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An Extension to the Economic Production Quantity Problem with Deteriorating Products Considering Random Machine Breakdown and Stochastic Repair Time

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ABSTRACT

The recent advances in manufacturing systems motivate several studies to focus on Economic Production Quantity (EPQ) problem. Althuogh there are several extentions to the EPQ, this paper provides a new extension by considering some of the real world parameters like: (a) shortages in the form of partial backordering, (b) inventory can deteriorate stochastically, (c) machine can break down stochastically, and (d) machine repair time may change stochastically based on the failure status of machine. As far as we know, there is no study treated all these suppositions in an EPQ framework. In addition to this development, two forms of uniformly- and exponentially-distributed repair times are formulated and necessary convexity conditions are discussed. Then, the corresponding optimality conditions are written that lead to finding the roots of two equations. Due to difficulty of achieving a closed-form solution, the solution is obtained numerically by means of Newton-Raphson method. Finally, some sensitivity analyses are provided to explain the models' applicability. The practicality and efficiency of the proposed method in this context lends weight to development of proposed EPQ with more complex elements and its application more broadly.

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1. INTRODUCTION

Production management is a difficult job when a manager should simultaneously consider many conflicting parameters through his/her decision making [1]. There are many models in the literature that each of them cast into specific working conditions of the plant. Economic Production Quantity (EPQ) problem is well-studied that can be extended into the many different cases in reality. For example, there is no inventory deterioration in the basic EPQ model. But, it can be observed in many products in the forms of decay, spoilage, vaporization, obsolescence, etc. In more details, many kinds of food and volatile liquids have a natural deteriorate-prone characteristic.

At first, as one of earliest studies, Misra [2] in 1975 developed an EPQ model with deterioration consideration. He studied the model with the fixed and variable deterioration rates and derived an estimation of expression for the size of production lot. Then, in 1999, Kim and Hong [3] studied EPQ with random process of the deteriorating production. They presented three deterioration models. Next, in 2004, Samanta [4] presented a production inventory model with allowable backlogging, where the items deterioration time follow the exponential distribution function.

In a similar manner, Baten and Kamil [5] developed the model of a production inventory considering Pareto distribution. They assumed that the time of deterioration follows from a generalized Pareto distribution with three parameters. They also assumed that a firm has no shortage and demand rate varies with time. In that model Pontryagin maximum principle is used to obtain an explicit solution under dealing with continuous review policy. In another integrated production-distribution system, Yang et. al., [6] did it for a deteriorating inventory item through a two-echelon supply chain with fixed deterioration rate. They derived the optimal

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solution based on an algebraic method for the case that deteriorating rate is very small.

Another realistic working condition is that there is partial backlogging instead of two extremes of lost sales and backlogs. This means that β% of shortages are to be satisfied in the future and (1-β)% of the shortages can be considered as lost sales. In another word, in real-world concepts, a percentage of customers is selected to wait for backlogging until next replenishment. This backlogging rate depends on waiting time diminished the waiting time length. Up to our knowledge, Mak [7] proposed partial backorder and formulated a model for a production-inventory system. During each period, the demand is partially backlogged and the items are replenished by assuming a uniform rate. He supposed that the willingness for a customer waiting for backorder during the period of shortage is declining regarding the waiting time length. Later, Balkhi [8] proposed an inventory-production system considering deterioration and learning effect with allowable partial backorders. Next, Lo et. al., [9] introduced a new integration of production and inventory model considering the Weibull distribution deteriorating items under partial backlogging and imperfect production process. Then, as mentioned above, Yang et. al., [6] extended the inventory models considering deteriorating items along with partial backlogging in an inventory lot-size model.

On the other hand, machine unavailability is an inevitable phenomenon in many production industries that is the result of three main things; preventive maintenance, corrective maintenance and machine suddenly breakdown. There are some studies in literature in which the machine unavailability has been explored. For example, Cheung and Hausman [10] introduced an unreliable production system to illustrate a balance between the costs of safety stock and preventive maintenance. They considered the time to failure by Weibull and Gamma distributions and solved it with exponentially and constant repair time. Independently, Abboud [11] considered a production-inventory system considering exponential machine breakdown and general machine repairing time. They supposed that no backordering is allowed.

Later, Arreola-Risa et al. [12] studied an inventory problem. They assumed that the supply of the products was arbitrarily disrupted with regrads to the random duration periods. They entered partial backorder and lost sales in to their proposed model. In a different research, Giri et al. [13] proposed two mathematical models for EPQ problem with unreliable production and maintenance cost. The main difference of their research was the consideration of the production rate as a decision variable. Afterward, Chakraborty et al. [14] introduced an EPQ considering machine breakdown and deterioration. They considered preventive and corrective maintenance, simultaneously. The same authors [15] one year later, proposed an integration of production, inventory and maintenance activities in a manufacturing system to study the joint impacts of inspections, machine breakdown and process deterioration on the decisions of the optimal lot sizing.

From the last decade, several studies have focused on various modifications and improvements of EPQ with regards to uncertain parameters. In 2010, Widyadana and Wee [16] studied an EPQ to deteriorate the items with stochastic repair time and machine breakdown. They determined the production time and the lost sales, optimality. They added a deteriorating product item and price dependent demand in to their model. They proposed two models with uniform and exponential distributions for machine repair time. Because of the complexity of the given problem a Genetic algorithm (GA) is employed to address it optimally. The same authors [17] one year later, extended their paper by using an optimization technique to derive an optimal solution but for the case of constant and non-price dependent demand rate. In the same way, in 2011, Chung et al. [18] expanded the work of Widyadana and Wee [17]. They assumed that the lost sales takes place when an urgent need is requested by the buyer who cannot wait for the next replenishment. They considered the backorder taking the place when the buyer can wait for the next replenishment. In 2012, for a production system, Chiu et al. [19] obtained the optimal replenishment run time considering failure for reworking and machine breakdown. They studied the effect of Poisson breakdown on the replenishment run time.

Furthermore, Taleizadeh et al. [20] suggested a multi-product single-machine inventory-production model with production capacity and service level limitation while considering partial backlogging. After that in 2013, Sarkar [21] developed a production-inventory model for a deteriorating item. He considered three types of statistical functions including the Uniform, Triangular, and Beta for deterioration. However, the shortage was not allowed. An algebraically solution is applied to find the minimum total cost of these models. Subsequently, Chang [22] extended the work of Sarkar and Sarker [23] by developing an improved solution method. They [23] presented an Economic Manufacture Quantity (EMQ) with deterioration consideration. They considered exponential demand and time varying the production rate. The Euler-Lagrange formula of control theory is used to derive the maximization procedure. Lastly, the same authos [24] developed an inventory model with regads to the diversification of the deteriorating items. They assumed that demand is stock-dependent and shortage is partial backlogging.

In 2014, Taleizadeh [25] contributed an Economic Order Quantity (EOQ) to evaporate the items with partial consecutive prepayments and backordering. They applied it to a real case of gasoline stations in Iran. Similarly, De et al., [26] considered an EOQ under uncertainty based on the backlogged. In their problem, the demand of customers is varied by sold price of the items. In addition, the demand in stock out is changed by duration of the shortage in each period. The total cost is included by set up cost, promotional effort costs, shortage cost and the inventory costs. Khan et al., [27], developed an integrated supply chain model under the learning curve effects in the production process. Also in their investigation defective products occur due to human errors. They assumed multi delivery EPQ policy for the vendor and traditional EOQ policy for the buyer.

In 2015, De et al., [28] in another work proposed EOQ with intuitionistic fuzzy inventory model with the possibility of backlogging. They used the score functions with the non-membership and membership functions. In addition, Palanivel and Uthayakumar [29] proposed an extension to the EOQ with non-instantaneous items. They contributed the price and advertisement dependent demand pattern. These suppositions are under the inflation factor and time value over a finite planning horizon.

In 2016, Pal et al., [30] developed an EPQ improvement by an integration of supplier, manufacturer and retailer. They considered the joint economic lot-sizing approach by considering imperfect quality and uncertain demand. Their main difference was to consider the production of supplier i.e., the raw material sending to the manufacturer as the decision variable. In addition, Mohanty et al., [31] offered another EPQ which is a two-warehouse inventory system with regards to non-instantaneous deteriorating products under uncertainty. They considered the shotages combining partial backlog and lost sales.

In 2017, Chanda and Kumar [32] developed an EOQ problem considering a firm selling the technology of the products in a finite planning horizon. The demand was dynamic and uncertain ceiling on the potential adoptions. This factor is sensitive with regards to the unit selling price and advertising expenditure. Manna et al., [33] extended an EPQ model with allowable shortages. In their model, imperfect items are reworked or disposed of then to reduce the defective items, the learning effect is considered in the production process.

In 2018, Qiu et al., [34] introduced closed loop production rooting problem with remanufacturing and reverse logistics activities under the vender managed inventory contract. They provided a comparison of system costs concerning the different remanufacturing parameters. In addition, Marchi et al., [35] offered a two-level (vendor-buyer) supply chain models considering two coordinated policies: classical and vendor-managed inventory with consignment stock, where the objective is to find the values of the decision variables that yield the minimum total supply chain cost. It includes the costs of holding inventory, green emissions and tax, energy usage, product and process quality, and transportation operations. The decision variables are the order quantity, the number of shipments, and the production rate.

More recently in 2019, Marchi et al., [36] investigated the learning effect in energy efficiency which is an essential factor in many manufacturing companies. Therefore, they proposed a lot-sizing problem to illustrate the interaction between learning in production and energy efficiency directly and indirectly and also an appropriate decision about the lot size quantity. Walid et al., [37] applied the learning effects in the mean and variance of non-conforming items which considered as a random variable. Therefore, they decrease under the effect of the learning process. Finally, Chen et al., [38] considered a firm (e.g., retailer) selling a single nonperishable product over a finite-period planning horizon. At the beginning of each period, the firm determines its selling price and inventory replenishment quantity with the objective of maximizing total profit, but it knows neither the average demand (as a function of price) nor the distribution of demand uncertainty a prior; hence, it has to make pricing and ordering decisions based on observed demand data.

Based on the reviewed studies the essential parameters should be considered in this problem are: shortage, machine breakdown and its repair time and deterioration. Also considering the stochastic nature of these parameters is an essential necessity to close the model to a reality which makes the model applicable to every industrial production company which is neglected in many previous studies. Our paper is a continuation of Chung et al. [18] and Widyadana and Wee [17]. The main difference of our paper with them is to consider the shortage and inventory levels stochastically for the first time. Taken together, as far as we know and based on the aforementioned papers, this study provides an extension to the EPQ problem by considering the following assumptions:

(a) Shortages are in the form of partial backordering,
(b) Inventory can deteriorate stochastically,
(c) Machine can break down stochastically due to its past workload, lubrication, etc.
(d) Machine repair time may change stochastically based on the failure status of machine. Also, two cases of uniformly- and exponentially-distributed repair times are formulated and necessary convexity conditions are discussed. Remaining parts of this paper is organized as follows: notation and proposed mathematical model is presented in section 2, computational results and sensitivity analysis are discussed in section 3, and finally some concluding remarks and guides for future research are given in section 4.

2. PROBLEM FORMULATION

The statement of our proposed problem is defined as follows: Figure 1 illustrates the variations of inventory/

shortage level during a working cycle for the proposed inventory-production model. During production time T_1, machine breakdown occurs stochastically. If machine doesn't break down over time segment T_1, the inventory reaches to its maximum level (I_m). When machine breaks down, the production process is stopped and a corrective repair process is started, where the repair time of T_p is a stochastic variable. In addition, products deteriorate continuously. Since production deterioration, machine breakdown and machine repair time are stochastic, shortage may occur in terms of partial backorders. In the partial backordering, backorders may change to the lost sales, stochastically. This section first provides a general formulation for the problem. Then, two models will be developed for the special cases of uniformly and exponentially distributed repair times.

Based on the above description, the following assumptions are existed in this paper:

- A plant produces a single product with constant rates of production and consumption (i.e. demand), where production rate is greater than the demand rate.
- There isn't any inventory at the start and the end of a working cycle.
- Machine breakdown can occur such that the time between two consecutive breakdowns follows from exponential distribution with parameter μ.
- After each breakdown, the machine needs repair and/or maintenance. Also, machine repair time is a stochastic variable with known distribution function. Also it is assumed that the machine repair time is independent of the time between two machine breakdowns.
- A fixed fraction of the inventory deteriorates per time unit. In addition, no repair or replacement of the deteriorated inventory is required during the planning period.
- Inventory deterioration can be considered in both production and consumption time of each cycle.
- The production manager aims to minimize the total costs of inventory holding, setups, lost sales, backorders and partial backorders.

The problem can be formulated by means of the following parameters and variables:

Input parameters:	
D	Demand rate
P	Production rate
Θ	Deterioration rate
K	Setup cost
H	Unit inventory holding cost
Π	Unit lost sales cost
$\hat{\pi}$	Unit backorder cost
C	Unit deterioration cost
M	Repair cost of machine
β	Parameter of partial backorder
μ	Breakdown rate of machine

Variables:	
I_m	Maximum inventory level of a cycle
I_1	Inventory level in a production period of a cycle
I_2	Inventory level in non-production period of a cycle
I	Total number of inventories per cycle
R	Total number of deteriorated items per cycle
I_L	Lost sales level of a cycle
I_b	Backorder level of a cycle
T_1	Production period of a cycle
T_2	Non-production period of a cycle
T_3	Shortage period of a cycle
T_p	Machine breakdown period of a cycle

The initial general calculations based on Widyadana and Wee [17] can be reviewed as follows:

At first, T_2 and T_3 are derived based on T_1. Equations (1) and (2) represent the inventory levels in production and non-production periods, respectively:

$$\frac{dI_1(t_1)}{dt_1} + \theta I_1(t_1) = p - d \qquad 0 \le t_1 \le T_1 \tag{1}$$

$$\frac{dI_2(t_2)}{dt_2} + \theta I_2(t_2) = -d \qquad 0 \le t_2 \le T_2 \tag{2}$$

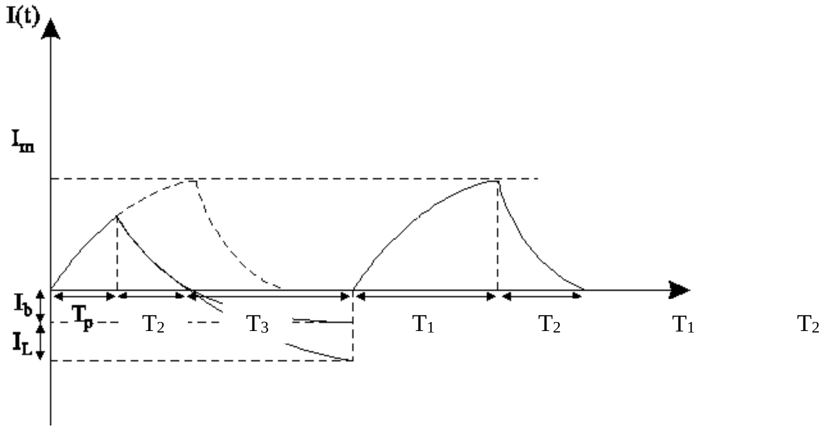

Figure 1. A working cycle of the studied inventory system

As shown in Figure 1, inventory level is zero at the end of T_2. So, inventory level in production and non-production periods can be formulated as follows:

$$I_1(T_1) = (\frac{p-d}{\theta})\left[1-e^{-\theta t_1}\right] \qquad 0 \le t_1 \le T_1 \tag{3}$$

$$I_2(T_2) = \frac{d}{\theta}(e^{\theta(T_2-t_2)}-1) \qquad 0 \le t_2 \le T_2 \tag{4}$$

At the top of inventory level, while t_1 equals to T_1 and t_2 is zero, $I_1(T_1)$ and $I_2(T_2)$ are equal. Then, we can write:

$$\frac{p-d}{\theta}(1-e^{-\theta T_1}) = \frac{d}{\theta}(e^{\theta T_2}-1) \tag{5}$$

Further, ΘT can be approximated by means of Taylor expansion. Since the estimation error of the Taylor series is negligible after the third term, we can approximate ΘT as follows:

$$e^{\theta T} \approx 1+\theta T+\frac{\theta T^2}{2} \tag{6}$$

According to Yang and Wee [39], the error of this estimation is lower than 0.2%. By replacing $e^{\theta T}$ approximation in Equation (5), the following equation is derived:

$$\frac{p-d}{\theta}(T_1-\frac{1}{2}\theta T_1^2) = \frac{d}{\theta}(T_2+\frac{1}{2}\theta T_2^2) \tag{7}$$

Since θT_2 is a small value, $\theta T_2{}^2$ can be eliminated in Equation (7). Hence, T_2 can be approximated in terms of T_1 as:

$$T_2 \approx \frac{(p-d)T_1(1-\frac{1}{2}\theta T_1)}{d} \tag{8}$$

Thus, the expected inventory level can be calculated as follows:

$$E(I) = \frac{(p-d)T_1^2}{2}+\frac{dT_2^2}{2} \tag{9}$$

By replacing T_2 from Equation (8) into Equation (9), we have:

$$E(I) = p-d(\frac{T_1^2}{2})+d\left(\frac{(p-d)T_1(1-\frac{1}{2}\theta T_1)}{2d}\right)^2 \tag{10}$$

Since θT_1 is very small, Equation (10) can be simplified as follows:

$$E(I) = \frac{p^2}{2d}(1-\frac{d}{p})T_1^2 \tag{11}$$

Obviously, machine breakdown can affect on the expected inventory level. Since expected inventory level depends on machine breakdown period, Equation (11) is formulated as follows:

$$E(I) \cong \begin{cases} \frac{p^2}{2d}(1-\frac{d}{p})T_p^2 & T_p \le T_1 \\ \frac{p^2}{2d}(1-\frac{d}{p})T_1^2 & T_p > T_1 \end{cases} \tag{12}$$

The distribution function of machine breakdown period is exponential with parameter µ. By getting the integral of the right hand side of Equation (12), the expected inventory level can be written as follows:

$$E(I) = \frac{p(p-d)(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1})}{d\mu^2} \tag{13}$$

According to Figure 1, the expected number of deteriorated items per cycle is:

$$E(R) = pT_1 - d(T_1+T_2) \tag{14}$$

By substituting T_2 from Equation (8) into relation (14), the expected number of deteriorated items per cycle is:

$$E(R) = pT_1 - d\left(T_1+\frac{(p-d)T_1(1-\frac{1}{2}\theta T_1)}{d}\right) \tag{15}$$

After simplifying Equation (15) and considering machine breakdown, the expected number of deteriorated items per cycle can be written as follows:

$$E(R) \cong \begin{cases} \frac{p}{2}(1-\frac{d}{p})\theta T_p^2 & T_p \le T_1 \\ \frac{p}{2}(1-\frac{d}{p})\theta T_1^2 & T_p > T_1 \end{cases} \tag{16}$$

As T_p follows from exponential distribution, the expected number of deteriorated items per cycle can be obtained by getting the integral from the right hand side of Equation (16) with the use of derivative function as follows:

$$E(R) = \frac{(p-d)\theta(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1})}{\mu^2} \tag{17}$$

In the next sub-sections, two cases of exponentially and uniformly distributed repair times are considered.

2. 1. Exponential Repair Time Let the repair time follow from an exponential distribution function with parameter of λ. The expected shortage time can be written as follows:

$$E(T_3) = \int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)\lambda e^{-\lambda t}\mu e^{-\mu T_p}\,dt\,dT_p \tag{18}$$

After computing the previous integral, expected shortage time is formulated as follows:

$$E(T_3) = \frac{e^{T_1\left(\frac{\lambda(p-d)}{2d}(-2+\theta T_1)+\frac{\lambda p}{d}+\mu\right)} - e^{\lambda T_1\left(\frac{T_1 p\theta}{2d}+1-\frac{\theta T_1}{2}\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} \quad (19)$$

Since a working cycle is the sum of production, non-production and shortage periods, then the expected time of a working cycle is:

$$E(T) = E(T_1+T_2) + E(T_3) \quad (20)$$

$$E(T) = \frac{p\left(1-e^{-\mu T_1}\right)}{d\mu} + \frac{e^{T_1}\left(\frac{\lambda(p-d)}{2d}(-2\theta T_1)+\frac{\lambda p}{d}+\mu\right) - e^{\lambda T_1}\left(\frac{T_1 p\theta}{2d}+1-\frac{\theta T_1}{2}\right)}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}} \quad (21)$$

Also, the probability of machine breakdown during production period is:

$$P(M_c) = (1-e^{-\mu T_1}) \quad (22)$$

So, it is clear that the expected repair cost can be formulated as follows:

$$E(M_c) = M(1-e^{-\mu T_1}) \quad (23)$$

Unsatisfied demand is divided into the backorder and lost sales. The probability of backlogging is a decreasing function of time *t*, where *t* is the waiting time up to the next replenishment. Conversely, the probability of lost sales is an increasing function of time *t*. The expected volume of backlogs and lost sales can be formulated as follows, respectively:

$$E(S_1) = d\int_{T_p=0}^{T_1}\int_{t=T_p}^{\infty}(t-T_2)\,f(t)e^{-\sigma(t-T_2)}\mu e^{-\mu T_p}\,dt\,dT_p \quad (24)$$

$$E(S_2) = d\int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)\,f(t)(1-e^{-\sigma(t-T_2)})\mu e^{-\mu T_p}\,dt\,dT_p \quad (25)$$

Thus, the total expected cyclic costs due to lost sales, backorders, setups, repair, inventory holding, and inventory deterioration can be written as follows:

$$TC(T_1,T_2) = E\left[K + M(1-e^{-\mu T_1}) + h\times E(I) + C\times E(R) + \pi E(S_1) + \hat{\pi}E(S_2)\right] \quad (26)$$

$$\begin{aligned} TC(T_1,T_2) = E\Bigg[& K + M(1-e^{-\mu T_1}) + h\times\frac{p(p-d)(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1})}{d\mu^2} + C\times\frac{(p-d)\theta(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1})}{\mu^2} \\ & + \pi d\int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)f(t)(1-e^{-\sigma(t-T_2)})\mu e^{-\mu T_p}\,dt\,dT_p + \hat{\pi}d\int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)f(t)e^{-\sigma(t-T_2)}\mu e^{-\mu T_p}\,dt\,dT_p)\Bigg] \end{aligned} \quad (27)$$

Consequently, the expected total cost per unit time can be written as:

$$\begin{aligned} TCT(T_1,T_2) = & \frac{k + \mu(1-e^{-\mu T_1}) + (h\frac{p}{d}+C\theta)(\frac{p-d)(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1})}{\mu^2})}{\frac{p(1-e^{-\mu T_1})}{d\mu} + \int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)f(t)\mu e^{-\mu T_p}\,dt\,dT_p} \\ & + \frac{\pi d\int_{T_p=0}^{T}\int_{t=T_2}^{\infty}(t-T_2)f(t)(1-e^{-\sigma(t-T_2)})\mu e^{-\mu T_p}\,dt\,dT_p + \hat{\pi}d\int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)f(t)e^{-\sigma(t-T_2)}\mu e^{-\mu T_p}\,dt\,dT_p}{\frac{p(1-e^{-\mu T_1})}{d\mu} + \int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)f(t)\mu e^{-\mu T_p}\,dt\,dT_p} \end{aligned} \quad (28)$$

Further, TCT is a convex function of T_1 under the following condition:

$$M < \frac{\left(\frac{hp}{d}+C\theta\right)(p-d) - \left(\frac{\pi d\left(\sigma^2+2\sigma\lambda\right)}{(\sigma+\lambda)^2} + \frac{\hat{\pi}d\lambda^2}{(\sigma+\lambda)^2}\right)\left(2\left(\frac{\lambda\mu(p-d)}{d}-\mu^2\right)\right)}{\mu^2}$$

A detailed proof is available upon request from authors. Then, the optimal value of T_1 can be found by getting the derivative of Equation (28) with respect to T_1 and equaling the result to zero:

$$\begin{aligned} \frac{dTCT}{dT_1} = & \frac{M\mu e^{-\mu T_1} + \left(\frac{hp}{d}+C\theta\right)\left(h(p-d)T_1 e^{-\mu T_1}\right)\left(e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}\right)}{\left(\frac{p\left(1-e^{-\mu T_1}\right)}{d\mu}\right)\left(e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}\right) - A_e + B_e} \\ & \frac{-\frac{\pi\left(\sigma^2+2\sigma\lambda\right)+\hat{\pi}\lambda^2}{(\sigma+\lambda)^2}d\left(\left(\lambda\left(\frac{T_1 p\theta}{d}\right)+1-\theta T_1+\frac{p}{d}+\frac{\mu^{T_1}}{\lambda}\right)+\left(\frac{\lambda(d-p)}{d}+\frac{(p-d)\lambda\theta T_1}{d}\right)B_e\right)}{\left(p\frac{\left(1-e^{-\mu T_1}\right)}{d\mu}\right)\left(e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}\right) - A_e + B_e} \end{aligned} \quad (29)$$

$$-\frac{C_e\left(\left(\frac{hp}{d}+C\theta\right)\left(\frac{(p-d)\left(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1}\right)}{\mu^2}\right)\left(e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}\right)\right)}{\left(\frac{p(1-e^{-\mu T_1})}{d\mu}e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}-A_e+B_e\right)}$$

$$-\frac{C_e\left(k+M\left(1-e^{-\mu T_1}\right)-\left(\left(\frac{\pi d\left(\sigma^2+2\sigma\lambda\right)}{(\sigma+\lambda)^2}+\frac{\hat{\pi} d\lambda^2}{(\sigma+\lambda)^2}\right)\times\frac{e^{\frac{-\lambda(p-d)T_1}{d}}}{\lambda}\right)\right)}{\left(\frac{p(1-e^{-\mu T_1})}{d\mu}e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}-A_e+B_e\right)}$$

where, A_e, B_e and C_e are:

$$A_e=e^{\lambda T_1\left(\frac{T_1\theta p}{d}+1-0/5\theta T_1\right)} \tag{30}$$

$$B_e=e^{T_1\left(\frac{\lambda(p-d)}{d}+\frac{\lambda(p-d)\theta T_1}{2d}+\frac{\lambda p}{d}+\mu\right)} \tag{31}$$

$$C_e=\frac{pe^{-\mu T_1}}{d}-\frac{\lambda\left(\frac{T_1 p\theta}{2d}+1-0/5\theta T_1-\frac{p}{d}-\frac{\mu}{\lambda}\right)e^{A_e}+\left(\frac{\lambda(p-d)}{d}+\frac{(p-d)\lambda\theta T_1}{d}\right)e^{B_e}}{\left(e^{\lambda_1\left(\frac{\lambda p}{d}+\mu\right)}\right)} \tag{32}$$

Unfortunately, it is not easy to obtain a closed-form solution by finding the parametric roots of Equation (29) with respect to T_1. Hence, an efficient method such as Newton-Raphson or bi-section method should be hired to find the solution numerically.

2. 2. Uniform Repair Time In this section, uniform distribution function with 0 and b parameter is supposed for repair time. Expected repair time can be calculated as follows:

$$E(T_3)=\frac{\left((b\mu d)^2-2b(p-d)d\mu+2(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2\mu T_1)-2b\mu d\right)}{2b\mu^2 d^2} \tag{33}$$

Combine Equations (20) and (33), expected time of planning period is as follows:

$$E(T)=\frac{p(1-e^{-\mu T_1})}{d\mu}+\frac{\left((b\mu d)^2-2b(p-d)d\mu+2(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2\mu T_1)-2b\mu d\right)}{2b\mu^2 d^2} \tag{34}$$

The probability of backlogging is β and the probability of lost sales is (1-β). The expected value of backlogging and lost sales can be formulated as follows:

$$E(S_1)=d\int_{T_p=0}^{T_1}\int_{t=T_p}^{\infty}(t-T_2)\,f(t)\beta\mu e^{-\mu T_p}\,dtdT_p \tag{35}$$

$$E(S_2)=d\int_{T_p=0}^{T_1}\int_{t=T_2}^{\infty}(t-T_2)\,f(t)(1-\beta)\mu e^{-\mu T_p}\,dtdT_p \tag{36}$$

The total cost function of t can be derived as follows:

$$\begin{aligned}TCT(T_1)=&\frac{k+M\left(1-e^{-\mu T_1}\right)+\frac{hp(p-d)\left(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1}\right)}{\mu^2 d}+C\theta(p-d)\theta\left(1-e^{-\mu T_1}-\mu T_1 e^{-\mu T_1}\right)}{\frac{p\left(1-e^{-\mu T_1}\right)}{d\mu}+\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2 d^2}}\\&+\frac{\hat{\pi}\beta d\,\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2 d^2}}{\frac{p\left(1-e^{-\mu T_1}\right)}{d\mu}+\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2 d^2}}\\&+\frac{\pi(1-\beta)d\,\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2 d^2}}{\frac{p\left(1-e^{-\mu T_1}\right)}{d\mu}+\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2 d^2}}\end{aligned} \tag{37}$$

Derivation of Equation (37) from T_1 is as follows:

$$\frac{dTCT(T_1)}{dT_1}=\frac{M\mu e^{-\mu T_1}+(h\frac{p}{d}+c\theta)\left((p-d)T_1e^{-\mu t_1}\right)-\left(\hat{\pi}\beta+\pi(1-\beta)\right)\left(\frac{\mu^3e^{-\mu T_1}\left(bd\left(-bd+2(p-d)T_1\right)-T_1(p-d)^2\right)}{2db\mu^2}\right)}{\frac{p\left(1-e^{-\mu T_1}\right)}{d\mu}+\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2d^2}}$$
$$-\frac{A\left(K+M\left(1-e^{-\mu t_1}\right)-\frac{(h\frac{p}{d}+c\theta)\left((p-d)\left(\mu t_1e^{-\mu t_1}+e^{-\mu t_1}-1\right)\right)}{\mu^2}\right)}{\left(\frac{p\left(1-e^{-\mu T_1}\right)}{d\mu}+\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2d^2}\right)^2}$$
$$-\frac{A\left((\hat{\pi}\beta+\pi(1-\beta))d\left(\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2d^2}\right)\right)}{\left(\frac{p\left(1-e^{-\mu T_1}\right)}{d\mu}+\frac{\left(\left((b\mu d)^2-2b(p-d)\mu d+(p-d)^2\right)\left(1-e^{-\mu T_1}\right)-e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1)-2b\mu d\right)\right)}{2b\mu^2d^2}\right)^2} \tag{38}$$

where A is as follows:

$$A=\frac{\left(\mu^3e^{-\mu T_1}\left(bd\left(-bd+T_1(p-d)\left(2-(p-d)\right)\right)\right)\right)}{2b\mu^2d^2} \tag{39}$$

The optimal value of T_1 value can be found by using Maple software with regards to Equation (38) to zero. Like Poisson repair time model, total cost function should be convex to deriving T_1 value if parameters are under this condition:

$$M<\frac{\left(\frac{hp}{d}+C\theta\right)(p-d)-\left(\frac{\pi d\left(\sigma^2+2\sigma\lambda\right)}{(\sigma+\lambda)^2}+\frac{\hat{\pi}d\lambda^2}{(\sigma+\lambda)^2}\right)\left(2\left(\frac{\lambda\mu(p-d)}{d}-\mu^2\right)\right)}{\mu^2} \tag{40}$$

The proof of total cost convexity is shown in appendixes.

3. NUMERICAL EXAMPLE

Here, we will do some sensitivities numerically on the proposed model. In this regard, a numerical example with the following input parameters will be solved in this section:

d=7500	*h=1*	$\lambda=2$
P=10000	$\hat{\pi}=2$	*M=200*
Θ=0.2	$\pi=3$	*C=5*
K=50	*b=0.1*	$\mu=0.2$
S=5		

Moreover, the Matlab software is used to solve the Equation (29) resulting in T_1=0.1443. The optimum total cost per unit time can be calculated by substituting T_1 in Equation (28) resulting in TCT= $692.0716. These are the result when repair time following Uniform distribution and for Exponential distribution according to Equation (38) T_1=0.2017 and by substituting Equation (37) resulting in TCT=$590.9. If the values do not apply to the unequal convexity condition of the numerical method, Newton -Raphson is used to solve Equations (29) and (38), respectively.

Figure 2 shows the comparison between partial backordering and lost sales during different repair time. It is illustrated that partial backordering case has less total cost, the difference between them are greater by increasing the value of production up time. It is concluded that partial backordering is more economical and pursuant to real world conditions.

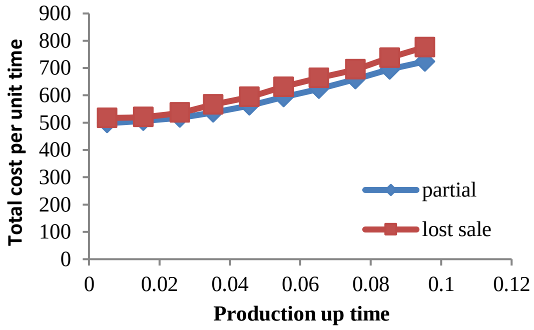

Figure 2. Total cost comparison for different production up time

4. CONCLUSION AND FUTURE RESEARCH

This paper studied an EPQ problem while considering new flavors from real world, including: (a) shortages in the form of partial backordering, (b) stochastic inventory deterioration, (c) stochastic machine breakdown, and (d) two types of stochastic repair time (i.e. uniform and exponential distributions). The two cases are formulated and necessary convexity conditions are discussed. Then, the corresponding optimality conditions are written that led to finding the roots of two equations. As it was hard to obtain a closed-form solution, the solution is obtained numerically by means of Newton-Raphson method. This research can be extended in various directions such as:

- other types of distributions (e.g. Normal) can be considered for repair times,
- the machine could experience different types of modes such as working normally, working abnormally by producing a number of defective items (i.e. a partial failure), and being unable to continue working (i.e. a complete failure) [40], the system can hire inspection procedures for forecasting future machine failures that can result in less number of emergency repairs/ maintenances [41]. Other types of inventory deterioration can be considered (e.g. fixed life time products) [42], multiple items can be produced on the machine, due to the complexity of the proposed model, various heuristics and meta-heuristics [43-45] and generally approximation methods [46-47] can be considered to address it optimality demand can be changed dynamically and/or stochastically [48-49].

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6. APPENDIXES

6. 1. Appendix A

In order to prove that the $TCT(t_1)$ is differentiable and also has a single root, firstly the following mathematical theory should be regarded:

Property: If the $f(x)=\dfrac{g(x)}{h(x)}$ and the domain and range values for both the functions are the same, the $f(x)$ is a convex function under the following conditions:

1. $g(x)$ is convex on the domain interval and $g(x)\geq 0$ for all x in this interval.
2. $h(x)$ is concave on the domain interval and $h(x)\geq 0$ for all x in this interval.
3. Both $g(x)$ and $h(x)$ functions are differentiable.

Here $f(x)$ is the total cost per unit time, $g(x)$ is the total cost and $h(x)$ is the total replenishment period. Basically if $g(x)$ convex and $h(x)$ is concave then the $f(x)$ will be convex. In the following the convexity in presence of the small values for λ,μ is investigated:

The expected replenishment time can be drived as follows:

$$E(T_E)=\frac{p\left(1-e^{-\mu T_1}\right)}{d\mu}+\left(\frac{-e^{+\lambda T_1\left(0/5\theta T_1\frac{(p-d)}{d}+1\right)}+e^{\lambda T_1\left(0/5\theta T_1\frac{(p-d)}{d}+\frac{\mu}{\lambda}+1\right)}}{e^{\lambda_1\left(\frac{\lambda p}{d}+\mu\right)}}\right)$$

Moreover, the second derivative of the expected replenishment time is:

$$\frac{dT_E^2}{d^2T_1}=\left(-\lambda\left(\frac{\theta(p-d)}{d}\right)e^{T_1\left(0/5\theta T_1\lambda\frac{(p-d)}{d}+\lambda\right)}-\left(\theta T_1\lambda\frac{(p-d)}{d}+\lambda\right)^2 e^{T_1\left(0/5\theta T_1\frac{\lambda(p-d)}{d}+\lambda\right)}+\frac{(p-d)}{d}\lambda\theta e^{T_1(0/5\theta T_1\lambda)\times\frac{(p-d)}{d}+\lambda+\mu}\right.$$

$$\left.+\left(\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu\right)^2 e^{T_1\left(0/5\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu\right)}\right)e^{\frac{1}{T_1\left(\frac{hp}{d}+\mu\right)}}$$

$$-2\left(-\left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)e^{T_1(0/5\theta T_1\lambda)\left(\frac{p-d}{d}\right)+\lambda}\right.$$

$$\left.+\left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)e^{T_1\left(0/5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\right)\frac{\left(\lambda\frac{p}{d}+\mu\right)}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}$$

$$+\left(-e^{T_1\left(0/5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda\right)}+e^{T_1\left(0/5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\right)\frac{\left(\lambda\frac{p}{d}+\mu\right)^2}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}<0$$

This term can be simplified as:

$$\frac{dT_E^2}{dT_1^2}=\frac{-p\mu e^{-\mu T_1}}{d}+\left(-\lambda\left(\frac{\theta(p-d)}{d}\right)A-(\theta T_1\lambda)\frac{(p-d)}{d}+\lambda\right)^2 A+\frac{(p-d)}{d}\lambda\theta Ae^{\mu T_1}$$

$$+\left(\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)^2 Ae^{\mu T_1}\Bigg)\frac{1}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}-2\left(-\left(\theta T_1\lambda\frac{(p-d)}{d}+\lambda\right)A+\left(\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu\right)Ae^{\mu T_1}\right)\times\frac{\left(\frac{\lambda p}{d}+\mu\right)}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}$$

$$+\left(-A+Ae^{\mu T_1}\right)\frac{\left(\frac{\lambda p}{d}+\mu\right)^2}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}<0 \qquad A=e^{T_1\left(0/5\theta T_1\frac{\lambda(p-d)}{d}+\lambda\right)}$$

So it can be rewritten as following equation:

$$\frac{d^2T_E}{d^2T_1}=dT_{E_1}+dT_{E_2}$$

$$dT_{E_1}=-\frac{p\mu e^{-\mu T_1}}{d}$$

$$dT_{E_2}=\left(\lambda\left(\frac{\theta(p-d)}{d}\right)\left(-1+e^{\mu T_1}\right)-\left(\theta T_1\lambda\left(\frac{(p-d)}{d}\right)+\lambda\right)^2 A+\left(\theta T_1\lambda\left(\frac{(p-d)}{d}\right)+\lambda+\mu^2\right)Ae^{\mu T_1}\right.$$

$$+2\left(\left(\theta T_1\lambda\frac{(p-d)}{d}+\lambda\right)A-\left(\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu\right)Ae^{\mu T_1}\right)\left(\frac{\lambda p}{d}+\mu\right)$$

$$+\left(-A+Ae^{\mu T_1}\right)\left(\frac{\lambda p}{d}+\mu\right)^2$$

The second term in the mentioned formula can be simplified to the following equation:

$$dT_{E_2}=\left(\frac{\lambda(\theta)(p-d)}{d}\right)\left(-1+e^{\mu T_1}\right)-\left(A_a-C\right)^2 A+\left(A_b-C\right)^2 Ae^{\mu T_1}$$

$$A_a=\theta T_1\lambda\frac{(p-d)}{d}+\lambda \qquad C=\left(\frac{\lambda p}{d}+\mu\right)$$

$$A_b=\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu$$

In this term if $A_b>A_a$ then $dT_{E_2}>0$. Accordingly the expected replenishment time is convex if :

$$\frac{p\mu e^{-\mu T_1}}{d}>\lambda\left(\frac{\theta(p-d)}{d}\right)\left(-1+e^{\mu T_1}\right)-\left(A_a-C\right)^2 A+\left(A_b-C\right)^2 Ae^{\mu T_1}$$

If θ and T_1 take small values the above condition can be rewritten as follows:

$$\frac{p\mu e^{-\mu T_1}}{d}>\lambda\left(\frac{\theta(p-d)}{d}\right)\left(-1+e^{\mu T_1}\right)-(\lambda-C)^2 e^{\lambda T_1}+(\lambda+\mu-C)^2 e^{(\lambda+\mu)T_1}$$

And in presence of small values for the μ,λ and θT_1 the condition can be simplified as:

$$2\left(\frac{p(\lambda-0/5)-\lambda d}{d}\right)-\mu<0$$

Here if $\lambda<0/5$ or $\lambda<\dfrac{d\mu+p}{2(p-d)}$ the expected replenishment time will be convex.

In the second part the concavity of the total cost function is investigated; its function is as follows:

$$TC_E = k + \mu\left(1 - e^{-\mu T_1}\right) + \frac{h(p)(p-d)\left(1 - e^{-\mu T_1} - \mu T_1 e^{-\mu T_1}\right)}{\mu^2 d} + \frac{C\theta(p-d)(\theta)\left(1 - e^{-\mu T_1} - \mu T_1 e^{-\mu T_1}\right)}{\mu^2}$$

$$+\left(\frac{\pi d\left(\sigma^2 + 2\sigma\lambda\right)}{(\sigma+\lambda)^2} + \frac{\hat{\pi} d\lambda^2}{(\sigma+\lambda)^2}\right)\left(\frac{e^{-\lambda T_1\left(0/5\theta T_1\frac{(p-d)}{d}+1\right)} + e^{\lambda T_1\left(0/5\theta T_1\frac{(p-d)}{d}+1+\frac{\mu}{\lambda}\right)}}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}\right)$$

Its second derivative is:

$$\frac{dTC_E^2}{dT_1^2} = -M\mu^2 e^{-\mu T_1} + \left(\frac{hp}{d} + C\theta\right)(p-d)\left(e^{-\mu T_1}\right)(1 - \mu T_1)$$

$$+\left(\frac{\pi d\left(\sigma^2+2\sigma\lambda\right)}{(\sigma+\lambda)^2} + \frac{\hat{\pi} d\sigma\lambda^2}{(\sigma+\lambda)^2}\right)\frac{1}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}\left(-\lambda\left(\theta\frac{(p-d)}{d}e^{T_1\left(0/5\theta T_1\lambda\frac{(p-d)}{d}+\lambda\right)} - \left(\theta T_1\frac{\lambda(p-d)}{d}+\lambda+\mu\right)^2 \times e^{T_1\left(0/5\theta T_1\lambda\frac{(p-d)}{d}+\lambda\right)}\right.\right.$$

$$\left. + \frac{(p-d)}{d}\lambda\theta e^{T_1\left(0/5\theta T_1\lambda\left(\frac{p-d}{d}\right)+\lambda+\mu\right)} + \left(\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu\right)^2 e^{T_1\left(0/5\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu\right)}\right)$$

$$-2\left(-\left(\theta T_1\lambda\frac{(p-d)}{d}\right)+\lambda\right)e^{T_1\left(0/5\theta T_1\left(\frac{p-d}{d}\right)+\lambda\right)} + \left(\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu\right)e^{T_1\left(0/5\theta T_1\left(\frac{p-d}{d}\right)+\lambda+\mu\right)}\Bigg)\frac{\left(\frac{\lambda p}{d}+\mu\right)}{e^{T_1\left(\frac{\lambda p-d}{d}\right)+\mu}}$$

$$+\left((-e^{T_1\left(0/5\theta T_1\lambda\frac{(p-d)}{d}+\lambda\right)} + -e^{T_1\left(0/5\theta T_1\lambda\frac{(p-d)}{d}+\lambda+\mu\right)})\frac{\left(\frac{\lambda p}{d}+\mu\right)^2}{e^{T_1\left(\frac{\lambda p}{d}+\mu\right)}}\right)$$

If μ , λ , T_1 are small we will have:

$$\frac{d^2TC_E}{d^2T_1} = -M\mu^2 + \left(\frac{hp}{d}+C\theta\right)(p-d) - \left(\frac{\pi d\left(\sigma^2+2\lambda\sigma\right)}{(\sigma+\lambda)^2} + \frac{\hat{\pi} d\lambda^2}{(\sigma+\lambda)^2}\right)\left(\left(\lambda - \left(\frac{\lambda p}{d}+\mu\right)^2 + \left(\lambda+\mu-\left(\frac{\lambda p}{d}+\mu\right)^2\right)\right)\right)$$

So the total cost functions is convex if:

$$M < \frac{\left(\frac{hp}{d}+C\theta\right)(p-d) - \left(\frac{\pi d\left(\sigma^2+2\sigma\lambda\right)}{(\sigma+\lambda)^2} + \frac{\hat{\pi} d\lambda^2}{(\sigma+\lambda)^2}\right)\left(2\left(\frac{\lambda\mu(p-d)}{d}-\mu^2\right)\right)}{\mu^2}$$

So based on the mentioned property, if this condition is verified by the problem parameters, the function is convex.

6. 2. Appendix B For the total cost function we have:

$$TC_u = K + M\left(1 - e^{-\mu t_1}\right) - \frac{(h\frac{p}{d} + c\theta)\left((p-d)\left(\mu t_1 e^{-\mu t_1} + e^{-\mu t_1} - 1\right)\right)}{\mu^2}$$

$$+(\hat{\pi}\beta + \pi(1-\beta))d\left(\frac{\left(\left((b\mu d)^2 - 2b(p-d)\mu d + (p-d)^2\right)\left(1-e^{-\mu T_1}\right) - e^{-\mu T_1}\mu T_1(p-d)\left((p-d)(2+\mu T_1) - 2b\mu d\right)\right)}{2b\mu^2 d^2}\right)$$

The second derivative of the total cost function is as follows:

$$\frac{dTC_U^2}{d^2T_1} = -M\mu^2 e^{-\mu T_1} + \left(\frac{hp}{d}+c\theta\right)\left((p-d)(1-b\mu)e^{-\mu T_1}\right)$$

$$-\left(\hat{\pi}\beta+\pi(1-\beta)\right)\left(\frac{\left(e^{-\mu T_1}\right)\mu^4\left(bd\left(bd+2T_1(p-d)+T_1^2(p-d)^2\right)+2e^{-\mu T_1}\right)\mu^3\left(bd\left(bd(p-d)-2T_1\left((p-d)^2\right)\right)\right)}{2db\mu^2}\right)$$

If $T_1 = 0$:

$$\frac{dTC_U^2}{d^2T_1} = -M\mu^2 + \left(\frac{hp}{d}+c\theta\right)(p-d) - \left(\hat{\pi}\beta+\pi(1-\beta)\right)\left(\frac{\mu bd\left(\mu bd + 2(p-d)\right)}{2}\right)$$

If the following condition will be satisfied this equation is convex:

$$M \prec \frac{\left(\frac{hp}{d}+c\theta\right)(p-d) - \left(\hat{\pi}\beta+\pi(1-\beta)\right)\left(\frac{\mu bd\left(\mu bd+2(p-d)\right)}{2}\right)}{\mu^2}$$ In

presence of the $T_1 = \frac{db\left(1-\frac{\theta b}{2}\right)}{(p-d)}$:

$$\frac{dTC_U^2}{d^2T_1} = \frac{1}{d}\left(h\left(8p(p-d) - 4pb\mu d(2+b\theta)\right) - \left(\hat{\pi}\beta+\pi(1-\beta)\right)\left(4\mu bd\theta(p-d) + b(\mu bd\theta)^2\right) + c\left(4\theta d\left(2p-2d-2pdb\mu-\mu^2 d\theta\right)\right) - M\mu^2 d\right) \times e^{\left(-\frac{\mu db(2+\theta b)}{2(p-d)}\right)}$$

This equation is convex if:

$$M < \frac{(hp+cd\theta)\left(8(p-d) - 4b\mu d(2+b\theta)\right) - \left(\hat{\pi}\beta+\pi(1-\beta)\right)\left(\mu bd\theta\left(4(p-d)+\mu b^2 d\theta\right)\right)}{\mu^2 d}$$

So based on the stated property, in presence of this condition the function is convex.

Persian Abstract

چکیده

پیشرفت های اخیر در سیستم های تولید، مطالعات متعددی را برای تمرکز بر مساله تولید اقتصادی (EPQ) ایجاد می کند. اگرچه چندین بخش EPQ وجود دارد، این مقاله با در نظر گرفتن برخی از پارامترهای دنیای واقعی مانند: (الف) کمبود به شکل بازسفارش جزئی، ب) موجودی که می تواند به صورت تصادفی رو به وخامت بگذارد، (ج) دستگاه می تواند به طور تصادفی شکسته شود، و (د) زمان تعمیر دستگاه ممکن است بر اساس وضعیت خرابی دستگاه تغییر کند. تا آنجا که می دانیم، هیچ مطالعه ای در مورد همه این فرضیات در چارچوب EPQ وجود ندارد. علاوه بر این توسعه، دو شکل از زمان تعمیر یکنواخت و نمایی توزیع شده تدوین شده و شرایط همرفت ضروری مورد بحث قرار گرفته است. سپس، شرایط بهینه مربوطه نوشته می شود که منجر به یافتن ریشه های دو معادله می شود. به دلیل دشواری در دستیابی به راه حل با فرم بسته، محلول با استفاده از روش نیوتن-رافسون به صورت عددی بدست می آید. سرانجام، برخی از تجزیه و تحلیل های حساسیت برای توضیح قابلیت کاربرد مدل ارائه شده است. عملی و کارایی روش پیشنهادی در این زمینه باعث می شود وزن EPQ پیشنهادی با عناصر پیچیده تر و کاربرد آن بطور گسترده تری ایجاد شود.