - 7C_{yo1} + C_{yo2}) + 30C_a(6C_{yo1} - 7C_{yo2}) - 180C_{yo2}]$ $g_m(4,4)$

$= \dfrac{12(M_{zi} + M_{zj})Ln(1+C_a)}{C_a^8L}[C_a^6C_{yo0} + C_a^5(C_{yo0} - C_{yo1}) - C_a^4(3C_{yo0} + C_{yo1} - C_{yo2}) - C_a^3(5C_{yo0} - 3C_{yo1} - C_{yo2}) - C_a^2$

$(2C_{yo2} - 5C_{yo1} + 3C_{yo2}) + C_a(2C_{yo1} - 5C_{yo2}) - 2C_{yo2}] - \dfrac{M_{zi}+M_{zj}}{35C_a^7 L}[C_a^6(147C_{yo0}+35C_{yo1}+13C_{yo2}) + 7C_a^5(80C_{yo0}$

$- 21C_{yo1} - 5C_{yo2}) - 7C_a^4(70C_{yo0}+ 80C_{yo1}- 21C_{yo2}) - 70C_a^3(24C_{yo0}- 71C_{yo1}- 8C_{yo2}) - 70C_a^2(12C_{yo0}- 24C_{yo1}+$

$7 C_{yo2}) + 840C_a (C_{yo1} - 2C_{yo2}) - 840C_{yo2}] + \beta \int_0^1 [M_{zi}(1-x) - M_{zj}x] \dfrac{b_E}{L}[36x^2(x-1)^2]dx \quad g_m (4,7)$

$= \dfrac{(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8}[C_a^7C_{yo0}+C_a^6(4C_{yo0}- C_{yo1}) - C_a^5(6C_{yo0}+4C_{yo1}- C_{yo2}) - 2C_a^4(16C_{yo0}- 3C_{yo1} - 2 C_{yo2})$

$- C_a^3 (35C_{yo0} - 32C_{yo1}+6C_{yo2}) - C_a^2(12C_{yo0}- 35C_{yo1}+32C_{yo2}) + C_a(12C_{yo1}- 35C_{yo2}) - 12C_{yo2}] - \dfrac{(M_{zi}+M_{zj})}{420C_a^7}$

$[C_a^6(1127C_{yo0}+ 22C_{yo2}) + 7C_a^5(80C_{yo0}- 161C_{yo1}) - 7C_a^4(1110C_{yo0}+ 80C_{yo1}- 161C_{yo2}) - 70C_a^3(174C_{yo0}$

$- 111C_{yo1}- 8C_{yo2}) - 210C_a^2(24C_{yo0}- 58C_{yo1}+37C_{yo2}) + 420C_a(12C_{yo1}- 29C_{yo2}) - 5040C_{yo2}] + \beta \int_0^1 [M_{zi}(1-x) -$

$M_{zj}x] b_E [6x(x-1)(3x^2- 4x+1)]dx, \quad g_m(4,11) = - \dfrac{6(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8 L}[C_a^6C_{yo0}+C_a^5(C_{yo0} - C_{yo1}) - C_a^4$

$(6C_{yo0} + C_{yo1} - C_{yo2}) - C_a^3(10C_{yo0}- 6C_{yo1}- C_{yo2}) - 2C_a^2(2C_{yo0}- 5C_{yo1}+ 3C_{yo2}) + 2C_a(2C_{yo1}- 5C_{yo2})$

$- 4C_{yo2}] + \dfrac{(M_{zi}+M_{zj})}{70C_a^7 L}[3C_a^6(28C_{yo0}- 3C_{yo2})+28C_a^5(25C_{yo0}- 3C_{yo1}) - 28C_a^4(35C_{yo0}+25C_{yo1}- 3C_{yo2}) - 140C_a^3$

$(24C_{yo0} - 7C_{yo1} - 5_{yo2}) - 140C_a^2(12C_{yo0}- 24C_{yo1}+ 7C_{yo2}) + 1680C_a(C_{yo1}- 2C_{yo2}) - 1680C_{yo2}]$

$+ \beta \int_0^1 [M_{zi}(1-x) - M_{zj}x]\dfrac{b_E}{L}[36x^2(x-1)^2]dx , \quad g_m(4,14) = \dfrac{(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8}[2C_a^6C_{yo0}+C_a^5(3C_{yo0}- 2C_{yo1})$

$- C_a^4(12C_{yo0}+3C_{yo1}- 2C_{yo2}) - C_a^3(25C_{yo0}- 12C_{yo1}- 3C_{yo2}) - C_a^2(12C_{yo0}- 25C_{yo1}+ 12C_{yo2}) + C_a(12C_{yo1}$

$- 25C_{yo2}) - 12C_{yo2}] - \dfrac{(M_{zi}+M_{zj})}{420C_a^7}[C_a^6(147C_{yo0}- 13C_{yo2})+7C_a^5(220C_{yo0}- 21C_{yo1}) - 7C_a^4(210C_{yo0}+220C_{yo1}$

$- 21C_{yo2}) - 70C_a^3(114C_{yo0} - 21C_{yo1}- 22_{yo2}) - 210C_a^2(24C_{yo0}- 38C_{yo1}+ 7C_{yo2}) + 420C_a(12C_{yo1}- 19C_{yo2})$

$- 5040C_{yo2}] + \beta \int_0^1 [M_{zi}(1-x) - M_{zj}x] b_E [6x^2(x-1)(3x-2)]dx, \quad g_m(5,7) = \dfrac{LF_{xj}Ln(1+C_a)}{C_a^7}$

$[C_a^6 C_{yo0}+ C_a^5(8C_{yo0} - C_{yo1}) + C_a^4(22C_{yo0}$

$- 8C_{yo1}+ C_{yo2}) + 2C_a^3(12C_{yo0}- 11C_{yo1}+ 4C_{yo2}) + C_a^2(9C_{yo0}- 24C_{yo1}+ 22C_{yo2}) - 3C_a(3C_{yo1} - 8C_{yo2}) + 9C_{yo2}]$

$- \dfrac{LF_{xj}}{60C_a^6}[C_a^5(165C_{yo0}- 8C_{yo1}- 2C_{yo2})+C_a^4(780C_{yo0}- 165C_{yo1}+8C_{yo2})+15C_a^3(78C_{yo0}- 5C_{yo1} +11C_{yo2}) +$

$30C_a^2(18C_{yo0} - 39C_{yo1}+26C_{yo2}) - 90C_a(6C_{yo1}- 13C_{yo2})+540C_{yo2}], \quad g_m(5,14) = g_m(7,12) = \dfrac{LF_{xj}Ln(1+C_a)}{C_a^7}$

$[2C_a^5C_{yo0}+ C_a^4(11C_{yo0}- 2C_{yo1}) + C_a^3(18C_{yo0}- 11C_{yo1}+ 2C_{yo2}) + C_a^2(9C_{yo0} - 18C_{yo1}+ 11C_{yo2}) - 9C_a$

$(C_{yo1}- 2C_{yo2})+9C_{yo2}] - \dfrac{LF_{xj}}{60C_a^6}[C_a^5(15C_{yo0}+2C_{yo1}+ C_{yo2}) +C_a^4(300C_{yo0}- 15C_{yo1} - 2C_{yo2}) +15C_a^3(54C_{yo0} -$

$20 C_{yo1} + C_{yo2}) + 30C_a^2(18C_{yo0} - 27C_{yo1} + 10C_{yo2}) - 270C_a(2C_{yo1} - 3C_{yo2}) + 540C_{yo2}] \quad g_m(7,7) =$

$- \dfrac{2L(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8}[C_a^6C_{yo0}+C_a^5(6C_{yo0}- C_{yo1})+C_a^4(12C_{yo0}- 6C_{yo1}+C_{yo2}) +2C_a^3(5C_{yo0}- 6C_{yo1}+3C_{yo2})$

$+ C_a^2(3C_{yo0}- 10C_{yo1}+12C_{yo2}) - C_a(3C_{yo1}- 10C_{yo2})+3C_{yo2}] + \dfrac{L(M_{zi}+M_{zj})}{210C_a^7}[2C_a^6(21C_{yo0}- C_{yo2}) + 7C_a^5$

$(155C_{yo0} - 6C_{yo1}) +7C_a^4(480C_{yo0}- 155C_{yo1}+6C_{yo2})+35C_a^3(102C_{yo0}- 96C_{yo1} +31C_{yo2}) + 210C_a^2 (6C_{yo0} -$

$17C_{yo1} + 16C_{yo2}) - 210C_a(6C_{yo1} - 17C_{yo2}) + 1260C_{yo2}] + \beta\int_0^1 [M_{zi}(1-x) - M_{zj}x]b_E L [(3x^2 - 4x + 1)^2]dx,$

$g_m(7,11) = \dfrac{(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8} [9C_a^5C_{yo0} + C_a^4(32C_{yo0} - 9C_{yo1}) + C_a^3(35C_{yo0} - 32C_{yo1} + 9C_{yo2}) + C_a^2$

$(12C_{yo0} - 35C_{yo1} + 32C_{yo2}) + C_a(35C_{yo0} - 12C_{yo1}) + 12C_{yo2}] + \dfrac{(M_{zi}+M_{zj})}{420C_a^7} [C_a^6(77C_{yo0} - 13C_{yo2}) - 7C_a^5$

$(100C_{yo0} + 11C_{yo1}) - 7C_a^4(1110C_{yo0} - 100C_{yo1} - 11C_{yo2}) - 70C_a^3(174C_{yo0} - 111C_{yo1} + 10C_{yo2}) - 210C_a^2(24$

$C_{yo0} - 58C_{yo1} + 37C_{yo2}) + 420C_a(12C_{yo1} - 29C_{yo2}) - 5040C_{yo2}] + \beta\int_0^1 [M_{zi}(1-x) - M_{zj}x]b_E [6x(x-1)(3x^2$

$- 4x + 1)]dx, \quad g_m(7,14) = -\dfrac{3L(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8} [C_a^5C_{yo0} + C_a^4(4C_{yo0} - C_{yo1}) + C_a^3(5C_{yo0} - 4C_{yo1} + C_{yo2})$

$+ C_a^2(2C_{yo0} - 5C_{yo1} + 4C_{yo2}) + C_a(5C_{yo2} - 2C_{yo1}) + 2C_{yo2}] - \dfrac{L(M_{zi}+M_{zj})}{140C_a^7} [C_a^6(7C_{yo0} - C_{yo2}) - 7C_a^5(10C_{yo0} +$

$C_{yo1}) - 7C_a^4(130C_{yo0} - 10C_{yo1} - C_{yo2}) - 70C_a^3(24C_{yo0} - 13C_{yo1} + C_{yo2}) - 70C_a^2(12C_{yo0} - 24C_{yo1} + 13C_{yo2}) +$

$840C_a(C_{yo1} - 2C_{yo2}) - 840C_{yo2}] + \beta\int_0^1 [M_{zi}(1-x) - M_{zj}x]b_E [x(2x-2)(3x^2 - 4x + 1)]dx, \quad g_m(11,11) =$

$-\dfrac{12(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8L} [3C_a^4C_{yo0} + C_a^3(5C_{yo0} - 3C_{yo1}) + C_a^2(2C_{yo0} - 5C_{yo1} + 3C_{yo2}) + C_a(5C_{yo2} - 2C_{yo1}) + 2$

$C_{yo2}] + \dfrac{(M_{zi}+M_{zj})}{35C_a^7L} [C_a^6(63C_{yo0} + 35C_{yo1} + 22C_{yo2}) - 7C_a^5(20C_{yo0} + 9C_{yo1} + 5C_{yo2}) + 7C_a^4(70C_{yo0} + 20C_{yo1} + 9$

$C_{yo2}) + 70C_a^3(24C_{yo0} - 7C_{yo1} - 2C_{yo2}) + 70C_a^2(12C_{yo0} - 24C_{yo1} + 7C_{yo2}) - 840C_a(C_{yo1} - 2C_{yo2}) + 840C_{yo2}] + \beta\int_0^1$

$[M_{zi}(1-x) - M_{zj}x]\dfrac{b_E}{L} [36x^2(x-1)^2]dx, \quad g_m(11,14) = \dfrac{(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8} [12C_a^4C_{yo0} + C_a^3(25C_{yo0} -$

$12C_{yo1}) + C_a^2(12C_{yo0} - 25C_{yo1} + 12C_{yo2}) + C_a(25C_{yo2} - 12C_{yo1}) + 12C_{yo2}] - \dfrac{(M_{zi}+M_{zj})}{420C_a^7} [C_a^6(63C_{yo0} - 22C_{yo2}) -$

$7C_a^5(40C_{yo0} + 9C_{yo1}) + 7C_a^4(210C_{yo0} + 40C_{yo1} + 9C_{yo2}) + 70C_a^3(114C_{yo0} - 21C_{yo1} - 4C_{yo2}) + 210C_a^2(24C_{yo0} -$

$38C_{yo1} + 7C_{yo2}) - 420C_a(12C_{yo1} - 19C_{yo2}) + 5040C_{yo2}] - \beta\int_0^1 [M_{zi}(1-x) - M_{zj}x]b_E [6x^2(x-1)(3x-2)]dx,$

$g_m(12,14) = \dfrac{LF_{xj}Ln(1+C_a)}{C_a^7} [4C_a^5C_{yo0} + 4C_a^3(3C_{yo0} - C_{yo1}) + C_a^2(9C_{yo0} - 12C_{yo1} + 4C_{yo2}) - 3C_a(3C_{yo1} - 4C_{yo2})$

$+ 9C_{yo2}] + \dfrac{LF_{xj}}{60C_a^6} [C_a^5(15C_{yo0} + 8C_{yo1} + 6C_{yo2}) - C_a^4(60C_{yo0} + 15C_{yo1} + 8C_{yo2}) - 15C_a^3(30C_{yo0} - 4C_{yo1} - C_{yo2}) -$

$30C_a^2(18C_{yo0} - 15C_{yo1} + 2C_{yo2}) + 90C_a(6C_{yo1} - 5C_{yo2}) - 540C_{yo2}], \quad g_m(14,14) = -\dfrac{2L(M_{zi}+M_{zj})Ln(1+C_a)}{C_a^8}$

$[2C_a^4C_{yo0} + C_a^3(5C_{yo0} - 2C_{yo1}) + C_a^2(3C_{yo0} - 5C_{yo1} + 2C_{yo2}) - C_a(3C_{yo1} - 5C_{yo2}) + 3C_{yo2}] + \dfrac{L(M_{zi}+M_{zj})}{210C_a^7} [C_a^6(7$

$C_{yo0} - 2C_{yo2}) - 7C_a^5(5C_{yo0} + C_{yo1}) + 7C_a^4(30C_{yo0} + 5C_{yo1} + C_{yo2}) + 35C_a^3(42C_{yo0} - 6C_{yo1} - C_{yo2}) + 210C_a^2(6$

$C_{yo0} - 7C_{yo1} + C_{yo2}) - 210C_a(6C_{yo1} - 7C_{yo2}) + 1260C_{yo2}] + \beta\int_0^1 [M_{zi}(1-x) - M_{zj}x]b_E L [x^2(3x-2)^2]dx$

If $C_a=0$, then the presented elements are replaced with:

$$g_m(3,4) = - g_m(3,11) = - g_m(4,10) = g_m(10,11) = - \frac{3F_{xj}}{35L}[14C_{yo0}+7C_{yo1}+4C_{yo2}), \quad g_m(3,7) = - g_m(4,5)=$$

$$g_m(5,11) = - g_m(7,10) = - \frac{F_{xj}}{70}(7C_{yo0}+7C_{yo1}+5C_{yo2}), \quad g_m(3,14) = - g_m(4,12) = g_m(10,14)= - g_m(11,12)=$$

$$-\frac{F_{xj}}{70}(7C_{yo0}- 2C_{yo1}), \quad g_m(4,4) = - \frac{(M_{zi}+M_{zj})}{35L}(35C_{yo0}+13C_{yo1}+6C_{yo2}) + \beta \int_0^1 [M_{zi}(1- x)- M_{zj}x]\frac{b_E}{L}[36x^2$$

$$(x- 1)^2] dx, \quad g_m(4,7) = - \frac{(M_{zi}+M_{zj})}{210}(11C_{yo1}+7C_{yo2}) + \beta \int_0^1 [M_{zi}(1- x)- M_{zj}x] b_E [6x(x- 1)(3x^2- 4x+1)] dx,$$

$$g_m(4,11) = - \frac{9(M_{zi}+M_{zj})}{70L}(C_{yo1}+C_{yo2}) + \beta \int_0^1 [M_{zi}(1- x)- M_{zj}x]\frac{b_E}{L}[36x^2(x- 1)^2] dx, \quad g_m(4,14) = \frac{(M_{zi}+M_{zj})}{420}$$

$$(13C_{yo1} + 12C_{yo2}) + \beta \int_0^1 [M_{zi}(1- x)- M_{zj}x] b_E [6x^2(x- 1)(3x- 2)]dx, \quad g_m(5,7) = \frac{LF_{xj}}{210}(28C_{yo0} + 7C_{yo1} +$$

$$4C_{yo2}), \quad g_m(5,14) = - \frac{LF_{xj}}{420}(14C_{yo0} + 7C_{yo1} + 6C_{yo2}), \quad g_m(7,7) = - \frac{L(M_{zi} + M_{zj})}{420}(4C_{yo1} + 3C_{yo2}) + \beta \int_0^1$$

$$[M_{zi}(1- x) - M_{zj}x] b_E L [(3x^2- 4x+1)^2] dx, \quad g_m(7,11) = - \frac{(M_{zi} + M_{zj})}{420}(13C_{yo1} + 14C_{yo2}) - \beta \int_0^1 [M_{zi}(1- x) -$$

$$M_{zj}x] b_E [6x(x- 1)(3x^2- 4x+1)]dx, \quad g_m(7,14) = \frac{L(M_{zi}+M_{zj})}{140}(C_{yo1}+C_{yo2}) + \beta \int_0^1 [M_{zi}(1- x)- M_{zj}x] b_E L [x$$

$$(3x- 2)(3x^2- 4x+1)] dx, \quad g_m(11,11) = \frac{(M_{zi}+M_{zj})}{35L}(35C_{yo0} + 22C_{yo1} + 15C_{yo2}) + \beta \int_0^1 [M_{zi}(1- x) - M_{zj}x]\frac{b_E}{L}[$$

$$36x^2(x- 1)^2] dx, \quad g_m(11,14) = \frac{(M_{zi} + M_{zj})}{210}(11C_{yo0}+15C_{yo2}) - \beta \int_0^1 [M_{zi}(1- x)- M_{zj}x] b_E$$

$$[6x^2(x- 1)(3x- 2)]dx, \quad g_m(12,14) = \frac{LF_{xj}}{210}(28C_{yo0}+ 21C_{yo1} + 18C_{yo2}) \quad g_m(14,14) = - \frac{L(M_{zi} + M_{zj})}{420}(4C_{yo1} +$$

$$5C_{yo2}) + \beta \int_0^1 [M_{zi}(1- x) - M_{zj}x] b_E L [x^2(3x- 2)^2] dx$$

NOTATION

M_w	Bimoment		x	Nondimensional coordinate
[N]	Shape function		l	Eigenvalue
o	Web midheight		s	Normal stress
r_p	Polar radius of gyration		t	Shear stress
s	Shear center		q	Rotation
[s]	Element stiffness matrix		A	Area
[S]	Global stiffness matrix		b	Flange width
t	Thickness		B	Bottom flange
T	Top flange		c	Centroid
u	Displacement		C	Taper constant
w	Web		{D}	Nodal displacements
W_i	Internal work		e	Linear strain
W_e	External work		E	Elasticity modulus
b_E	Monosymmetry parameter		f	Flange
h	Nonlinear strain		F_x	Axial force
g	Taper coefficient		F_y, F_z	Shear force
			G	Shear modulus

[g]	Element stability matrix
[G]	Global stability matrix
h	Flange centroids distance
I_y, I_z	Moment of inertia
I_p	Polar moment of inertia
I_w	Warping constant
$I_y, I_{wy}, I_{wz}, I_{yz}$	Section constants
J	Torsion constant
L	Element length
M_y, M_z	Flexural moment

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