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# Change Point Estimation of the Stationary State in Auto Regressive Moving Average Models, Using Maximum Likelihood Estimation and Singular Value Decomposition-based Filtering

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*ABSTRACT*

In this paper, for the first time, the subject of change point estimation has been utilized in the stationary state of auto regressive moving average (ARMA) (1, 1). In the monitoring phase, in case the features of the question pursue a time series, i.e., ARMA(1,1), on the basis of the maximum likelihood technique, an approach will be developed for the estimation of the stationary state's change point. To estimate unidentified parameters following the change point, the dynamic linear model's filtering was utilized on the basis of the singular decomposition of values. The proposed model has wide applications in several fields such as finance, stock exchange marks and rapid production. The results of simulation showed the suggested estimator's effectiveness. In addition, a real example on stock exchange market is offered to delineate the application.

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## 1. INTRODUCTION

Auto regressive moving average (ARMA) is considered a kind of prevalent time series utilized in various areas, including quality control and financial markets. For the first time, the concept of change point in the field of quality control is mixed with the concept of ARMA (1, 1) in time series analysis. Analyzing ARMA could be different in non-stationary and stationary states. Hence, detecting the change point on the basis of the stationary state could be significant in enhancing the features of decision making and production processes, as well as decreasing costs.

The time that control charts demonstrate an uncontrolled signal would not often be the very time that a change occurs. In reality, a change occurs prior to the signal time of the control chart. The change point is the very time that a change occurs [1].

A model of change point which would detect either a trend change or a mean shift, or when autocorrelation is accounted for in short-term time series, could be examined by the use of simulations or a technique as Sturludottir et al. [2] suggested. Slama and Saggou [3] investigated the Bayesian assessment of a probable change in the variables of the autoregressive time series in the determined order of p, AR (p) in an isolated method.

Bodnar [4] generated some multivariable control diagrams to check the multivariate GARCH processes' mean vector at the time of the changes, through the maximization of the ratio of the generalized likelihood. A monitoring structure based on Shewhart control chart is used to monitor financial processes modeled with ARMA-GARCH time series structure by Doroudyan et al. [5]. A combined algorithm of time variable parameter (TVP) model, dynamic auto regressive exogenous variable (DARX) approach, nonlinear correlation analysis and criterion-based elimination method is presented by Moghadam et al. [6].

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To remove the autocorrelation impact in phase-II of the monitoring of polynomial auto-correlated profiles, in the case of AR(1), Keramatpour et al. [7] introduced the GLT/R diagram .

Vakilian et al. [8] utilized an estimator of the maximum likelihood so as to measure the monotonous change point in linear simple profiles with the 1st order autocorrelation autoregressive framework in every profile. In order of monitoring the financial trends modeled using the time series framework of ARMA-GARCH, A method was suggested by Doroudyan et al. [5] on the basis of the Shewhart control chart. Tian et al. [9] suggested a monitoring plan to identify a basic change in the randomized coefficient autoregressive model of time series of the order p (RCA(p)) in sequence, following the training duration of size T.

Chang et al. [10] proposed a new nonparametric analytical model for identifying heterogeneous segments in time-series data for data-abundant processes. Ebadi et al. [11] applied robust methods to estimate the parameters of multivariate processes in the absence and presence of outliers. Their numerical analysis has shown that the robust estimators have the same (in the absence of contamination) or better performance (in the presence of contamination) than the classical methods.

Nishina [12] proposed the built-in EWMA estimator, and Page [13] introduced the built-in CUSUM estimator in order of measuring the change point. Utilizing the maximum likelihood estimator (MLE), artificial neural networks (ANNs) and clustering were introduced to measure the change point alluded to in the literature. To measure a bivariate process's drift change point, ANN was utilized by Noorossana and Atashgar [14]. In addition, ANN was utilized by Ahmadzadeh [15] so as to measure the step change point in the multivariable process. Noorossana et al. [16], Pignatiello Jr, and Samuel [17], as well as Samuel et al. [18, 19] regarded the estimation of the step change point making use of the maximum likelihood approach. Daryabari et al. [20] used a time□dependent learning effect algorithm along with measurement errors and incorporate them into the Bernoulli CUSUM control chart statistic.

Perry et al. [21] along with Noorossana and Shadman [22] utilized MLE to detect the monotonous change point. Perry and Pignatiello Jr [23] in addition to Perry et al. [24] introduced the maximum likelihood approach so as to measure the change point drift. Besides, Ghazanfari et al. [25] accompanied by Alaeddini et al. [26] developed a clustering method so as to measure the step change point. To get a full overview of the literature on change point measurement, refer to Amiri and Allahyari [27].

Kazemzadeh et al. [28] introduced Maximum likelihood estimators (MLE) for both linear and step drift changes in the regression variables of linear multivariable profiles. Making use of maximum likelihood estimation, a significantly efficient model of change point was introduced by Doğu and Kocakoç [29] connected with the control chart of generalized variance where statistics needed were measured using relevant distributional features. MLE was applied by Lee and Park [30] to the point of the process change, where the control chart of the fixed sampling rate (FSR) scheme or the variable sampling rate (VSR) scheme was utilized to monitor a procedure for identifying the changes to the variance of a variable with a normal quality or the process mean. Likewise, Chang and Lu [31] suggested a combined likelihood approach to concurrently identify shifts in the variance and mean of a regular process.

Asghari Torkamani et al. [32] suggested the estimators of the change point for the variables of the correlated processes of the Poisson count. For this purpose, the Newton's method was utilized to estimate the process parameters. Next, the estimators of maximum likelihood were developed for the process change point .

To identify the mean of the multivariable normal processes making use of DLM, Ayoubi et al. [33] measured the occasional change point and obviated the necessity of possessing the advance knowledge of the type of the change. In the same vein, Ayoubi et al. [34] utilized DLM to estimate multivariable profile parameters following the change point, when the modeling changes were permitted in every direction. DLM is capable of estimating variables in an irregular or a regular way. Ayoubi et al. [35] suggested MLE, the monotonous change point, concerning the mean of multivariable linear profiles. In order of estimating the change point where the AR(1) stationary model would change into a non-stationary model, Sheikhrabori et al. [36] developed a maximum likelihood techniqu.

Poisson distribution for count processes and the first-order integer-valued autoregressive (INAR (1)) model are presented by Ashuri and Amiri [37], where that a combined EWMA and C control chart are utilized to monitor the process. They used Newton's method to estimate the parameters of the process after the change. Then, the maximum likelihood estimators be used to estimate the real time of change in the parameters [37].

Hawkins and Zamba [38] developed a control diagram in order to identify the shifts in the process variance, where the variance's regular value was unidentified.

Malela□Majika and Rapoo [39] suggested a control diagram of the exponentially weighted moving average-cumulative sum (EWMA-CUSUM) and a distribution-free combined control diagram of the cumulative sum exponentially weighted moving average (CUSUM-EWMA). The Wilcoxon rank sum test was the basis of the diagrams alluded to in identifying the shifts of the process mean with no distributional presumption of the basic quality process. Amiri and Zolfaghari [40] introduced a two-step technique, through which the quality features' mean values changed on the basis of the linear drift and step shift.

Daryabari et al. [41] studied effects of measurement errors on performance of the Max EWMAMS control chart, in terms of average time to signal (ATS) criterion. They showed that measurement errors adversely affect the performance of the control chart.

Making use of the autoregressive (AR(1)) model of the first order and the model of the autoregressive moving average (ARMA(1,1)), Safaeipour and Niaki [1] modelled a multistep process of an individual feature monitored at every step. They suggested MLE for the estimation of the change points, i.e. the stage number and the sample number where the drift change was applied to the location variable of multistep processes. They utilized EWMA and CUSUM control diagrams in order of monitoring the process. The "fuzzy shift change-point algorithm" was utilized by Lu et al. [42] that required neither the process parameter nor the distribution knowledge in order of identifying the shift change points of the process mean .

By monitoring the original observations' ARMA statistics, Jiang et al. [43] presented a control diagram for the autoregressive moving average (ARMA). Tsay and Tiao [44] proposed an integrated approach for the provisional particularization of the order of the combined non-stationary and stationary ARMA models.

To assess the time series, Shao and Yang [45] suggested a theoretically rationalized two-stage method, including the ARMA error term and the smooth process function, which was mathematically effective and simple for professionals to execute. The process was measured making use of the maximum likelihood estimator and the B-spline regression on the basis of residuals demonstrated to be oracally effective, and it was asymptotically as efficacious as the cases where the real process function was known and then eliminated to acquire ARMA errors.

Ghanbarzadeh and Aminghafari [46] presented a technique for predicting the time series of the non-stationary mode. It was on the basis of predicting non-decimated wavelet (NDW) signals via SSA, and then predicting the residuals making use of the wavelet regression.

MLE, in the present paper, was developed to estimate the the stationary state of the ARMA's change point (1,1), and following the change point, DLM was exerted to the parameter estimation. This led to the model change in every direction. DLM was able to estimate the varying parameters in irregular or regular manners. The proposed model has wide applications in several fields such as finance, stock exchange marks and rapid production .

The rest of the paper is structured as follows: Section 2 explains the stationary state concept in the models of time series. Section 3 discusses the monitoring approach. The explanation of DLMs is presented in section 4. Section 5 describes filtering as a DLM method of parameter estimation. Section 6 introduces The MLE proposed. Section 7 presents simulation results. A real example has been presented in section 8 to describe how to apply the method of the present paper. Conclusion is provided in the final part.

## 2. STATIONARY

Strictly speaking, a process is stationary if the function of its distribution is not time-dependent, simply put:

$$F(x_{t_1},\ldots,x_{t_n}) = F(x_{t_1+K},\ldots,x_{t_n+K}) \tag{1}$$

A time-series process is a weak stationary one, in case:

$$E(x_t) = \mu \tag{2}$$

$$E[(x_t-\mu)^2] = E[(x_{t-s}-\mu)^2] = \sigma^2 < \infty \tag{3}$$

$$E[(x_{t_1}-\mu)(x_{t_2}-\mu)] = \lambda_{t_1-t_2} \tag{4}$$

In the present paper, it was postulated that the ARMA (1,1) model was situated between the control chart's sample statistics. Therefore, the model of ARMA (1,1) ( $\bar{x}_j = \phi_1\bar{x}_{j-1} + \varepsilon_j + \psi_1\varepsilon_{j-1}$ ) was stationary in essence in case $|\phi_1|<1$, on the contrary, it would be non-stationary.

## 3. MONITORING METHOD

In the current study, it was presumed that there was a correlation among the sample statistics, namely, among $\bar{x}$ statistics. One can say that if a process is controlled (or stationary, i.e. $|\phi_1|<1$), at the time $j$, $\bar{x}_j$, the quality feature may be implied by the items that follow:

$$\bar{x}_j = \phi_1\bar{x}_{j-1} + \varepsilon_j + \psi_1\varepsilon_{j-1}, \qquad \varepsilon_j \sim NID(0,\sigma^2) \tag{5}$$

$\tau$ is the change point. If the process is uncontrollable, or non-stationary, value $\phi_j$ is produced from parameter $\phi_1$, which equals 1 or greater than one.

In the present study, the change point estimators of maximum likelihood were employed if the Shewhart control diagrams emitted an uncontrollable signal. The model residuals' Shewhart control diagram ( $e_1=0$ *and* $e_j=\bar{x}_j-\phi_1\bar{x}_{j-1}-\psi_1\varepsilon_{j-1}$ , *for* $j=2,3,\ldots$) presented by Montgomery [47] was utilized in the present paper, in which $\varepsilon_j$ is independently and normally distributed with $\sigma$ being the standard deviation, and the zero mean as follows:

$$UCL = 3\sigma \quad LCL = -3\sigma \tag{6}$$

In case the control diagram above emitted an uncontrollable signal at the time of *T*, it was resulted that a non-stationary model was produced from the stationary ARMA (1,1) process.

## 4. DELINEATION OF THE DYNAMIC LINEAR MODEL

In the present study, dynamic linear model (DLM) was applied in order to measure the parameters of the model following the change point. Therefore, this part is associated with the non-stationary model which follows the change point. The Gaussian linear state space model or the dynamic linear model is a type of state-space model where a normal distribution is followed by the findings. State-space models could be applied to the time series of a non-stationary nature. Further information about DLMs is discussed by Petris et al. [48].

Linear dynamic models are comprised of two equations; the 1st one is an observation equation and the 2nd one is a system or a state equation. The DLMs' primary goal is the obtaining of the state's posterior distribution, making use of the existing information. The Bayesian process is accomplished by taking into account an earlier guess at the 1st time ( $j = 0$ ) in the state equation, and together with the observed data so as to acquire the state's posterior distribution. The DLM equations in case of $j \geq 1$ are presented as such:

$$\begin{aligned} \mathbf{y}_j &= \mathbf{F}_j\boldsymbol{\theta}_j + \boldsymbol{\upsilon}_j, \quad \boldsymbol{\upsilon}_j \sim MN_m(\mathbf{0}, \mathbf{V}_j) \\ \boldsymbol{\theta}_j &= \mathbf{G}_j\boldsymbol{\theta}_{j-1} + \boldsymbol{\omega}_j, \quad \boldsymbol{\omega}_j \sim MN_p(\mathbf{0}, \mathbf{W}_j) \end{aligned}, \quad j \geq 1 \tag{7}$$

In which, $\mathbf{y}_j$ implies the *m*-variate vector, comprised of regular observations, $\boldsymbol{\theta}_j$ denotes the *p*-dimensional state vector, $\mathbf{F}_j$ is the identified $m \times p$ matrix, with $\mathbf{G}_j$ being the identified p × p matrix, and $\boldsymbol{\omega}_j$ and $\boldsymbol{\upsilon}_j$ represent multivariable regular random vectors with the covariance matrices and zero mean of $\mathbf{W}_j$ and $\mathbf{V}_j$, respectively.

The state space model's primary goal is the estimation of the non-observed state vector at any time, making use of the available data. In the DLM literature, the existing data ( $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_j$ ) are implied by $D_j$, with the state estimation being performable using three varied conditional probability density methods of $p(\boldsymbol{\theta}_s \mid \mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_j)$. In case $s = j$, filtering would be utilized. In case $s < j$, the smoothing of the problem would be employed. In the end, $s > j$ is related to the predication of the problem [48].

In the present study, the goal followed by the application of DLM was the estimation of the process parameters after the change point. Hence, after a change point, unidentified parameters might be estimated only via filtering.

### 4. 1. Modifying $\overline{x}$ Control's Chart ARMA (1,1) Model Into The Dynamic Linear Model

In this part, the equations of DLM were modified into the ARMA (1,1) model's observations mentioned in Equation (5) in the following manner:

$$\begin{aligned} \overline{x}_j &= \mathbf{F}_j * \boldsymbol{\theta}_j, \\ \boldsymbol{\theta}_j &= \mathbf{G}_j\boldsymbol{\theta}_{j-1} + \boldsymbol{\omega}_j, \quad \boldsymbol{\omega}_j \sim N(0, \mathbf{W}_j), \end{aligned} \quad \tau + 1 \leq j \leq T \tag{8}$$

In which, $m = p = 1$ with $\boldsymbol{\theta}_j = \begin{bmatrix} \overline{x}_j \\ \psi_1\varepsilon_j \end{bmatrix}$, $\boldsymbol{\omega}_j = \begin{bmatrix} \varepsilon_j \\ \psi_1\varepsilon_j \end{bmatrix}$ where $\varepsilon_j \sim N(0, \sigma^2)$, $\mathbf{F}_j = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\mathbf{G}_j = \begin{bmatrix} \phi_j & 1 \\ 0 & 0 \end{bmatrix}$, $V_j = 0$ and $\mathbf{W}_j = \sigma^2 * \begin{bmatrix} 1 & \psi_1 \\ \psi_1 & \psi_1^2 \end{bmatrix}$. As the estimation of the change point was addressed in the second phase of the present study, parameters $\sigma^2, \psi_1$, and $\phi = \phi_1$ were identified before the change point, based on the equation model (5). Subsequent to the change point, equation model (8) was applied, assuming that the variance of the process did not change, and just parameter $\phi = \phi_j$ was modified into a value equal to one or greater than it, thereby causing a non-stationary process to exist.

Change point $\tau$ was regarded as the DLM's starting point, for DLM was utilized following the change point until the signal time *T* of the control diagram. Thus, we would see $\tau + 1 \leq j \leq T$ in Equation (8). The available data about DLM, utilized for estimating the parameters, were presented as $D_j : \overline{x}_{\tau+1}, \overline{x}_{\tau+2}, ..., \overline{x}_j$. As the data were present until the signal time *T*, for the estimation of parameters at time $j$ ($\tau + 1 \leq j \leq T$), the predicting problem could be regarded. In the present study, parameter estimation was decided to happen following the change point, making use of the task of filtering (i.e. $p(\theta_j \mid \overline{x}_{\tau+1}, ..., \overline{x}_j)$ for $j = \tau + 1, \tau + 2, ..., T$ ). Thus, the following part was assigned to describing DLM filtering.

## 5. THE DYNAMIC LINEAR MODEL's FILTERING

In DLMs, the famous Kalman filter was assigned to the problem of filtering, having delineated in the following parts. By replacing $m = p = 1$, $\mathbf{F}_j = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\mathbf{G}_j = \begin{bmatrix} \phi_j & 1 \\ 0 & 0 \end{bmatrix}$, $V_j = 0$, and $\mathbf{W}_j = \sigma^2 * \begin{bmatrix} 1 & \psi_1 \\ \psi_1 & \psi_1^2 \end{bmatrix}$ in Kalman filter equations of the present study [48], the following part was described as the modification of the equation model (8).

**5. 1. Kalman Filter** To resolve the filtering issue, the starting point's succeeding distribution was regarded as the preceding guess, as follows:

$$\boldsymbol{\theta}_{\tau} \mid D_{\tau} \sim N(\mathbf{m}_{\tau}, \mathbf{C}_{\tau}) \tag{9}$$

Next, the variance and mean of the state vector's one-step-ahead normal predictive density was obtained for $j \geq \tau+1$, making use of the data $D_{j-1} : \bar{x}_{\tau+1}, \ldots, \bar{x}_{j-1}$, as follows:

$$\begin{aligned} \mathbf{a}_j &= E(\boldsymbol{\theta}_j \mid D_{j-1}) = \mathbf{G}_j \mathbf{m}_{j-1} \\ \mathbf{R}_j &= \mathrm{var}(\boldsymbol{\theta}_j \mid D_{j-1}) = \mathbf{G}_j \mathbf{C}_{j-1} \mathbf{G}'_j + \mathbf{W}_j \end{aligned} \tag{10}$$

The variance and mean of the observations' one-step-ahead normal predictive density were calculated making use of $D_{j-1}$ by utilizing the next equations:

$$\begin{aligned} f_j &= E(\bar{x}_j \mid D_{j-1}) = \mathbf{F}_j \mathbf{a}_j \\ Q_j &= \mathrm{var}(\bar{x}_j \mid D_{j-1}) = \mathbf{F}_j \mathbf{R}_j \mathbf{F}'_j + V_j \end{aligned} \tag{11}$$

Finally, utilizing the accurate formulas developed by Petris et al. [48] regarding filtering density, an individual matrix was obtained that could be utilized by the algorithm related to the SVD-based Kalman filter. To resolve this issue, the normal filtering density formulas of the states were confirmed by utilizing the procedure introduced by Meinhold and Singpurwalla [49] as applied to the $\bar{x}_j$ control chart's ARMA(1,1) model, in the present study (Appendix A).

Thus, the variance and mean were changed in the following manner:

$$\begin{aligned} \mathbf{m}_j &= E(\boldsymbol{\theta}_j \mid D_j) = \mathbf{a}_j + \begin{bmatrix} 1 \\ \dfrac{\psi_1 \sigma^2}{\mathbf{F}_j \mathbf{R}_j \mathbf{F}'_j} \end{bmatrix} * (\bar{x}_j - \mathbf{F}_j \mathbf{a}_j) \\ \mathbf{C}_j &= \mathrm{var}(\boldsymbol{\theta}_j \mid D_j) = \mathbf{R}_j - \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' \\ \psi_1 \sigma^2 \end{bmatrix} * (\mathbf{F}_j \mathbf{R}_j \mathbf{F}')^{-1} * \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' & \psi_1 \sigma^2 \end{bmatrix}. \end{aligned} \tag{12}$$

**5. 2. The Kalman Filter based On Singular Value Decomposition** The main drawback of Equations (12) and (10) was the round off errors or the numerical instability in measuring all matrices of covariance, i.e. $\mathbf{C}_j$ and $\mathbf{R}_j$. The problem could produce a matrix of covariance which was not positively determined or at least positively half decided. Thus, the algorithms of higher stability were developed to resolve this issue. The SVD-based Kalman filter was one of those algorithms that could be observed in the studies done by Zhang and Li [50] as well as Wang et al. [51]. The next part is concerned with the delineation of the SVD-based Kalman filter as utilized in the current study.

Concerning $j \geq \tau+1$, it was supposed that the SVD related to the covariance matrix $\mathbf{C}_{j-1}$ existed as such:

$$\mathbf{C}_{j-1} = \mathbf{U}_{j-1} \boldsymbol{\Lambda}^2_{j-1} \mathbf{U}'_{j-1} \tag{13}$$

Thus, $\mathbf{R}_j$'s value in Equation (10) was:

$$\mathbf{R}_j = \mathbf{G}_j \mathbf{U}_{j-1} \boldsymbol{\Lambda}^2_{j-1} \mathbf{U}'_{j-1} \mathbf{G}'_j + \mathbf{W}_j \tag{14}$$

Identifying the factors of $\mathbf{U}_{R_j}$ and $\mathbf{S}^2_{R_j}$, i.e. $\mathbf{R}_j = \mathbf{U}_{R_j} \boldsymbol{\Lambda}^2_{R_j} \mathbf{U}'_{R_j}$ was the following task. Hence, the next matrix could be described as follows:

$$\begin{bmatrix} \boldsymbol{\Lambda}_{j-1} \mathbf{U}'_{j-1} \mathbf{G}'_j \\ \sqrt{\mathbf{W}'_j} \end{bmatrix} \tag{15}$$

By measuring the above matrix's SVD, we would have:

$$\begin{bmatrix} \boldsymbol{\Lambda}_{j-1} \mathbf{U}'_{j-1} \mathbf{G}'_j \\ \sqrt{\mathbf{W}'_j} \end{bmatrix} = \mathbf{U}^k_{R_j} \begin{bmatrix} \boldsymbol{\Lambda}^k_{R_j} \\ 0 \end{bmatrix} (\mathbf{V}^k_{R_j})' \tag{16}$$

The pre-multiplication of the two sides of the above equation by their transposes would result in:

$$\mathbf{G}_j \mathbf{U}_{j-1} \boldsymbol{\Lambda}^2_{j-1} \mathbf{U}_{j-1} \mathbf{G}_j + \sqrt{\mathbf{W}_j} (\sqrt{\mathbf{W}_j})' = \mathbf{V}^k_{R_j} \begin{bmatrix} (\boldsymbol{\Lambda}^k_{R_j})' & 0 \end{bmatrix} (\mathbf{U}^k_{R_j})' \mathbf{U}^k_{R_j} \begin{bmatrix} \boldsymbol{\Lambda}^k_{R_j} \\ 0 \end{bmatrix} (\mathbf{V}^k_{R_j})' \tag{17}$$

Or similarly:

$$\mathbf{G}_j \mathbf{U}_{j-1} \boldsymbol{\Lambda}^2_{j-1} \mathbf{U}'_{j-1} \mathbf{G}'_j + \mathbf{W}_j = \mathbf{V}^k_{R_j} (\boldsymbol{\Lambda}^k_{R_j})^2 (\mathbf{V}^k_{R_j})' \tag{18}$$

Upon the comparing of Equation (14) and Equation (18), we would have:

$$\begin{cases} \mathbf{U}_{R_j} = \mathbf{V}^k_{R_j} \\ \boldsymbol{\Lambda}_{R_j} = \boldsymbol{\Lambda}^k_{R_j} \end{cases} \tag{19}$$

At this moment, the covariance matrix $\mathbf{R}_j$'s SVD was attainable. Hence, the definite symmetric positive matrix $\mathbf{R}_j$ could be rephrased in the following form:

$$\mathbf{R}_j = \mathbf{U}_{R_j} \boldsymbol{\Lambda}^2_{R_j} \mathbf{U}'_{R_j} \tag{20}$$

Thus, by utilizing the matrix inversion lemma [ $(\mathbf{A} + \mathbf{UBV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{B}^{-1} + \mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1}$ ], the filtering density updated could be measured by SVD in the following manner: (for further information, refer to Wang et al. [51])

$$\begin{aligned} &\mathbf{C}_j^{-1} = \mathbf{R}_j^{-1} + \mathbf{R}_j^{-1} \times \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' \\ \psi_1 \sigma^2 \end{bmatrix} \times (A)^{-1} \times \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' & \psi_1 \sigma^2 \end{bmatrix} \mathbf{R}_j^{-1}, \\ &\text{in } which\ A = \mathbf{F}_j \mathbf{R}_j \mathbf{F}' - \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' & \psi_1 \sigma^2 \end{bmatrix} \times \mathbf{R}_j^{-1} \times \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' \\ \psi_1 \sigma^2 \end{bmatrix} \end{aligned} \tag{21}$$

Using Equations (21) and (20), we could result in:

$$\mathbf{C}_j^{-1} = \mathbf{U}^{-1}_{R_j} \boldsymbol{\Lambda}^{-2}_{R_j} \mathbf{U}'^{-1}_{R_j} + \mathbf{U}'_{R_j} \mathbf{U}'^{-1}_{R_j} \times \mathbf{R}_j^{-1} \times \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' \\ \psi_1 \sigma^2 \end{bmatrix} \times (A)^{-1} \times \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' & \psi_{11} \sigma^2 \end{bmatrix} \mathbf{R}_j^{-1} \mathbf{U}_{R_j} \mathbf{U}^{-1}_{R_j} \tag{22}$$

Hence,

$$\mathbf{C}_j^{-1} = \mathbf{U}'^{-1}_{R_j} [\boldsymbol{\Lambda}^{-2}_{R_j} + \mathbf{U}'_{R_j} \mathbf{R}_j^{-1} \times \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' \\ \psi_1 \sigma^2 \end{bmatrix} \times (A)^{-1} \times \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' & \psi_1 \sigma^2 \end{bmatrix} \mathbf{R}_j^{-1} \mathbf{U}_{R_j} ] \mathbf{U}^{-1}_{R_j} \tag{23}$$

Thus, the succeeding matrix could be defined in this way:

$$\begin{bmatrix} 1/\sqrt{A}\left[\mathbf{F}_j\mathbf{R}_j\mathbf{F}' \quad \psi_1\sigma^2\right]\mathbf{R}_j^{-1}\mathbf{U}_{R_j} \\ \mathbf{\Lambda}_{R_j}^{-1} \end{bmatrix}$$

The measuring of the previous matrix's SVD led to the subsequent result in:

$$\begin{bmatrix} 1/\sqrt{A}\left[\mathbf{F}_j\mathbf{R}_j\mathbf{F}' \quad \psi_1\sigma^2\right]\mathbf{R}_j^{-1}\mathbf{U}_{R_j} \\ \mathbf{\Lambda}_{R_j}^{-1} \end{bmatrix} = \mathbf{U}^*\begin{bmatrix}\mathbf{\Lambda}^* \\ 0\end{bmatrix}(\mathbf{V}^*)' \tag{24}$$

Upon the pre-multiplication of the two sides of the earlier equation by their transposes, the next conclusions were made:

$$\begin{aligned}&\mathbf{\Lambda}_{R_j}^{-2} + \mathbf{U}'_{R_j}\mathbf{R}_j^{-1}\times\begin{bmatrix}\mathbf{F}_j\mathbf{R}_j\mathbf{F}' \\ \psi_1\sigma^2\end{bmatrix}\times(A)^{-1}\times\left[\mathbf{F}_j\mathbf{R}_j\mathbf{F}' \quad \psi_1\sigma^2\right]\mathbf{R}_j^{-1}\mathbf{U}_{R_j} \\ &= \mathbf{V}^*\left[(\mathbf{\Lambda}^*)' \quad 0\right](\mathbf{U}^*)'\mathbf{U}^*\begin{bmatrix}\mathbf{\Lambda}^* \\ 0\end{bmatrix}(\mathbf{V}^*)' = \mathbf{V}^*(\mathbf{\Lambda}^*)^2(\mathbf{V}^*)'\end{aligned} \tag{25}$$

Thus, Equation (23) could be rephrased in this manner:

$$\mathbf{C}_j^{-1} = (\mathbf{U}'_{R_j})^{-1}\mathbf{V}^*(\mathbf{\Lambda}^*)^2(\mathbf{V}^*)'\mathbf{U}_{R_j}^{-1} = [(\mathbf{U}_{R_j}\mathbf{V}^*)']^{-1}(\mathbf{\Lambda}^*)^2[(\mathbf{U}_{R_j}\mathbf{V}^*)]^{-1} \tag{26}$$

Hence, the next equation could be obtained as such:

$$\begin{cases}\mathbf{U}_j = \mathbf{U}_{R_j}\mathbf{V}^* \\ \mathbf{\Lambda}_j = (\mathbf{\Lambda}^*)^{-1}\end{cases} \tag{27}$$

## 6. THE DERIVATION OF THE MAXIMUM LIKELIHOOD ESTIMATOR

In the method of MLE, the change point was the point which maximized the probability function or its respective logarithm. In the present paper, it was presumed that a shift happened in the ARMA process's stationary state. Therefore, $V_j = 0$ and $\mathbf{W}_j = \begin{bmatrix}1 & \psi_1 \\ \psi_1 & \psi_1^2\end{bmatrix}\sigma^2$ were fixed for every sample of $j = 1, 2\ldots\tau, \tau+1, \ldots, T$, having been identified in Phase II. Nevertheless, variable $\phi_1$ was identified for the samples before the change point and identified for them following the change point (written in the form of $\phi_j$). To estimate unidentified variable $\phi_j$, filtering was utilized.

The ARMA (1,1) process's probability function was in the form of the next joint density:

$$L(\phi_1, \phi_j, \tau \mid \bar{x}) = f_{\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_T}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_T) \tag{28}$$

With the change point being $\tau$, the function above could be rephrased in this form:

$$\begin{aligned}&L(\phi_1,\phi_j,\tau|\bar{x}) = f_{\bar{X}_1,\bar{X}_2,\ldots,\bar{X}_T}(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_T) = \\ &f(\bar{x}_T|\bar{x}_{T-1})\times f(\bar{x}_{T-1}|\bar{x}_{T-2})\times\ldots\times f(\bar{x}_{\tau+2}|\bar{x}_{\tau+1})\times f(\bar{x}_{\tau+1}|\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_\tau)\times f(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_\tau) \\ &= \prod_{j=\tau+2}^{T} f(\bar{x}_j|\bar{x}_{j-1})\times f(\bar{x}_{\tau+1}|\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_\tau)\times f(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_\tau)\end{aligned} \tag{29}$$

As in this paper the ARMA (1,1) model $f(\bar{x}_{\tau+1} \mid \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_\tau) = f(\bar{x}_{\tau+1} \mid \bar{x}_\tau)$ was utilized, it was replaced in (29). In addition, section $f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_\tau)$ in Equation (29) could be equal to:

$$\begin{aligned}&f(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_\tau) = \\ &f(\bar{x}_\tau|\bar{x}_{\tau-1})\times f(\bar{x}_{\tau-1}|\bar{x}_{\tau-2})\times\ldots\times f(\bar{x}_2|\bar{x}_1)\times f(\bar{x}_1) \\ &= \prod_{j=2}^{\tau} f(\bar{x}_j|\bar{x}_{j-1})\times f(\bar{x}_1)\end{aligned} \tag{30}$$

Thus, Equation (29) could be rephrased in this manner:

$$\begin{aligned}&L(\phi_1,\phi_j,\tau|\bar{x}) = f_{\bar{X}_1,\bar{X}_2,\ldots,\bar{X}_T}(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_T) = \\ &\prod_{j=2}^{\tau} f(\bar{x}_j|\bar{x}_{j-1})\times f(\bar{x}_1)\times\prod_{j=\tau+1}^{T} f(\bar{x}_j|\bar{x}_{j-1})\end{aligned} \tag{31}$$

Where $j \le \tau, f(\bar{x}_j \mid \bar{x}_{j-1}) \sim N(\phi_1\bar{x}_{j-1}, \sigma^2(1+\psi_1^2))$, and $f(\bar{x}_1) \sim N(0, B)$, in which $B = \frac{(1+2\phi_1\psi_1+\psi_1^2)\sigma^2}{1-\phi_1^2}$

Hence, the likelihood function could be written as follows:

$$\begin{aligned}&L(\phi_1,\phi_j,\tau|\bar{x}) = \\ &\left(\left(\frac{1}{\sigma\sqrt{1+\psi_1^2}\sqrt{2\pi}}\right)^{\tau-1}\times e^{\left(-\frac{1}{2(1+\psi_1^2)\sigma^2}\right)\times\sum_{j=2}^{\tau}(\bar{x}_j-\phi_1\bar{x}_{j-1})^2}\times\frac{1}{\sqrt{2B\pi}}\times e^{\left(-\frac{1}{2B}\times\bar{x}_1^2\right)}\right)\times\prod_{j=\tau+1}^{T} f(\bar{x}_j|\bar{x}_{j-1})\end{aligned} \tag{32}$$

Hence, considering Equation (32)'s natural logarithm, we would have:

$$\begin{aligned}&Ln(L(\phi_1,\phi_j,\tau|\bar{x})) = \\ &((\tau-1)\times\ln\left(\frac{1}{\sigma\sqrt{1+\psi_1^2}\sqrt{2\pi}}\right) - \frac{1}{2(1+\psi_1^2)\sigma^2}\times\sum_{j=2}^{\tau}(\bar{x}_j-\phi_1\bar{x}_{j-1})^2 - Ln\sqrt{2B\pi} - \\ &\frac{1}{2B}\times\bar{x}_1^2 + \sum_{j=\tau+1}^{T} Ln(f(\bar{x}_j|\bar{x}_{j-1}))\end{aligned} \tag{33}$$

Considering $j > \tau$, $\left(\text{i.e.} \sum_{j=\tau+1}^{T} Ln(f(\bar{x}_j \mid \bar{x}_{j-1}))\right)$, the variance and mean of normal distributions were measured through filtering as explained in part 5.

Making use of filtering, section $Ln(f(\bar{x}_j \mid \bar{x}_{j-1})$ of Equation (33) would be written as follows: (refer to Petris et al. [48])

$$\begin{aligned}&Ln(f(\bar{x}_j|\bar{x}_{j-1},\mathrm{D}_{j-1}) = \\ &-\frac{1}{2}\sum_{j=\tau+1}^{T}\ln|Q_j| - \frac{1}{2}\sum_{j=\tau+1}^{T}(\bar{x}_j - f_j)^2 Q_j^{-1}\end{aligned} \tag{34}$$

Thus, the estimator of the filtering change point was provided in this way:

$$\hat{\tau}_{filtering} = \underset{0\le\tau\le T-2}{\operatorname{argmax}}\begin{Bmatrix}((\tau-1)\times\ln\left(\frac{1}{\sigma\sqrt{1+\psi_1^2}\sqrt{2\pi}}\right) - \frac{1}{2(1+\psi_1^2)\sigma^2}\times\sum_{j=2}^{\tau}(\bar{x}_j-\phi_1\bar{x}_{j-1})^2 - Ln\sqrt{2B\pi} - \frac{1}{2B}\times\bar{x}_1^2 \\ -\frac{1}{2}\sum_{j=\tau+1}^{T}\ln|Q_j| - \frac{1}{2}\sum_{j=\tau+1}^{T}(\bar{x}_j-f_j)^2Q_j^{-1}\end{Bmatrix} \tag{35}$$

In order to conduct filtering for Equation (35), firstly, $\phi_j$ had to be identified. As following the change point, $\phi_j$ changed into the value equaling 1 or greater than it (improving the stationary trend into a non-stationary one), the value was not identified, so it had to be estimated. In order of estimating $\phi_j$ at the time of $j$, the next equation could be used:

$$\hat{\phi}_j = \frac{\sum_{j=t+2}^{T-1} \bar{x}_j (\bar{x}_{j-1} - \bar{\bar{x}}_{j-1})}{\sum_{j=t+2}^{T-1} (\bar{x}_{j-1} - \bar{\bar{x}}_{j-1})^2}, \quad for \quad 0 \le t \le T-3$$
$$and, \quad \hat{\phi}_j = \frac{\bar{x}_t}{\bar{x}_{t-1}}, \quad for \quad t = T-2 \tag{36}$$

where, $\tau$ was presumed to have been the change point in $0 \le \tau \le T-2$ and in $\tau \neq T-1$; in addition, since $\phi_j$ could not be measured by utilizing an uncontrolled point , in order to measure $\hat{\tau}$, it was presumed that $0 \le t \le T-2$ .

## 7. PERFORMANCE EVALUATION

In this part, the model's performance suggested for the estimation of the change point was assessed. To make the assessment, Monte Carlo simulation was utilized with 10000 iterations. In every iteration, if the monitoring technique of the Shewhart control chart emitted a signal implying an uncontrolled condition, the estimators of the change point would be applied in order of estimating the actual change time.

In the present simulation, each point was a sample's mean $(n = 5)$ and $\sigma^2 = 1$. In the controlled model of ARMA, the $\psi_1$ value was regarded to be equal to 0.5. In addition, the change point was presumed as $\tau = 25$. Due to the use of a Shewhart-type control diagram, in case a false alarm went off prior to time $\tau = 25$, a controlled sample would be presented for the false alarm's parallel sample.

In order to trigger Kalman filtering, it was presumed that $\mathbf{m}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, with the matrix of the primary covariance of $\boldsymbol{\theta}_j = \begin{bmatrix} \bar{x}_j \\ \psi_1 \varepsilon_j \end{bmatrix}$ proved as:

$$\mathbf{C}_0 = \begin{bmatrix} \frac{(1 + 2\phi_1\psi_1 + \psi_1^2)\sigma^2}{1 - \phi_1^2} & \psi_1 \\ \psi_1 & \psi_1^2 \end{bmatrix}.$$

For the stationary state, the model is run with different parameters. We take $\phi_1 < 1$ as the state is stationary $\phi_1 = 0.2, 0.5, 0.8$ and $\phi_j \ge 1$, $\phi_j = 1.1, 1.3, 1.5, 1.8, 2.2, 2.7$ for non-stationary state uncontrolled process following $\tau = 25$. The basis of the selection of various amounts of $\phi_j \ge 1$, is that these amount are equispaced. In the part related to accuracy, $E(T)$ was present as the value expected for the uncontrolled signal time of the parallel control diagram of $A\hat{R}L = E(T) - \tau$, $\hat{\tau}_{filtering}$, demonstrating the exactness of the estimator of filtering; the numbers inside the parentheses demonstrate the estimates' MSE (mean squared error).

In the part related to precision, $\hat{P}_0 = P(|\hat{\tau} - \tau| = 0)$, $\hat{P}_1 = P(|\hat{\tau} - \tau| \le 1)$, $\hat{P}_2 = P(|\hat{\tau} - \tau| \le 2)$, $\hat{P}_3 = P(|\hat{\tau} - \tau| \le 3)$, $\hat{P}_4 = P(|\hat{\tau} - \tau| \le 4)$, $\hat{P}_5 = P(|\hat{\tau} - \tau| \le 5)$, and $\hat{P}_{10} = P(|\hat{\tau} - \tau| \le 10)$, and $\hat{P}_{15} = P(|\hat{\tau} - \tau| \le 15)$ demonstrate the estimator's precisions. Table 1 shows the simulation results. This table implys that the filtering estimator suggested gave a satisfactory performance in identifying the change point. In addition, with an increase in the shifts, the filtering's performance got improved.

## 8. A REAL EXAMPLE

In this part, a real example is offered to describe the method used in the present research. The real instance of the Standard and Poor's 500 index has been presented in the research conducted by Francq et al. [52]. The sample of the data obtained since January 3rd, 1979 until December 31st, 2001 included 5808 observations in total, with the log-return having been shown by $\{\bar{x}_j\}_{j=1}^{5807}$ from which it was identified that at a 5% significance level, the powerful white noise hypothesis was repudiated, while the weaker one was not repudiated. This was in conformity with the results of the research conducted by Francq et al. [52] in the following manner:

$$\tilde{x}_j = 0.828\tilde{x}_{j-1} - 0.723\varepsilon_{j-1} + \varepsilon_j \tag{37}$$

As $\phi_0 = 0.828$ was lower than one, the model ARMA (1, 1) above was stationary. In addition, the statistics of the Shewhart control diagram being the model residuals, i.e. $e_1 = 0$ , $e_j = \bar{x}_j - 0.828\bar{x}_{j-1} + 0.723\varepsilon_{t-1}$, $for \quad j = 2, ..., T$, they were measured in order of monitoring the process.

Given the controlled model, 25 controlled samples were produced utilizing the process of simulation $(\tau = 25)$ and following sample 26, the uncontrolled samples were produced making use of model $\tilde{x}_j = 1.7\tilde{x}_{j-1} - 0.723\varepsilon_{j-1} + \varepsilon_j$, i.e. $\phi = 1.7 > 1$ up until the time of the control diagram signal. The relevant results have been provided in Figure 1.

**TABLE 1.** The precision and accuracy performances of the suggested estimators of the filtering change point, where $\tau = 25$, and $N = 10000$ replications for $\phi_1 = 0.2, 0.5, 0.8$

| $\phi_1 = 0.2$ | | $\phi_j$ 1.1 | 1.3 | 1.5 | 1.8 | 2.2 | 2.7 |
|---|---|---|---|---|---|---|---|
| Accuracy | $E(T)$ | 32.54 | 29.95 | 28.72 | 27.91 | 27.40 | 27.02 |
| | $ARL = E(T) - \tau$ | 7.54 | 4.95 | 3.72 | 2.91 | 2.40 | 2.02 |
| | $\hat{\tau}_{filtering}(MSE)$ | **29.90** (61.34) | **27.38** (18.36) | **26.22** (7.49) | **25.52** (3.55) | **25.11** (2.20) | **24.79** (1.57) |
| Precision | $\hat{P}_0 = P(\lvert\hat{\tau} - \tau\rvert = 0)$ | 0.1402 | 0.2110 | 0.2900 | 0.3392 | 0.3698 | 0.3940 |
| | $\hat{P}_1 = P(\lvert\hat{\tau} - \tau\rvert \le 1)$ | 0.3351 | 0.4944 | 0.6208 | 0.7178 | 0.7678 | 0.7921 |
| | $\hat{P}_2 = P(\lvert\hat{\tau} - \tau\rvert \le 2)$ | 0.4553 | 0.6442 | 0.7835 | 0.8811 | 0.9316 | 0.9632 |
| | $\hat{P}_3 = P(\lvert\hat{\tau} - \tau\rvert \le 3)$ | 0.5355 | 0.7331 | 0.8568 | 0.9332 | 0.9689 | 0.9881 |
| | $\hat{P}_4 = P(\lvert\hat{\tau} - \tau\rvert \le 4)$ | 0.6064 | 0.8006 | 0.9077 | 0.9620 | 0.9866 | 0.9951 |
| | $\hat{P}_5 = P(\lvert\hat{\tau} - \tau\rvert \le 5)$ | 0.6633 | 0.8492 | 0.9384 | 0.9795 | 0.9931 | 0.9981 |
| | $\hat{P}_{10} = P(\lvert\hat{\tau} - \tau\rvert \le 10)$ | 0.8557 | 0.9616 | 0.9923 | 0.9992 | 0.9998 | 1.0000 |
| | $\hat{P}_{15} = P(\lvert\hat{\tau} - \tau\rvert \le 15)$ | 0.9373 | 0.9913 | 0.9991 | 1.0000 | 1.0000 | 1.0000 |

| $\phi_1 = 0.5$ | | $\phi_j$ 1.1 | 1.3 | 1.5 | 1.8 | 2.2 | 2.7 |
|---|---|---|---|---|---|---|---|
| Accuracy | $E(T)$ | 35.19 | 30.63 | 28.98 | 27.99 | 27.39 | 27 |
| | $ARL = E(T) - \tau$ | 10. 19 | 5.63 | 3.98 | 2.99 | 2.39 | 2 |
| | $\hat{\tau}_{filtering}(MSE)$ | **32.36** (119.07) | **27,93** (23.84) | **26.38** (8.53) | **25.50** (3.89) | **25** (2.4) | **24.68** (1.92) |
| Precision | $\hat{P}_0 = P(\lvert\hat{\tau} - \tau\rvert = 0)$ | 0.0767 | 0.1576 | 0.2169 | 0.2888 | 0.3226 | 0.3422 |
| | $\hat{P}_1 = P(\lvert\hat{\tau} - \tau\rvert \le 1)$ | 0.2187 | 0.4104 | 0.5623 | 0.6757 | 0.7119 | 0.7261 |
| | $\hat{P}_2 = P(\lvert\hat{\tau} - \tau\rvert \le 2)$ | 0.3222 | 0.5698 | 0.7442 | 0.8697 | 0.9204 | 0.9481 |
| | $\hat{P}_3 = P(\lvert\hat{\tau} - \tau\rvert \le 3)$ | 0.4001 | 0.6689 | 0.8336 | 0.9317 | 0.9706 | 0.9844 |
| | $\hat{P}_4 = P(\lvert\hat{\tau} - \tau\rvert \le 4)$ | 0.4702 | 0.7470 | 0.8912 | 0.9624 | 0.9874 | 0.9948 |
| | $\hat{P}_5 = P(\lvert\hat{\tau} - \tau\rvert \le 5)$ | 0.5290 | 0.8044 | 0.9281 | 0.9797 | 0.9939 | 0.9985 |
| | $\hat{P}_{10} = P(\lvert\hat{\tau} - \tau\rvert \le 10)$ | 0.7480 | 0.9487 | 0.9911 | 0.9983 | 0.9999 | 1.0000 |
| | $\hat{P}_{15} = P(\lvert\hat{\tau} - \tau\rvert \le 15)$ | 0.8636 | 0.9866 | 0.9989 | 0.9996 | 1.0000 | 1.0000 |

| $\phi_1 = 0.8$ | | $\phi_j$ 1.1 | 1.3 | 1.5 | 1.8 | 2.2 | 2.7 |
|---|---|---|---|---|---|---|---|
| Accuracy | $E(T)$ | 40.02 | 31.27 | 29.11 | 27.86 | 27.16 | 26.60 |
| | $ARL = E(T) - \tau$ | 15.02 | 6.27 | 4.11 | 2.86 | 2.16 | 1.60 |
| | $\hat{\tau}_{filtering}(MSE)$ | **36.83** (253.62) | **28.35** (29.40) | **26.30** (9.72) | **25.12** (4.79) | **24.48** (3.54) | **24.60** (1.08) |
| P | $\hat{P}_0 = P(\lvert\hat{\tau} - \tau\rvert = 0)$ | 0.0298 | 0.1073 | 0.1623 | 0.2152 | 0.2406 | 0.2528 |
| | $\hat{P}_1 = P(\lvert\hat{\tau} - \tau\rvert \le 1)$ | 0.0899 | 0.3201 | 0.4720 | 0.5551 | 0.5662 | 0.5490 |
| | $\hat{P}_2 = P(\lvert\hat{\tau} - \tau\rvert \le 2)$ | 0.1514 | 0.4942 | 0.6995 | 0.8206 | 0.8705 | 0.8867 |
| | $\hat{P}_3 = P(\lvert\hat{\tau} - \tau\rvert \le 3)$ | 0.2049 | 0.6065 | 0.8097 | 0.9091 | 0.9465 | 0.9577 |
| | $\hat{P}_4 = P(\lvert\hat{\tau} - \tau\rvert \le 4)$ | 0.2623 | 0.6934 | 0.8718 | 0.9549 | 0.9770 | 0.9846 |
| | $\hat{P}_5 = P(\lvert\hat{\tau} - \tau\rvert \le 5)$ | 0.3174 | 0.7633 | 0.9146 | 0.9767 | 0.9889 | 0.9931 |
| | $\hat{P}_{10} = P(\lvert\hat{\tau} - \tau\rvert \le 10)$ | 0.5609 | 0.9352 | 0.9922 | 0.9988 | 0.9995 | 0.9998 |
| | $\hat{P}_{15} = P(\lvert\hat{\tau} - \tau\rvert \le 15)$ | 0.7261 | 0.9823 | 0.9992 | 1.0000 | 1.0000 | 1.0000 |

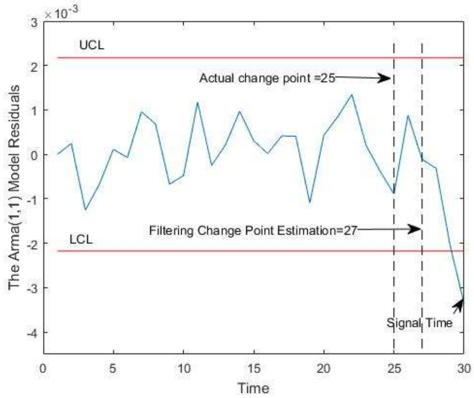


**Figure 1.** Change point estimators' results presented for the S&P500 index's real case

As it is shown, the Shewhart control diagram emitted an uncontrolled signal at the 30th sample. In order of estimating the real change point, the estimator could be used by filtering. The estimator presented the 28th sample as the filtering change point estimator's change point.

## 9. CONCLUSION

In this study, for the first time, the subject of change point estimation has been utilized in the stationary state of ARMA (1, 1). In the monitoring phase, in case the features of the question pursue a time series, i.e., ARMA(1,1), on the basis of the maximum likelihood technique, an approach is developed for the estimation of the stationary state's change point. To estimate unidentified parameters followed the change point, the Dynamic Linear Model's Filtering is utilized on the basis of the singular decomposition of values. The proposed model has wide applications in several fields such as finance, stock exchange marks and rapid production. Results of simulation showed the suggested estimator's effectiveness.

In the present study, MLE is introduced in order of estimating the changes of a stationary nature to the $\overline{X}$ control diagram's ARMA (1, 1) model with the correlation existing among $\overline{X}$ statistics. Filtering, having been considered a DLM estimation process, was utilized in order of estimating unidentified variables following the change point. The performance assessments were carried out by simulating studies on the model of ARMA (1, 1) with a varied $\phi$ coefficient. Simulation findings demonstrated that the estimators suggested a satisfactory performance in order of estimating the change point. In the meantime, with an increase in the shift size, the estimator's performance was enhanced. In conclusion, a real instance was presented to manifest the application of the uncontrolled method.

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## APPENDIX A:

In ARMA(1,1) we have $F = \begin{bmatrix}1 & 0\end{bmatrix}$

$$Q = FRF' = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = R_1$$

$$C = R - RF'Q^{-1}FR = \begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix} - \begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} * \frac{1}{R_1} * \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}$$

$$= \begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix} - \begin{bmatrix}R_1 & R_2\\ R_3 & \frac{R_2R_3}{R_1}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & R_4 - \frac{R_2R_3}{R_1}\end{bmatrix}$$

C is not reversible to use of SVD algorithm. So we couldn't use of Petris model.

Meinhold and Singpurwalla [49] stated that if $X_1$ and $X_2$ have a bivariate normal distribution with the means $\mu_1$ and $\mu_2$, respectively, and a covariance matrix of $\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}$ denoted by $\begin{bmatrix}X_1\\ X_2\end{bmatrix} \sim MN\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}\right)$, the conditional distribution of $X_1$ given $X_2$ will be put as follows:

$$(X_1 \mid X_2 = x_2) \sim N\left[\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right] \quad \text{(A1)}$$

Let $X_1$ correspond to $\boldsymbol{\theta}_j = \begin{bmatrix} \bar{x}_j \\ \psi_1 \varepsilon_j \end{bmatrix}$ and $X_2$ correspond to $\bar{x}_j$.

Hence,

$$X_1 = \boldsymbol{\theta}_j \Rightarrow \begin{cases} \boldsymbol{\mu}_1 = E(\boldsymbol{\theta}_j \mid D_{j-1}) = \mathbf{a}_j \\ \boldsymbol{\Sigma}_{11} = Var(\boldsymbol{\theta}_j \mid D_{j-1}) = \mathbf{R}_j \end{cases},$$

$X_2 = \bar{x}_j \Rightarrow \mu_2 = E(\bar{x}_j \mid D_{j-1}) = \mathbf{F}_j \mathbf{a}_j$, Since $V_j = 0$ in this paper, we have $\Sigma_{22} = Var(\bar{x}_j \mid D_{j-1}) = \mathbf{F}_j \mathbf{R}_j \mathbf{F}'$.

Due to the ARMA (1,1) model of $\bar{x}_j = \phi_0 \bar{x}_{j-1} + \varepsilon_j + \varphi_1 \varepsilon_{j-1}$

$$\boldsymbol{\Sigma}_{12} = Cov(\boldsymbol{\theta}_j, \bar{x}_j \mid D_{j-1}) = Cov(\begin{bmatrix} \bar{x}_j \\ \psi_1 \varepsilon_j \end{bmatrix}, \bar{x}_j \mid D_{j-1})$$

$$= E(\begin{bmatrix} \bar{x}_j^2 \\ \psi_1 \varepsilon_j \bar{x}_j \end{bmatrix} \mid D_{j-1}) - E(\begin{bmatrix} \bar{x}_j \\ \psi_1 \varepsilon_j \end{bmatrix} \mid D_{j-1}) \times E(\bar{x}_j \mid D_{j-1})$$

and $E(\varepsilon_j) = E(\varepsilon_{j-1}) = 0$, we can substitute $E(\begin{bmatrix} \bar{x}_j^2 \\ \psi_1 \varepsilon_j \bar{x}_j \end{bmatrix} \mid D_{j-1})$ with

$$\begin{bmatrix} \mathrm{var}(\bar{x}_j \mid \mathrm{D}_{j-1}) + \left[E(\bar{x}_j \mid \mathrm{D}_{j-1})\right]^2 \\ E[\psi_1 \varepsilon_j (\phi_j \bar{x}_{j-1} + \psi_1 \varepsilon_j \varepsilon_{j-1} + \varepsilon_j^2) \mid D_{j-1}] \end{bmatrix} = \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}_j' + (\mathbf{F}_j \mathbf{a}_j)^2 \\ \psi_1 \sigma^2 \end{bmatrix}$$

Also, it is realized that $E(\begin{bmatrix} \bar{x}_j \\ \psi_1 \varepsilon_j \end{bmatrix} \mid D_{j-1}) = \begin{bmatrix} \mathbf{F}_j \mathbf{a}_j \\ 0 \end{bmatrix}$.

So,

$$\boldsymbol{\Sigma}_{12} = \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}_j' + (\mathbf{F}_j \mathbf{a}_j)^2 \\ \psi_1 \sigma^2 \end{bmatrix} - \begin{bmatrix} \mathbf{F}_j \mathbf{a}_j \\ 0 \end{bmatrix} \times \mathbf{F}_j \mathbf{a}_j = \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' \\ \psi_1 \sigma^2 \end{bmatrix}$$

As a result, the following equation can be obtained according to equation (A1):

$$\mathbf{C}_j = \mathrm{var}(\boldsymbol{\theta}_j \mid \bar{x}_j, D_{j-1}) =$$
$$\mathrm{var}(\boldsymbol{\theta}_j \mid D_j) = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21} =$$
$$\mathbf{R}_j - \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' \\ \psi_1 \sigma^2 \end{bmatrix} * (\mathbf{F}_j \mathbf{R}_j \mathbf{F}')^{-1} * \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' & \psi_1 \sigma^2 \end{bmatrix}$$

$$\mathbf{C}_j = \mathrm{var}(\boldsymbol{\theta}_j \mid \bar{x}_j, D_{j-1}) =$$
$$\mathrm{var}(\boldsymbol{\theta}_j \mid D_j) = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21} =$$
$$\mathbf{R}_j - \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' \\ \psi_1 \sigma^2 \end{bmatrix} * (\mathbf{F}_j \mathbf{R}_j \mathbf{F}')^{-1} * \begin{bmatrix} \mathbf{F}_j \mathbf{R}_j \mathbf{F}' & \psi_1 \sigma^2 \end{bmatrix}$$

# Change Point Estimation of the Stationary State in Auto Regressive Moving Average Models, Using Maximum Likelihood Estimation and Singular Value Decomposition-based Filtering

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**چکیده**

در این مقاله، برای اولین بار، موضوع تخمین نقطه تغییر در حالت مانایی سری زمانی آرما مرتبه اول بکار رفته است. در فاز کنترل، تکنیک حداکثر درستنمایی برای تخمین نقطه تغییر حالت مانایی، توسعه پیدا کرده است. در این مدل به منظور تخمین پارامترهای نامعلوم بعد از نقطه تغییر، از روش فیلترینگ (نوعی از مدل خطی پویا) بر مبنای روش تجزیه مقادیر منفرد استفاده شده است. مدل ارایه شده در زمینه‌های زیادی همچون بازار سهام، فایننانس، تولید انبوه و ... کاربرد دارد. نتایج شبیه‌سازی حاکی از کارایی، مدل پیشنهادی است. همچنین یک مثال از کاربرد واقعی مدل پیشنهادی در بازار سهام ارایه شده است.

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