FINDING THE BEST COEFFICIENTS IN THE OBJECTIVE FUNCTION OF A LINEAR QUADRATIC CONTROL PROBLEM

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ABSTRACT Finding the best weights of the state variables and the control variables in the objective function of a linear-quadratic control problem is considered. The weights of these variables are considered as two diagonal matrices with appropriate size and so the objective function of the control problem becomes a function of the diagonal elements of these matrices. The optimization problem which is discussed in this paper is to minimize the objective function of the control problem as a functión of these diagonal elements, when these elements are positive and the their sum in each matrix is a constant. This problem is named "the substitution between objectives" in economic planning literature. In this paper, it is proved that the optimal solution of this problem is in one of the extreme points of the feasible set subject to the positive definiteness of these diagonal matrices. A method for selecting this extreme point is offered and then the unconstrained optimization problem is solved by the steepest descent method. The resulting optimal solution is the selected extreme point. Finally, the procedure is used to solve a numerical example.

چکیده تعیین بهترین اوزان متغیرهای حالت و متغیرهای تصمیم در تابع هدف یک مسئله کنترل خطی - درجه دو در این مقاله مورد بررسی قرار گرفته است. به این ترتیب که اوزان متغیرها به صورت دوماتریس قطری با ابعاد مناسب در نظر گرفته شده و در نتیجه تابع هدف مسأله کنترل تابعی از عناصر قطری این ماتریسها می شود. مسأله بهینه سازی مورد بحث در این مقاله عبارتست از به حداقل رساندن تابع هدف مسأله کنترل به عنوان تابعی از این عناصر قطری، وقتیکه این عناصر مثبت بوده و مجموع آنها در هر ماتریس مقدار ثابتی است. این مسأله در متون برنامه ریزی اقتصادی به «جانشینی بین اهداف» موسوم است. در این مقاله ثابت می شود که جواب بهینه این مسأله، با شرط مثبت معین باقی ماندن این دوماتریس قطری، در یکی از نقاط فرین مجموعه قابل قبول است. روشی برای حدس این نقطه فرین ارائه شده و سپس مسأله بهینه سازی با تغییراتی بوسیله روش سریعترین نزول حل می شود. جواب بهینهٔ حاصل همان نقطه فرین حدس زده شده است. در انتها، روش مقاله برای حل یک مثال عددی به کاربرده می شود.

INTRODUCTION

The objective function of a linear-quadratic tracking problem is usually considered as the sum of squares of the differences between the state and control variables from their desired values with different weights. These weights are entered in the model as two diagonal matrices with appropriate size, one for the state variables and the other for the control variables. The diagonal elements of these matrices show the relative importance of their corresponding variables in the objective function; i.e., the more this coefficient, the less the difference between the corresponding variable from its desired value in the optimal solution. So changing these coefficients will change the optimal solution and the minimum value of the objective function of the control problem, and the question is how the optimal values of these weights are computed. This problem is named the substitution between objectives in economic planning literature, since the diagonal elements of these matrices determine for reaching one of the objectives, how much we are ready to be far from the other objectives. In practice, this problem is solved by the trial and error method; i.e., the diagonal elements of these matrices are selected according to the policy-maker preferences and if the solutions are not favorable, the appropriate change will take place in the coefficients of the variables which have unfavorable values. This procedure is repeated as much as all of the resulting solutions become favorable from the policy-maker's point of view.

In this paper, the problem of finding the best diagonal elements of these weighting matrices is formulated and solved as an optimization problem. In this manner, the decision variables of this optimization problem are the same as the stated diagonal elements, its objective function is the same as the control problem objective function, and its constraints are the sum of the diagonal elements of each matrix being a constant and these elements being nonnegative. Since the objective function of this problem is a concave function of its variables which should be minimized and the problem feasible set is a convex set, so the optimal solution is located at one of the extreme points of the feasible set. This extreme point is guessed using a method. Then using necessary changes, the optimization problem will change to an unconstrained problem and will be solved by the steepest descent method. The resulting optimal solution agrees with the guessed extreme point.

PROBLEM STATEMENT

Few papers relating to this subject can be found in the literature. Here, we refer to some of them which have a limited relation to the corresponding subject. Sengupta [1] and Pindyck [2] formulate and solve the economic stabilization policy as a linear-quadratic control problem. The problem of optimal economic stabilization policies under decentralized control and conflicting objectives is discussed in Pindyck [3], and the same problem with multiple objectives is considered in Deissenberg [4]. The problem of the substitution between objectives is studied more than elsewhere in Rustem, Velupillai and Westcott [5]. They start with an initial weighting matrix and offer a method to update this matrix according to the policy-maker points of view. This iterative procedure continues until the optimal solution which is acceptable to the policy-maker is found for the problem. The method of updating the objective function coefficients is developed in Rustem and Velupillai [6].

Consider the linear-quadratic control problem for a discrete-time system in steady state with the system equation

$$x(k+1) = Ax(k) + Bu(k) \qquad (1)$$

and with the objective function

$$J = \sum_{k=0}^{\infty} [x'(k)\, Q\, x(k) + u'(k)\, R\, u(k)] \qquad (2)$$

which should be minimized, when x (k) is the n-dimensional state vector, u(k) is the m-dimensional control vector, and A, B, Q and R are constant matrices of appropriate size. Suppose x (0) is given, the system (1) is controllable, and both of Q and R are diagonal positive definite matrices. Then as it is known [7], the optimal control in stage k is

$$u(k) = -Kx(k)$$

when K is an mxn constant gain matrix which satisfies

$$K = (R + B'PB)^{-1} B'PA$$

and P is an nxn symmetric positive definite matrix which is the solution of the following Riccati equation:

$$P = Q + A'PA - A'PB(R + B'PB)^{-1} B'PA \qquad (3)$$

Equation (3) can also be written as a function of the matrix K; i.e.,

$$P = Q + A'PA - A'PBK \qquad (4)$$

Also, the minimum value of the objective function can be written in terms of the matrix P as

$$\text{Min. } J = x'(o)\, Px(o) \qquad (5)$$

Now, we want to obtain the best values of the diagonal elements of the matrices Q and R by solving an optimization problem. Suppose the i-th element of the matrix Q main diagonal is denoted by α_i and the j-th element of the matrix R main diagonal is denoted by β_j ; i.e.,

$$Q = \text{diag}\,[\alpha_1, \alpha_2, \ldots, \alpha_n],$$
$$R = \text{diag}\,[\beta_1, \beta_2, \ldots, \beta_m].$$

In addition, the column vectors α and β with the dimensions n and m, respectively, are defined as

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)',$$
$$\beta = (\beta_1, \beta_2, \ldots, \beta_m)'.$$

Then the matrices P and K, and so the objective function J are all functions of the vectors α and β and the optimization problem of this paper is to compute the vectors α and β in the following:

$$\text{Min. } J(\alpha, \beta) = x'(o)\, P(\alpha, \beta)\, x(o)$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n} \alpha_i = a \\ \sum_{j=1}^{m} \beta_j = b \\ \alpha_i \geq o,\ \beta_j \geq o, \text{ for all } i, j \end{cases} \tag{6}$$

when a and b are positive constant values.

As seen in the model (6), the sum of the diagonal elements of each of the matrices Q and R is considered equal to a positive constant. Because without adding these two constraints, the problem has the trivial solution $\alpha_i = \beta_j = 0$. Also, for modeling the problem of substitution between objectives; i.e., for reaching one of the objectives how much we are ready to be far from the other objectives, it is necessary to add these two constraints. On the other hand, since the matrices Q and R should be positive definite, the sign constraints should be stated as $\alpha_i > 0$, $\beta_j > 0$. But since the feasible set of the problem is an open set in that case, and the extreme point is not defined for such a set, at present the problem is exactly considered like the model (6) for the mathematical solution in the next section. Then, the positiveness of α_i and β_j will be provided in the numerical solution of the problem.

MATHEMATICAL SOLUTION OF THE OPTIMIZATION PROBLEM

Since the objective function of the model (6) should be minimized and its feasible set is convex, if we show that the objective function is concave respect to α and β then it is proved that the optimal solution will be located at one of the extreme points of the feasible set.

Proposition- The objective function of the model (6) or $J(\alpha, \beta)$ is a concave fucntion of α and β.

Proof- Suppose the feasible set of the model (6) is denoted by F. Then by definition, $J(\alpha, \beta)$ is a concave function of α and β if for every α^1, $\beta^1 \in F$ and α^2, $\beta^2 \in F$, and any real number $0 \leqslant \theta \leqslant 1$, we have

$$x'(0)\, P(\theta\alpha^1 + (1-\theta)\alpha^2, \theta\beta^1 + (1-\theta)\beta^2)\, x(0)$$
$$\geq \theta x'(0)\, P(\alpha^1, \beta^1) x(0) + (1-\theta) x'(0)\, P(\alpha^2, \beta^2) x(0) \tag{7}$$

Suppose the diagonal matrices corresponding to the coefficient vectors α^1, α^2, β^1 and β^2 are denoted by Q_1, Q_2, R_1 and R_2 respectively. Also, the solutions of the Riccati equation (3) for the weighting matrices (Q_1, R_1) and (Q_2, R_2) are P_1 and P_2, respectively. In addition, the solution of this Riccati equation is denoted by P for the weighting matrices $(Q_1 + Q_2,\ R_1 + R_2)$. By the Riccati equation properties, its solutions will be θP_1 and $(1-\theta) P_2$ for the matrices $(\theta Q_1, \theta R_1)$ and $((1-\theta) Q_2, (1-\theta) R_2)$, respectively. Therefore, the inequality (7) will be satisfied if we can prove

$$P \geqslant P_1 + P_2 \tag{8}$$

The inequality (8) is proved by induction. In practice, the solution of the Riccati equation is obtained by using an iterative procedure; i.e., the last value of the matrix P is put on the right hand side of this equation in order to compute the next value of this matrix in each iteration, starting with $P^1 = Q$, when P^k denotes the value of the matrix P in the k-th iteration. Then according to the induction method, assuming

$$P^k \geqslant P_1^k + P_2^k,$$

we want to prove

$$P^{k+1} \geqslant P_1^{k+1} + P_2^{k+1}$$

The following equations show P^{k+1}, P_1^{k+1} and P_2^{k+1} in terms of P^k, P_1^k and P_2^k, respectively:

$$P^{k+1} = Q_1 + Q_2 + A'P^kA - A'P^kB(R_1 + R_2 + B'P^kB)^{-1} B'P^kA,$$
$$P_1^{k+1} = Q_1 + A'P_1^kA - A'P_1^kB(R_1 + B'P_1^kB)^{-1} B'P_1^kA,$$
$$P_2^{k+1} = Q_2 + A'P_2^kA - A'P_2^kB(R_2 + B'P_2^kB)^{-1} B'P_2^kA,$$

So for proving the inequality (8), it is sufficient to show the following inequality:

$$A'P^kB(R_1 + R_2 + B'P^kB)^{-1} B'P^kA$$
$$\leqslant A'P_1^kB(R_1 + B'P_1^kB)^{-1}B'P_1^kA$$
$$+ A'P_2^kB(R_2 + B'P_2^kB)^{-1} B'P_2^kA \tag{9}$$

According to the induction assumption, we have

$$P^k \geqslant P_1^k + P_2^k,$$

and so the following inequalities are resulted one after the other:

$$B'P^kB \geqslant B'P_1^kB + B'P_2^kB,$$

$$R_1 + R_2 + B'P^kB \geqslant (R_1 + B'P_1^kB) + (R_2 + B'P_2^kB),$$

$$(R_1 + R_2 + B'R^kB)^{-1} \leqslant [(R_1 + B'P_1^kB) + (R_2 + B'P_2^kB)]^{-1} \leqslant (R_1 + B'P_1^kB)^{-1} + (R_2 + B'P_2^kB)^{-1},$$

$$A'P^kB(R_1 + R_2 + B'P^kB)^{-1}B'P^kA \leqslant A'(-P^k)B(R_1 + B'P_1^kB)^{-1}B'(-P^k)A + A'(-P^k)B(R_2 + B'P_2^kB)^{-1}B'(-P^k)A \quad (10)$$

On the other hand, we can write $P^k \geqslant P_1^k$ and $P^k \geqslant P_2^k$, which result $-P^k \leqslant -P_1^k$ and $-P^k \leqslant -P_2^k$. In the first and second terms on the right hand side of the inequality (10), we put $-P_1^k$ and $-P_2^k$ instead of $-P^k$, respectively:

$$A'P^kB(R_1 + R_2 + B'P^kB)^{-1}B'P^kA \leqslant A'(-P_1^k)B(R_1 + B'P_1^kB)^{-1}B'(-P_1^k)A + A'(-P_2^k)B(R_2 + B'P_2^kB)^{-1}B'(-P_2^k)A \quad (11)$$

The inequality (9) is obtained by multiplying the negative signs by each other in (11). In addition, the following equalities hold in the first iteration of the iterative procedure for solving the Riccati equation: $P^1 = Q_1 + Q_2$, $P_1^1 = Q_1$ and $P_2^1 = Q_2$; i.e., for $k = 1$, the inequality $P^1 \geqslant P_1^1 + P_2^1$ holds. So the proof of induction is complete.

To the best of the knowledge of the author, the formulation of the problem (6), the mathematical and the numerical solutions of this problem, and the above proposition with its proof are the original contributions of this paper. As stated before, the substitution between objectives problem is considered in [5] and [6], but their approach to the problem is different from the one we used above; i.e., instead of solving an optimization problem like (6), they used an iterative procedure to improve the values of the objective function coefficients in each iteration, converging to the best values of these coefficients. Therefore, they did not need to prove the concavity of the objective function respect to α and β.

Therefore as stated before, the optimal solution will be at one of the extreme points of the problem (6) feasible set. of course, in the numerical method for finding this extreme point, the positive definiteness of the matrices Q and R should be taken into account; i.e., none of the diagonal elements of these two matrices can be equal to zero. In practice, a very small positive number will be used instead of zero for these diagonal elements.

NUMERICAL SOLUTION OF THE PROBLEM

For solving problem (6) by the usual numerical methods of solving unconstrained problems like steepest descent, its constraints should be eliminated somehow. First, we can be sure of satisfying the practical form of the sign constraits; i.e., $\alpha_i > o$ and $\beta_j > o$, for all i and j, by adding a conditional statement to our computer program. Second, for eliminating the equality constraints, one of the coefficients in each constraint is computed in terms of the other coefficients and this coefficient in company with the corresponding constraint is sent out of the model; for instance, if we want to send the coefficients α_k and β_1 out of the model, we will write the following equations:

$$\alpha_k = a - \sum_{i \neq k} \alpha_i \quad (12)$$

$$\beta_1 = b - \sum_{j \neq 1} \beta_j \quad (13)$$

In this manner, the constraints of the problem are eliminated and the minimization of the problem objective function is taken place respect to the coefficients α_i, $i \neq k$ and β_j, $j \neq 1$. Then out of the model, the optimal values of α_k and β_1 are computed from equations (12) and (13), respectively.

About the selection of the eliminated coefficients from the model, it can be argued that: any extreme point of the problem (6) feasible set has the property in which in the first constraint, one of the coefficients α_i is equal to a, while the other coefficients are equal to zero, and in the second constraint, one of the coefficients β_j is equal to b, while the other coefficients are equal to zero. These nonzero coefficients are actually the same as the eliminated coefficients from the model and the selection of them determines the corresponding extreme point. Now, if the numerical value of the gradient vector of $J(\alpha, \beta)$ respect to α and β can be computed somehow, the eliminated coefficients from the model or α_k and β_1 will be selected such that

$$J'(\alpha_k)=\min_i J'(\alpha i) \quad (14)$$

$$J'(\beta 1)=\min_j J'(\beta j) \quad (15)$$

when $J'(\alpha_i)$ denotes the first partial derivative of J with respect to α_i, and so on. Because, for example, $J'(\alpha_i)$ shows the change in the value of the objective function for a unit change in α_i. Since we want to increase the value of α_k up to the constant a, for minimizing J, it is better to select the element of the gradient vector which has the minimum value. The same argument applies to the gradient vector of J with respect to β. Of course as stated before, for preserving the positive definiteness of the matrices Q and R, any zero coefficient is taken to be equal to a small positive number.

For making sure of the optimality of the selected extreme point, the unconstrained optimization problem is solved by the steepest descent method. As known, the numerical values of the objective function, the gradient vector and the step size are needed in each iteration of this method, which are computed as follows. First, the Riccati equation (3) should be solved for computing the numerical value of the objective function. As stated, this equation is solved using an iterative procedure in the following manner: initially, the matrix Q is put instead of the matrix P in the righthand side of this equation, then in each iteration, the computed value of P is put instead of this matrix in the righthand side and this procedure is continued until the convergence is obtained. The solution of the equation is the matrix to which P converges, and the minimum value of the objective function is computed by setting this matrix in the equation (5).

Second, for computing the numerical value of the gradient vector, the Riccati equation (4) is written in terms of α and β as

$$P(\alpha,\beta)=Q(\alpha)+A'P(\alpha,\beta)A-A'P(\alpha,\beta)BK(\alpha,\beta) \quad (16)$$

If the derivative is taken with respect to α_i from both sides of equation (16), when the derivative of the matrices P and Q with respect to α_i are respectively denoted by $P_{\alpha i}$ and $Q_{\alpha i}$, it will result in

$$P\alpha_i=Q\alpha_i+A'P\alpha_i(A-BK) \\ -A'PB(R+B'PB)^{-1}B'P\alpha_i(A-BK) \quad (17)$$

Similarly, if the derivative is taken with respect to β, from both sides of equation (16), when the derivative of the matrices P and R with respect to β_j are respectively denoted by $P_{\beta j}$ and $R_{\beta j}$, the result will be

$$P\beta j=A'P\beta j(A-BK) \\ -A'PB(R+B'PB)^{-1}[B'P\beta j(A-BK)-R\beta jK] \quad (18)$$

For computing the matrices $P_{\alpha i}$ and $P_{\beta j}$, from equations (17) and (18), an iterative procedure similar to the procedure of computing the matrix P from the Riccati equation (3) is used, when the initial conditions are $P_{\alpha i} = Q_{\alpha i}$ in (17) and $P_{\beta j} = O$ in (18). Taking the derivative from the objective function of the problem (6) with respect to α_i and β_j, it results in

$$J\alpha_i=x'(O)P\alpha_i\,x(O) \quad (19)$$

$$J\beta j=x'(O)P\beta j\,x(O) \quad (20)$$

when $J_{\alpha i}$ and $J_{\beta j}$ denote the derivatives of the objective function J with respect to α_i and β_j, respectively. The numerical values of the gradient vectors are computed by setting the matrices $P_{\alpha i}$ and $P_{\beta j}$ in equations (19) and (20).

Third, the cubic fit method [8] is used for computing the step size in the steepest descent method. Since in this method, in addition to the numerical values of the objective function at two different points, the numerical values of this function first derivative at these two points are also needed, the method of computing the objective function first derivative with respect to the step size at a particular point is stated here. In the steepest descent method, the vector α proceeds as follows:

$$\alpha_{k+1}=\alpha_k - a_k g_k$$

when α_k, g_k and a_k are the values of the vector α, the gradient vector and the step size in iteration k, respectively. The objective function J can be expressed as a function of the step size a; i.e.,

$$J(a)=x'(0)P(\alpha_k - ag_k)x(0)$$

Then, J(a) is differentiated with respect to a and the numerical value of the derivative at a particular point a_o or $J'(a_o)$ is computed as

$$J'(a_o)=-\nabla J(a_o)g_k \qquad (21)$$

Now, it is time to state the numerical solution algorithm of the problem using the steepest descent method. The following steps are used in each iteration of this method:

Step 0- The numerical values of the matrices A, B, Q, R, $Q_{\alpha i}$ for all i, $R_{\beta j}$ for all j, the vector $x(0)$ and the constants a and b in the model (6) are introduced. In this step, Q and R are considered to be the $n \times n$ and $m \times m$ identity matrices, respectively.

Step 1- Solving the Riccati equation (3) and equations (17) and (18), the numerical values of the matrices P, $P_{\alpha i}$ and $P_{\beta j}$ are computed. Then, the values of the objective function and the gradient vector are obtained from equations (5), (19) and (20). Also, the numerical value of the objective function first derivative with respect to step size at $a_o = 0$ is calculated using (21).

Step 2- The minimum elements of the gradient vector with respect to α and β are computed from (14) and (15), and the indices k and l are determined. A step is taken with the step size $a_o = 1$, and the new values of the matrices Q and R diagonal elements for $i \neq k$ and $j \neq l$ are obtained. The values of α_k and β_1 are calculated by equations (12) and (13), respectively. If any of these diagonal elements become less than or equal to zero, it will be set equal to the small positive number 10^{-6}.

Step 3- The computations of Step 1 for the new values of the matrices Q and R are repeated again, only this time, the derivative of J with respect to the step size in (21) is obtained at $a_o = 1$.

Step 4- The optimal step size is determined using the cubic fit method between the two points $a_o = 0$ and $a_o = 1$. It should be noted that for the step size corresponding to each iteration of the cubic fit method, the calculations of Step 2 and then Step 1 are exactly performed and this process is continued until the convergence occurs in the step size value.

Step 5- The diagonal elements of the matrices Q and R are updated using the optimal step size and it is returned to Step 1.

Steps 1 to 5 are repeated as long as the diagonal elements of the matrices Q and R are converged.

NUMERICAL EXAMPLE

A numerical example with the dimensions n=5 and m=3 is considered which its input quantities are

$$A=\begin{bmatrix} 2 & -1 & -2 & 3 & 5 \\ 1 & 4 & 3 & -1 & 2 \\ 3 & 5 & 6 & 4 & -1 \\ 1 & 2 & 4 & 3 & -2 \\ 5 & -5 & -4 & 6 & 1 \end{bmatrix}, B=\begin{bmatrix} 1 & 2 & -2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \\ 4 & -2 & 3 \\ 5 & 1 & -3 \end{bmatrix}$$

$$x(O)=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad Q=I_5,\ R=I_3,\ a=5 \text{ and } b=3.$$

The output of this example is given in Table 1.

CONCLUSIONS

The substitution between objectives problem or determination of the best coefficients in the objective function of a linear-quadratic control problem was discussed. This optimization problem was formulated as the model (6). Then, proving the concavity of this model objective function, it was shown that the corresponding optimal solution was located at one of the extreme points of its feasible set. In the problem

Table 1. The Numerical Results of the Example

i	α_1	α_2	α_3	α_4	α_5	β_1	β_2	β_3	j
1	1	1	1	1	1	1	1	1	89.5035
2	10^{-6}	10^{-6}	10^{-6}	0.5743	4.4257	10^{-6}	2.999998	10^{-6}	8.3270
3	10^{-6}	10^{-6}	10^{-6}	0.3333	4.6667	10^{-6}	2.999998	10^{-6}	8.2854
4	10^{-6}	10^{-6}	10^{-6}	0.2053	4.7947	10^{-6}	2.999998	10^{-6}	8.2394
5	10^{-6}	10^{-6}	10^{-6}	0.1356	4.8644	10^{-6}	2.999998	10^{-6}	8.1900
6	10^{-6}	10^{-6}	10^{-6}	0.0925	4.9074	10^{-6}	2.999998	10^{-6}	8.1318
7	10^{-6}	10^{-6}	10^{-6}	0.0171	4.9828	10^{-6}	2.999998	10^{-6}	7.5639
8	10^{-6}	10^{-6}	10^{-6}	10-0	4.999996	10^{-6}	2.999998	10^{-6}	5.0253
9	10^{-6}	10^{-6}	10^{-6}	10-0	4.999996	10^{-6}	2.999998	10^{-6}	5.0253

numerical solution, a method of selecting this extreme point was introduced first. Then, the problem constraints were eliminated somehow, and the unconstrained problem was solved by the steepest descent method. The numerical example of the paper showed the selected extreme point was actually the optimal solution of the problem.

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