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The Effect of Measurement Errors on the Performance of Variable Sample Size and Sampling Interval $\bar{X}$ Control Chart

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A B S T R A C T

The effect of measurement errors on adaptive and non-adaptive control charts has been occasionally considered by researchers throughout the years. However, that effect on the variable sample size and sampling interval (VSSI) $\bar{X}$ control charts has not so far been investigated. In this paper, we evaluate the effect of measurement errors on the VSSI $\bar{X}$ control charts. After a model development, the effect of measurement errors and multiple measurements on the performance of VSSI scheme are evaluated in terms of the out-of-control average time to signal (ATS) criterion, which is obtained using a Markov Chain approach. At last, a real case is presented to show the application of the proposed scheme.

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1. INTRODUCTION[1]

Since the introduction of the Shewhart $\bar{X}$ control chart in 1924, which is very effective in detecting large process mean shifts, many developments have been made to improve the performance of the original $\bar{X}$ control chart, by allowing the detection of small or moderate process mean shifts. One of the main approaches to improve the performance of Shewhart charts, is using the adaptive control charts. An adaptive control chart is a chart that at least one of its parameters (sample size, sampling interval and control limit coefficient) is variable throughout the process.

Reynolds et al. [1] and Runger and Pignatiello [2] considered control charts with variable sampling intervals with two types of sampling intervals. Prabhu et al. [3], Costa [4] and Zimmer et al. [5] are among those who considered variable sample size (VSS) control charts. Costa [6, 7] considered variable sample size and sampling interval (VSSI) and variable parameters (VP) $\bar{X}$ control charts. Tagaras [8] presented a comprehensive survey about adaptive control charts, covering all types of adaptive charts; including VSS, VSI, variable control limit coefficients, and also different combination of design parameters. Zimmer et al. [9] presented the performance study of several adaptive control charts. Carot et al. [10] studied a combined double sampling variable sampling interval (DSVSI) $\bar{X}$ chart. They showed that their adaptive chart is better in performance than the CUSUM and EWMA control charts. Jensen et al. [11] considered the issues of evaluating, designing and implementing adaptive $\bar{X}$ control charts. Recently, Lim et al. [12] presented the optimal designs of a VSSI $\bar{X}$ control chart when the process mean and standard deviation are estimated (Phase I) and compared them to when they are assumed known (Phase II).

All of the above studies have shown that the adaptive control charts outperform the classical Shewhart control chart in detecting small to moderate mean shifts. Moreover, researches such as Prabhu et al. [13] have shown that VSSI $\bar{X}$ charts have better performance than VSI $\bar{X}$ and VSS $\bar{X}$ control charts. Most of the researchers for simplicity assume that there are no measurement errors in the process. However, in

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reality, considering measurement errors in the control charts is a more reasonable scenario. With considering measurement errors, the performance of the control charts deteriorates. Bennet [14] investigated the effect of measurement errors on the original $\bar{X}$ chart. They used the following model throughout their investigation: $Y = X + \varepsilon$, where X is the true value of the quality characteristic, Y is the observed value and ε is the random measurement error. Kanazuka [15] showed that the performance of $\bar{X}$ and R control charts deteriorates in the presence of measurement errors. Linna and Woodall [16] used a linear covariate model $Y = A + BX + \varepsilon$ between the observed value *Y* and the true value *X*, where *A* and *B* are known constants. Cocchi and Scagliarini [17] investigated the effect of the two-component measurement errors on the performance of Shewhart control chart. Maleki et al. [18] investigated the effect of measurement errors with linearly increasing variance on the performance of the ELR control chart for simultaneous monitoring of multivariate process mean vector and covariance matrix.

The effect of measurement errors on the adaptive control charts has rarely been investigated. Hu et al.[19] investigated the effect of measurement errors on the performance of VSS $\bar{X}$ scheme. They showed that the performance of the control chart is significantly affected by measurement errors. Hu et al.[20] explored the performance of variable sampling interval (VSI) $\bar{X}$ control charts in the presence of measurement errors and showed that the statistical performance of VSI charts will be significantly affected by measurement errors.

Maleki et al. [21] performed a review on the effect of measurement errors on the control charts. According to this review paper, the effect of measurement errors on the VSSI $\bar{X}$ control charts has not been investigated yet.

In this work, we use Linna and Woodall 's [16] linear covariate error model and combine it with a two types sample sizes and two types sampling intervals VSSI scheme. Then, we investigate the effect of measurement errors, multiple measurements and constant *B* on the main properties of this adaptive control chart.

The structure of this paper is as follows: we introduce the linearly covariate error model in the next section. In Section 3, we combine the mentioned error model and the VSSI $\bar{X}$ control chart. In Section 4, the effect of measurement errors on the VSSI $\bar{X}$ chart is thoroughly investigated. In Section 5, an illustrative example is presented. Finally, in Section 6, conclusions and future remarks are included.

2. LINEARLY COVARIATE ERROR MODEL

For $i = 1,2,3\ldots$ which are the sampling times, each having sample size of n_s ($s \in \{1,2\}$, n_1 is the small sample size and n_2 is the large sample size), with sampling interval of t_s ($s \in \{1,2\}$, t_1 is the short sampling interval and t_2 is the long sampling interval), we assume that Y_{ij} is the quality characteristic of those n_s consecutive items. Y_{ij} follows a normal distribution $(\mu_0 + \delta\sigma_0, \sigma_0)$, where μ_0 and σ_0 are the in-control mean and standard deviation, respectively, and both are assumed known, and δ is the standard mean shift (in case of in-control process $\delta = 0$). We also assume that Y_{ij} is not directly observable and can only be estimated by using X_{ijk}, where k is the index of the number of measurements for each item and j is the sample size index. X_{ijk} is obtained by using a linearly covariate model as follows:

$$X_{ijk} = A + BY_{ij} + \varepsilon_{ijk}, \tag{1}$$

where *A* and *B* are constants, and ε_{ijk} is the random error term which is normally distributed (0, σ_{0M}). Note that σ_M can be constant and independent of the in-control mean (μ_0) or as pointed out by Montgomery and Runger [22] and Linna and Wooddall [16], can be $\sigma_M = C + D\mu$, where *C* and *D* are two constants and $\mu = \mu_0 + \delta\sigma_0$. In this paper, for simplicity we assume that ε_{ijk} has a constant variance. In the case of linearly increasing measurement error variance, σ_M should be replaced by $C + D\mu$ in the model, control chart and the Markov chain probabilities.

The sample mean for each sample (i=1,2,3,…..) is:

$$\bar{X}_i = \frac{1}{mn_s}\sum_{j=1}^{n_s}\sum_{k=1}^{m} X_{ijk}. \tag{2}$$

Substituting Equation (1) in Equation (2), we have:

$$\bar{X}_i = A + \frac{1}{n_s}\left(B\sum_{j=1}^{n_s} Y_{ij} + \frac{1}{m}\sum_{j=1}^{n_s}\sum_{k=1}^{m}\varepsilon_{ijk}\right). \tag{3}$$

Then, after basic computations, we have:

$$E\left(\bar{X}_i\right) = A + B\left(\mu_0 + \delta\sigma_0\right), \tag{4}$$

$$V(\bar{X}_i) = \frac{1}{n_s}(B^2\sigma_0^2 + \frac{\sigma_M^2}{m}). \tag{5}$$

3. VSSI $\bar{X}$ CONTROL CHART WITH MEASUREMENT ERRORS

When the sample sizes and the sampling intervals are variable, the control chart for this scheme is called VSSI control chart. In this paper, we assume that there are two types of sample sizes ($n_1 < n_2$) and also two types of sampling intervals ($t_1 < t_2$). We also assume that there are only one set of warning limits, alongside the usual control limits.

In case of in-control process, $\delta = 0$, the control limits, *LCL* and *UCL* of the VSSI, $\bar{X}$ chart with linearly covariate error model (discussed in the previous section), are as follows:

$$LCL = A + B\mu_0 - K\sqrt{\frac{1}{n_s}\left(B^2\sigma_0^2 + \frac{\sigma_M^2}{m}\right)}, \quad (6)$$

$$UCL = A + B\mu_0 + K\sqrt{\frac{1}{n_s}\left(B^2\sigma_0^2 + \frac{\sigma_M^2}{m}\right)}. \quad (7)$$

Similarly, the warning limits *LWL* and *UWL* are:

$$LWL = A + B\mu_0 - W\sqrt{\frac{1}{n_s}\left(B^2\sigma_0^2 + \frac{\sigma_M^2}{m}\right)} \quad (8)$$

$$UWL = A + B\mu_0 + W\sqrt{\frac{1}{n_s}\left(B^2\sigma_0^2 + \frac{\sigma_M^2}{m}\right)}, \quad (9)$$

where $K > W > 0$.

For simplicity, we define as follows:

$$Z_i = \frac{\bar{X}_i - (A + B\mu_0)}{\sqrt{\frac{1}{n_s}\left(B^2\sigma_0^2 + \frac{\sigma_M^2}{m}\right)}}. \quad (10)$$

Therefore, Z_i has a standard normal distribution (0,1). Using Z_i instead of $\bar{X}_i$ for each sample, we have:

$$LCL = -K, \quad (11)$$

$$UCL = K, \quad (12)$$

$$LWL = -W, \quad (13)$$

$$UWL = W. \quad (14)$$

The VSSI strategy for choosing the next sampling interval and size is as follows:

First let:

$I_1 = [LWL, UWL]$,

$I_2 = [LCL, LWL) U (UWL, UCL]$ and

$I_3 = [LCL, UCL]$.

Then:

- If Z_i falls in I_1, then the process is called as in-control and we choose the small sample size and the long sampling interval for the next sample (n_1, t_2).
- If Z_i falls in I_2, then the process is also called as in-control, but we choose the large sample size and the short sampling interval for the next sample (n_2, t_1).
- If Z_i falls out of I_3, then the process is declared as out-of-control, and corrective actions are needed.

We also have:

$$\Pr(Z \in I_1) = \Phi(W) - \Phi(-W) = 2\Phi(W) - 1, \quad (15)$$

$$\Pr(Z \in I_2) = \Phi(K) - \Phi(W) + \Phi(-W) - \Phi(-K) = 2(\Phi(K) - \Phi(W)), \quad (16)$$

$$\Pr(Z \in I_3) = \Phi(K) - \Phi(-K) = 2\Phi(K) - 1, \quad (17)$$

where $\Phi(x)$ is the c.d.f. (cumulative distribution function) of the normal (0, 1) distribution. When the process is in-control, we can define:

$$E(n_s) = n_1\frac{\Pr(Z \in I_1)}{\Pr(Z \in I_3)} + n_2\frac{\Pr(Z \in I_2)}{\Pr(Z \in I_3)}, \quad (18)$$

$$E(t_s) = t_2\frac{\Pr(Z \in I_1)}{\Pr(Z \in I_3)} + t_1\frac{\Pr(Z \in I_2)}{\Pr(Z \in I_3)}. \quad (19)$$

4. THE EFFECT OF MEASUREMENT ERRORS ON THE VSSI CHART

We use the average run length (ARL) and the average time to signal (ATS) criteria throughout this paper. ARL is the average number of samples taken before an off-target signal occurs and ATS is the mean time needed until a control chart signals an off-target situation.

In the standard $\bar{X}$ charts, where sampling intervals (*t*) are constant, obtaining ARL for evaluating the effectiveness of the control chart, is enough. Multiplying it by *t* would simply give us ATS. In order to evaluate the effectiveness of VSSI control charts, in which the sampling intervals are varied (t_s), we should compute ATS. When the process is not in-control ($\delta > 0$), shorter ATS is preferable, because the continuation of the process in an out-of-control state is not logical. On the contrary, when the process is in-control, longer ATS allows the process to run longer on target.

Let us define ARL_0 and ATS_0 as the ARL and ATS values when the process is in-control and ARL_1 and ATS_1, as the ARL and ATS values when the process mean has shifted ($\mu_0 \to \mu_0 + \delta\sigma_0$). Then, in the VSSI control charts, where both sample sizes and sampling intervals are variable, we can use a Markov Chain approach as used in Prabhu et al. [13].

In order to use this Markov Chain approach, we should define the transition probability matrix with the following three states:

State 1: $[LWL, UWL]$,

State 2: $[UCL, UWL) U (LWL, LCL]$,

State 3: $(-\infty, LCL) U (UCL, \infty)$.

The first two states are transient and the third state is absorbing, so that the process stops in the third state.

Then, we have:

$$P_1 = \begin{bmatrix} p_{11}{}^1 & p_{12}{}^1 & p_{13}{}^1 \\ p_{21}{}^1 & p_{22}{}^1 & p_{23}{}^1 \\ p_{31}{}^1 & p_{32}{}^1 & p_{33}{}^1 \end{bmatrix}$$

where $p_{ij}{}^1$ is the transition probability from the previous state, *i*, to the current state, *j*, when the process mean has shifted ($\mu_0 \to \mu_0 + \delta\sigma_0$).

Assume that $\gamma = \frac{\sigma_M}{\sigma_0}$ is the measurement errors ratio, then, for the transient states we have:

$$p_{s1}{}^1 = \Pr\left[-W \le Z_{i-1} \le W \mid n_s;\delta\right] = \Phi\left(W - \frac{\delta\sqrt{n_s}}{\sqrt{1+\frac{\gamma^2}{B^2 m}}}\right) - \Phi\left(-W - \frac{\delta\sqrt{n_s}}{\sqrt{1+\frac{\gamma^2}{B^2 m}}}\right) \quad (20)$$

$$p_{s2}{}^1 = \Pr\left[W < Z_{i-1} \le K \mid n_s;\delta\right] + Pr\left[-K \le Z_{i-1} < -W \mid n_s;\delta\right] = \Phi\left(K - \frac{\delta\sqrt{n_s}}{\sqrt{1+\frac{\gamma^2}{B^2 m}}}\right) - \Phi\left(W - \frac{\delta\sqrt{n_s}}{\sqrt{1+\frac{\gamma^2}{B^2 m}}}\right) + \Phi\left(-W - \frac{\delta\sqrt{n_s}}{\sqrt{1+\frac{\gamma^2}{B^2 m}}}\right) - \Phi\left(-K - \frac{\delta\sqrt{n_s}}{\sqrt{1+\frac{\gamma^2}{B^2 m}}}\right). \quad (21)$$

Note that when the process is out-of-control, and the measurement errors variance is constant, replacing μ_0 with $\mu_0 + \delta\sigma_0$ in Equation (10) would simply conclude that Z_i has a normal distribution with the mean of $\frac{\delta\sqrt{n_s}}{\sqrt{1+\frac{\gamma^2}{B^2 m}}}$ and the standard deviation of 1.

Also, for the absorbing state we have:

$p_{31}{}^1 = p_{32}{}^1 = 0$ and $p_{33}{}^1 = 1$,

$p_{13}{}^1 = 1 - p_{11}{}^1 - p_{12}{}^1$ and $p_{23}{}^1 = 1 - p_{21}{}^1 - p_{22}{}^1$.

Then, from Prabhu et al. [13], we have:

$$\text{ARL}_1 = \mathbf{b}^T(\mathbf{I} - \mathbf{Q}_1)\mathbf{1}, \quad (22)$$

$$\text{ATS}_1 = \mathbf{b}^T(\mathbf{I} - \mathbf{Q}_1)\mathbf{t}, \quad (23)$$

where $\mathbf{b}^T = (b_1, b_2)$ is the vector of starting probabilities such that $b_1 + b_2 = 1$, *I* is the identity matrix of order 2, Q_1 is the 2×2 transition probability matrix for transient states, 1 is a 2×1 unit column vector and $\boldsymbol{t}^T = (t_2, t_1)$ is the vector of sampling intervals.

At the beginning, when the process is running on target, b_1 and b_2 are obtained as follows:

$$b_1 = \frac{p_{11}{}^0}{p_{11}{}^0 + p_{12}{}^0}, \quad (24)$$

$$b_2 = \frac{p_{22}{}^0}{p_{21}{}^0 + p_{22}{}^0}, \quad (25)$$

where $p_{ij}{}^0$ is the transition probability from the previous state, *i*, to the current state, *j*, when the process mean has not shifted ($\delta = 0$). The chart parameters (n_1, t_2) or (n_2, t_1) for the first subgroup can be chosen with b_1 and b_2 probabilities, and they can be obtained using:

$$b_1 n_1 + b_2 n_2 = E(n_s), \quad (26)$$

$$b_1 t_2 + b_2 t_1 = E(t_s). \quad (27)$$

For a fair comparison, all charts must have the same in-control performance. Therefore, ARL_0 and ATS_0 should be equal for all in-control charts and this can be achieved by fixing the average sample size ($E(n_s)$), the average sampling interval ($E(t_s)$) and control limit coefficient (*K*). In order to have $K = 3$, we let $\text{ARL}_0 = 370.4$ and we also assume that $E(t_s) = 1$ (hour), therefore $\text{ATS}_0 = \text{ARL}_0 \times E(t_s) = 370.4$, for all charts.

By fixing *K*, $E(t_s)$ and $E(n_s)$ for an in-control scheme, as mentioned in Prabhu et al. [13], from Equations (15) to (19) we have:

$$W = \Phi^{-1}\left[\frac{2\Phi(K)\left(E(n_s) - n_2\right) + n_1 - E(n_s)}{2(n_1 - n_2)}\right] = \Phi^{-1}\left[\frac{2\Phi(K)\left(E(t_s) - t_1\right) + t_2 - E(t_s)}{2(t_2 - t_1)}\right]. \quad (28)$$

After fixing n_1, n_2 and t_1, by using Equation (28), we can obtain t_2. Note that, because t_1 is more dependent on adopted sampling and inspection methods, we prefer to fix t_1 and then obtain t_2, not the other way around. Therefore, we have:

$$t_2 = \frac{E(t_s)(n_1 - n_2) - b - t_1 c}{n_1 - n_2 - c}, \quad (29)$$

where

$$b = 2\left(E(t_s) - t_1\right)(n_1 - n_2)\Phi(K),$$

$$c = 2\left(E(n_s) - n_2\right)\Phi(K) + n_1 - E(n_s).$$

After having all of the needed parameters, now we compare the control charts using ARL_1 and ATS_1. Note that throughout this paper we use MATLAB 2015 for our computations.

In Table 1, by using Minitab 16, the significance hypothesis tests of three main factors (γ , *m* and *B*), the three two-factor interactions and the one three-factor interactions on the response variable ($ATS_{1(\delta=0.1)}$) has been performed based on 70 samples (combinations of γ , m and B values). The last column shows the p-value. When the p-value is less than 0.05, the effect of factors on the response variable is significant. As expected, the result show that only the effect of γ and any interaction involving γ ($\{\gamma\}$, $\{\gamma,B\}$, $\{\gamma,m\}$ and $\{\gamma,m,B\}$) is significant on ATS_1 .

Three assumptions including normality of residuals, constant variance of the residuals and independency of the residuals in ANOVA (analysis of variance) table are also investigated which are illustrated in Figure 1.

In Table 2, assuming ARL_0 and $ATS_0 = 370.4$, $K = 3$, $E(t_s) = 1$, $E(n_s) = 5$, $m = 1$, $B = 1$, $n_1 \in \{1,3\}$ $n_2 \in \{6,7,10\}$, $t_1 \in \{0.01,0.1,0.25,0.5\}$ and $\delta \in \{0.1,0.5,1,2\}$, we have compared ARL_1 and ATS_1 in terms of different measurement errors ratios; $\gamma \in \{0,0.3,0.7,1\}$. From the results in Table 2, we can see that when there is no measurement errors ($\gamma = 0$) , the out-of-control ARLs and ATSs , are smaller, in comparisons to with measurement errors cases ($\gamma \in \{0.3,0.7,1\}$) . We can also see that, larger mean shift (δ), will result in smaller ARL_1 and ATS_1 . These results are completely logical, because when there is measurement errors in the process, ARL_1 and ATS_1 are supposed to become worse and when we have a shift in the process mean, we want ARL_1 and ATS_1 to be smaller, allowing the signal to occur sooner with bigger mean shifts.

Remember that smaller ARL_1 and ATS_1 are always better. Based on Table 2, we can easily see that all of these conditions have been met for different combinations of the parameters 'values.

In Table 3, having all of the mentioned assumptions, except $\gamma = 1$, we have compared ARL_1 and ATS_1 in terms of the number of each items' measurements; $m \in \{1,2,3,4\}$.

From the results of this table, we can easily see that when the number of measurements increases, the negative effect of measurement errors decreases (we will have smaller ARL_1 and ATS_1), for different combinations of the parameters' values in the VSSI scheme.

In Table 4, we have performed sensitivity analysis for the parameter *B*. Note that since the parameter *A* has automatically been eliminated from our calculations (see the pervious section), its' value is irrelevant to our evaluations. As for parameter *B*, again with the same assumptions as the first analysis, except $\gamma = 1$ and $B \in \{1,2,3,4\}$, we can easily see in Table 4 that larger *B* will result in better performance in detecting a signal (we will have smaller ARL_1 and ATS_1).

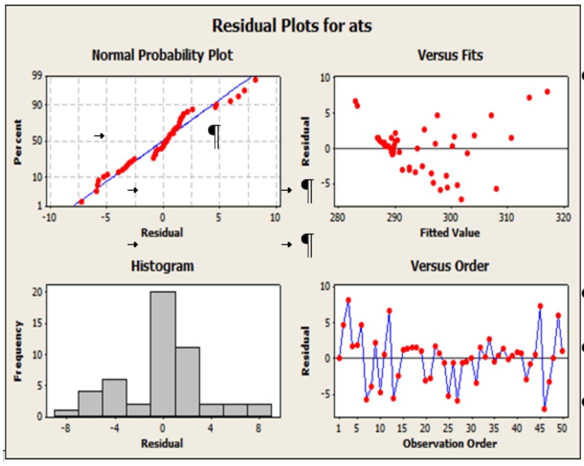

Figure 1. ANOVA

TABLE 1. ANOVA which shows the effect of factors on the response variable

Source	DF	Seq SS	Adj SS	Adj MS	F	P
γ	1	873.52	532.93	532.93	49.51	0.00
m	1	299.60	5.89	5.89	0.55	0.462
B	1	1107.47	9.36	9.36	0.87	0.355
γ **m*	1	131.49	133.52	133.52	12.40	0.001
γ **B*	1	514.29	219.61	219.61	20.40	0.000
*m***B*	1	173.88	2.06	2.06	0.19	0.664
γ **m***B*	1	52.85	52.85	52.85	4.91	0.030
Error	62	667.39	667.39	10.76		
Total	69	3820.48				

TABLE 2. (ARL_1 and ATS_1) when $ARL_0 = ATS_0 = 370.4$, $K = 3$, $m = 1$, $B = 1$

n_1, n_1, t_1 t_2, W	$n_1 = 1, n_2 = 6$ $t_1 = 0.01, t_2 = 4.96$ $W = 0.2526$	$n_1 = 1, n_2 = 6$ $t_1 = 0.1, t_2 = 4.6$ $W = 0.2526$	$n_1 = 1, n_2 = 10$ $t_1 = 0.1, t_2 = 1.72$ $W = 0.7622$	$n_1 = 3, n_2 = 7$ $t_1 = 0.25, t_2 = 1.75$ $W = 0.6724$	$n_1 = 3, n_2 = 10$ $t_1 = 0.25, t_2 = 1.3$ $W = 1.0633$	$n_1 = 3, n_2 = 10$ $t_1 = 0.5, t_2 = 1.2$ $W = 1.0633$
δ			$\gamma = 0$			
0.1	(295.24, 288.37)	(295.24, 288.99)	(293.39, 288.16)	(295.02, 290.45)	(294.17, 290.58)	(294.17,291.78)
0.5	(29.05, 16.34)	(29.05, 17.49)	(18.30, 11.49)	(26.34, 18.06)	(19.82, 14.27)	(19.82, 16.12)
1	(3.68, 1.37)	(3.68, 1.58)	(2.63, 1.72)	(3.23, 1.78)	(2.58, 1.68)	(2.58, 1.98)
2	(1.20, 1.03)	(1.20, 1.05)	(1.52, 1.16)	(1.17, 1.04)	(1.23, 1.06)	(1.23, 1.12)
δ			$\gamma = 0.3$			
0.1	(300.42, 294.01)	(300.42, 294.59)	(298.81, 293.93)	(300.23, 295.97)	(299.49, 296.15)	(299.49, 297.26)
0.5	(32.65, 19.20)	(32.65, 20.42)	(21.07, 13.68)	(29.83, 21.03)	(22.85, 16.87)	(22.85, 18.86)
1	(4.16, 1.47)	(4.16, 1.72)	(2.85, 1.79)	(3.62, 1.95)	(2.81, 1.80)	(2.81, 2.14)
2	(1.22, 1.04)	(1.22, 1.05)	(1.55, 1.18)	(1.20, 1.05)	(1.27, 1.08)	(1.27, 1.14)
δ			$\gamma = 0.7$			
0.1	(316.85, 311.88)	(316.85, 312.34)	(315.89, 312.12)	(316.74, 313.44)	(316.30, 313.72)	(316.30, 314.58)
0.5	(48.62, 32.80)	(48.62, 34.24)	(34.39, 24.94)	(45.59, 35.11)	(37.26, 29.79)	(37.26, 32.28)
1	(6.63, 2.22)	(6.63, 2.62)	(4.00, 2.22)	(5.66, 2.94)	(4.07, 2.46)	(4.07, 2.99)
2	(1.37, 1.06)	(1.37, 1.09)	(1.63, 1.26)	(1.36, 1.10)	(1.4290, 1.14)	(1.43, 1.23)
δ			$\gamma = 1$			
0.1	(329.18, 325.34)	(329.18, 325.69)	(328.62, 325.69)	(329.12, 326.57	(328.86, 326.86)	(328.86, 327.53)
0.5	(68.05, 50.69)	(68.05, 52.27)	(52.29, 41.19)	(65.07, 53.48	(56.16, 47.62)	(56.16, 50.47)
1	(10.45, 3.89)	(10.45, 4.48)	(5.98, 3.14)	(8.96, 4.87	(6.24, 3.77)	(6.24, 4.59)
2	(1.67, 1.09)	(1.67, 1.14)	(1.76, 1.36)	(1.62, 1.19	(1.61, 1.22)	(1.61, 1.35)

TABLE 3. (ARL_1 and ATS_1) when $ARL_0 = ATS_0 = 370.4$, $K = 3$, $\gamma = 1$, $B = 1$

n_1, n_1, t_1 t_2, W	$n_1 = 1, n_2 = 6$ $t_1 = 0.01, t_2 = 4.96$ $W = 0.2526$	$n_1 = 1, n_2 = 6$ $t_1 = 0.1, t_2 = 4.6$ $W = 0.2526$	$n_1 = 1, n_2 = 10$ $t_1 = 0.1, t_2 = 1.72$ $W = 0.7622$	$n_1 = 3, n_2 = 7$ $t_1 = 0.25, t_2 = 1.75$ $W = 0.6724$	$n_1 = 3, n_2 = 10$ $t_1 = 0.25, t_2 = 1.3$ $W = 1.0633$	$n_1 = 3, n_2 = 10$ $t_1 = 0.5, t_2 = 1.2$ $W = 1.0633$
δ			$m = 1$			
0.1	(329.18, 325.34	(329.18, 325.69)	(328.62, 325.69)	(329.12, 326.57)	(328.86, 326.86)	(328.86, 327.53)
0.5	(68.05, 50.69	(68.05, 52.27)	(52.29, 41.19)	(65.07, 53.48)	(56.16, 47.62)	(56.16, 50.47)
1	(10.45, 3.89	(10.45, 4.48)	(5.98, 3.14)	(8.96, 4.87)	(6.24, 3.77)	(6.24, 4.59)
2	(1.67, 1.09	(1.67, 1.14)	(1.76, 1.36)	(1.62, 1.19)	(1.61, 1.22)	(1.61, 1.35)
δ			$m = 2$			
0.1	(317.16, 312.23	(317.16, 312.67)	(316.22, 312.46)	(317.05, 313.77)	(316.62, 314.05)	(316.62, 314.91)
0.5	(49.02, 33.16	(49.02, 34.60)	(34.74, 25.25)	(45.98, 35.47)	(37.63, 30.14)	(37.63, 32.63)
1	(6.70, 2.24	(6.70, 2.65)	(4.04, 2.24)	(5.73, 2.97)	(4.11, 2.48)	(4.11, 3.02)
2	(1.38, 1.06	(1.38, 1.09)	(1.64, 1.27)	(1.37, 1.10)	(1.43, 1.14)	(1.43, 1.24)
δ			$m = 3$			
0.1	(311.43, 305.98	(311.43, 306.47)	(310.27, 306.13)	(311.29, 307.67)	(310.77, 307.93)	(310.77, 308.87)
0.5	(42.41, 27.36	(42.41, 28.73)	(29.02, 20.29)	(39.42, 29.49)	(31.49, 24.53)	(31.49, 26.85)
1	(5.60, 1.87	(5.60, 2.21)	(3.51, 2.03)	(4.80, 2.50)	(3.53, 2.17)	(3.53, 2.62)
2	(1.31, 1.05	(1.31, 1.07)	(1.60, 1.23)	(1.30, 1.08)	(1.37, 1.11)	(1.37, 1.20)
δ			$m = 4$			
0.1	(308.07, 302.33	(308.07, 302.85)	(306.79, 302.41)	(307.92, 304.10)	(307.33, 304.33)	(307.33, 305.33)
0.5	(39.07, 24.51	(39.07, 25.83)	(26.25, 17.93)	(36.12, 26.54)	(28.48, 21.82)	(28.48, 24.04)
1	(5.09, 1.71	(5.09, 2.02)	(3.27, 1.94)	(4.38, 2.29)	(3.27, 2.03)	(3.27, 2.44)
2	(1.27, 1.05	(1.27, 1.07)	(1.58, 1.21)	(1.26, 1.07)	(1.34, 1.10)	(1.34, 1.18)

TABLE 4. (ARL_1 and ATS_1) when $\text{ARL}_0 = \text{ATS}_0 = 370.4$, $K = 3$, $\gamma = 1$, $m = 1$

n_1, n_1, t_1 t_2, W	$n_1 = 1, n_2 = 6$ $t_1 = 0.01, t_2 = 4.96$ $W = 0.2526$	$n_1 = 1, n_2 = 6$ $t_1 = 0.1, t_2 = 4.6$ $W = 0.2526$	$n_1 = 1, n_2 = 10$ $t_1 = 0.1, t_2 = 1.72$ $W = 0.7622$	$n_1 = 3, n_2 = 7$ $t_1 = 0.25, t_2 = 1.75$ $W = 0.6724$	$n_1 = 3, n_2 = 10$ $t_1 = 0.25, t_2 = 1.3$ $W = 1.0633$	$n_1 = 3, n_2 = 10$ $t_1 = 0.5, t_2 = 1.2$ $W = 1.0633$
δ	$B = 1$					
0.1	(329.19, 325.34)	(329.19, 325.69)	(328.62, 325.69)	(329.12, 326.57)	(328.86, 326.86)	(328.86, 327.53)
0.5	(68.05, 50.69)	(68.05, 52.27)	(52.29, 41.19)	(65.07, 53.480)	(56.16, 47.62)	(56.16, 50.47)
1	(10.45, 3.89)	(10.45, 4.48)	(5.98, 3.14)	(8.96, 4.87)	(6.24, 3.77)	(6.24, 4.59)
2	(1.67, 1.09)	(1.67, 1.14)	(1.77, 1.36)	(1.62, 1.19)	(1.61, 1.22)	(1.61, 1.35)
δ	$B = 2$					
0.1	(308.07, 302.33)	(308.07, 302.85)	(306.79, 302.41)	(307.92, 304.10)	(307.33, 304.34)	(307.33, 305.33)
0.5	(39.07, 24.51)	(39.07, 25.83)	(26.24, 17.93)	(36.12, 26.54)	(28.48, 21.82)	(28.48, 24.04)
1	(5.09, 1.71)	(5.09, 2.02)	(3.27, 1.94)	(4.38, 2.29)	(3.27, 2.03)	(3.27, 2.44)
2	(1.27, 1.05)	(1.27, 1.07)	(1.58, 1.21)	(1.26, 1.07)	(1.34, 1.10)	(1.34, 1.18)
δ	$B = 3$					
0.1	(301.54, 295.22)	(301.54, 295.79)	(299.98, 295.17)	(301.35, 297.16)	(300.64, 297.34)	(300.64, 298.44)
0.5	(33.50, 19.88)	(33.50, 21.12)	(21.73, 14.22)	(30.66, 21.74)	(23.58, 17.50)	(23.58, 19.52)
1	(4.28, 1.50)	(4.28, 1.75)	(2.90, 1.81)	(3.71, 1.99)	(2.87, 1.83)	(2.87, 2.17)
2	(1.23, 1.04)	(1.23, 1.06)	(1.55, 1.18)	(1.21, 1.06)	(1.28, 1.08)	(1.28, 1.15)
δ	$B = 4$					
0.1	(298.91, 292.37)	(298.91, 292.96)	(297.24, 292.26)	(298.72, 294.37)	(297.95, 294.53)	(297.95, 295.67)
0.5	(31.55, 18.31)	(31.55, 19.52)	(20.21, 12.10)	(28.76, 20.11)	(21.91, 16.06)	(21.91, 18.01)
1	(4.01, 1.44)	(4.01, 1.67)	(2.78, 1.77)	(3.50, 1.90)	(2.74, 1.76)	(2.74, 2.09)
2	(1.22, 1.03)	(1.22, 1.05)	(1.54, 1.17)	(1.19, 1.05)	(1.26, 1.07)	(1.26, 1.14)

We also investigate the effect of measurement errors on the standard deviation of time to signal shown as Jensen et al. [11]:

$$\text{SDTS}_1 = \sqrt{\mathbf{b}^T \left(\mathbf{I} - \mathbf{Q}_1\right)^{-1} (2\mathbf{D}_t \left(\mathbf{I} - \mathbf{Q}_1\right)^{-1} \mathbf{t} - \mathbf{t}^{(2)}) - (ATS_1)^2}\ , \quad (30)$$

where $\mathbf{b}^T = (b_1, b_2)$ is the vector of starting probabilities, **I** is the identity matrix of order 2, Q_1 is the 2×2 transition probability matrix, $\mathbf{t}^T = (t_2, t_1)$ is the vector of sampling intervals, $\mathbf{D}_t$ is the 2×2 diagonal matrix with the diagonal elements of **t** and $\mathbf{t}^{(2)}$ contains the squares of the elements of the **t** vector. The results for different shift sizes and for *K*=3, *m*=1, *B*=1, $E(t_s) = 1$, $E(n_s) = 5$, $n_1 = 2$, $n_2 = 10$, $t_1 = 0.1$ can be seen in Figure 2.

Since the value of parameter *B* is determined when the measurement system is set up, now we evaluate the value of parameter *m*, for different values of parameter *B* and γ. The result for the case of $\delta = 0.5$, *K*=3, $E(t_s) = 1$, $E(n_s) = 3$, $n_1 = 2$ $n_2 = 5$ and $t_1 = 0.1$ are displayed in Figures 3 and 4.

As we can see, in all cases, multiple measurements is only effective up to *m*=4, and more than that, it has negligible effect on the chart's performance. Also, in the cases of *B*=4 and *B*=5 ($B \geq 4$), multiple measurements has no effect on the chart's performance.

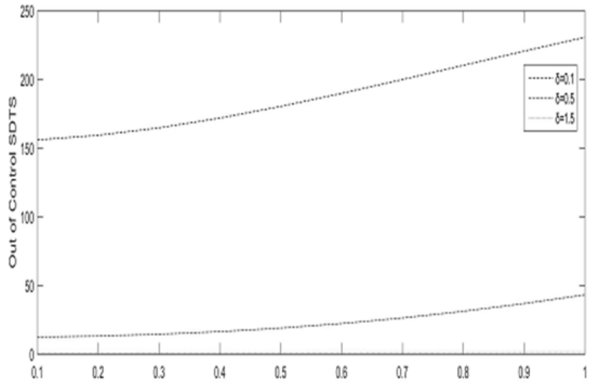

Figure 2. $\text{SDTS}_1 vs\ \gamma$

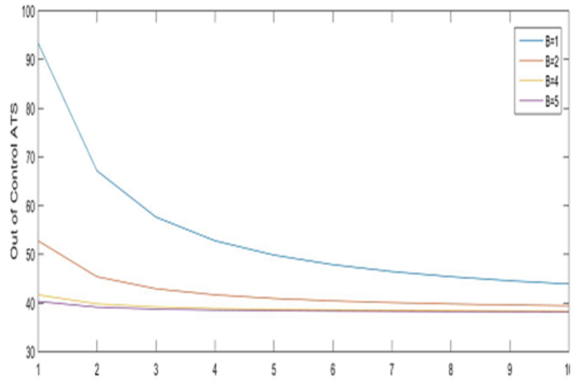

Figure 3. $\text{ATS}_1 vs\ m.\ \gamma = 1$

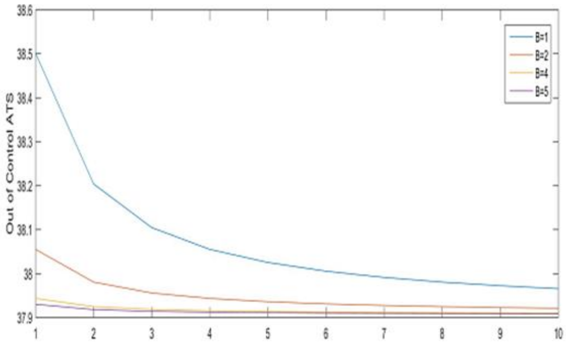

Figure 4. $\mathrm{ATS}_1 vs\ m.\ \gamma = 0.1$

5. A REAL CASE

In order to show an application for our VSSI scheme, we use the data from Costa and Castagliola [23]. Consider a 125 gr yogurt cup filling process. The quality characteristic *Y* is the weight of each cup.

After a long time study (Phase I), we know that $\mu_0 = 124.9$ and $\sigma_0 = 0.76$. From another study, we have: $\sigma_M = 0.24$. Therefore: $\gamma = \frac{\sigma_M}{\sigma_0} = \frac{0.24}{o.76} = 0.316$. In this example, we assume that, $t_1 = 0.3hr$, $E(t_s) = 1hr$, $E(n_s) = 3$, $n_1 = 2$, $n_2 = 5$, *K*=3, *A*=0, *B*=1 and *m*=2. Having these assumptions and by using Equations (28) and (29),we have: *W*=0.9638 and $t_2 = 1.35hrs$.

We would like to take twenty samples in total. For the first sample size and sampling interval we choose: $n = n_1 = 2$ and $t = t_2 = 1.35$. Having calculated Z_i for each sample using Equation (10), for other sample sizes and sampling intervals we use the mentioned methodology in Section 3. You can see the final results in Table 5 and also graphically in Figure 5. As it is clear from the results, from sample 12 on, after twelve hours, the process mean will shift below *LCL*, meaning a signal of an out-of-control situation. Therefore, corrective actions are required.

TABLE 5. Twenty samples of size 2 or 5 with $m = 2$, $t_1 = 0.3$, $t_2 = 1.35$, $k = 3$, $W = 0.9638$, *B=1*, $\gamma = 0.316$ and $\mu_0 = 124.9$

i	n_i										$\bar{X}_i$	Z_i	t_i	$\sum t_i$	Status
	m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2	m_1	m_2					
1	124.9	124.8	125.9	125.9	-	-	-	-	-	-	125.375	0.86	1.35	1.35	In-control
2	124.9	125.2	125.5	125.0	-	-	-	-	-	-	125.15	0.45	1.35	2.7	In-control
3	125.1	125.1	125.2	124.8	-	-	-	-	-	-	125.05	0.27	1.35	4.05	In-control
4	126.1	125.9	124.6	124.8	-	-	-	-	-	-	125.35	0.82	1.35	5.4	In-control
5	125.8	125.7	122.6	122.6	-	-	-	-	-	-	124.175	-1.32	1.35	6.75	In-control
6	124.9	125.3	125.5	124.8	124.6	125.2	124.9	124.8	124.8	124.2	124.9	0	0.3	7.05	In-control
7	124.2	124.6	125.8	125.3	-	-	-	-	-	-	124.975	0.14	1.35	8.4	In-control
8	124.9	124.9	123.8	123.2	-	-	-	-	-	-	124.2	-1.27	1.35	9.75	In-control
9	125.9	125.8	124.4	124.8	126.3	125.7	124.9	125.2	125.2	125.1	125.33	1.23	0.3	10.05	In-control
10	124.2	124.3	126.2	125.5	125.6	125.0	124.4	124.4	124.1	124.3	124.8	-0.29	0.3	10.35	In-control
11	123.7	123.6	123.4	123.3	-	-	-	-	-	-	123.5	-2.54	1.35	11.7	In-control
12	124.0	124.1	122.6	122.4	123.6	123.6	124.4	124.5	123.6	123.1	123.59	-3.76	0.3	12	Out-of-control
13	122.0	122.5	123.9	124.0	123.7	124.1	124.3	124.4	121.9	122.9	123.37	-4.39	0.3	12.3	Out-of-control
14	122.4	123.0	122.8	123.1	123.7	124.2	123.7	124.1	122.8	123.1	123.29	-4.62	0.3	12.6	Out-of-control
15	123.9	123.6	124.1	124.5	123.4	122.9	123.1	123.1	124.5	125.1	123.82	-3.10	0.3	12.9	Out-of-control
16	121.9	122.3	123.4	123.3	123.5	123.3	125.3	125.5	123.3	123.6	123.54	-3.91	0.3	13.2	Out-of-control
17	123.3	122.9	123.6	123.5	124.2	123.8	123.4	123.6	123.5	123.4	123.52	-3.96	0.3	13.5	Out-of-control
18	122.0	122.2	123.6	123.4	124.7	125.0	122.6	122.5	124.5	123.9	123.44	-4.19	0.3	13.8	Out-of-control
19	124.0	123.9	123.1	123.4	123.9	124.5	122.6	122.8	124.2	123.5	123.59	-3.76	0.3	14.1	Out-of-control
20	125.5	124.9	122.2	122.3	123.2	123.2	123.2	123.3	123.2	123.2	123.42	-4.25	0.3	14.4	Out-of-control

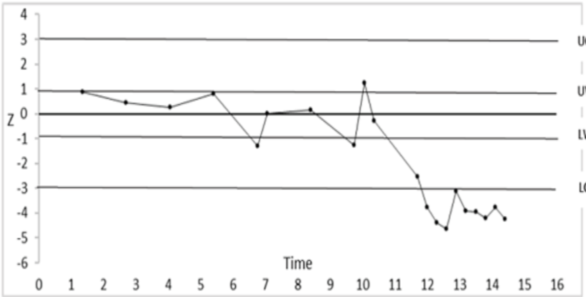

Figure 5. Z Control Chart based on the VSSI scheme

6. CONCLUSION AND FUTURE RESEARCHES

In this paper, the measurement errors under single and multiple measurement cases have been considered in a VSSI $\bar{X}$ control chart**.** Using a Markov Chain approach, we obtained out-of-control ARLs and ATSs for this scheme.

First, we performed ANOVA to test the effect of γ , *m*, *B*, and their interactions on the ATS and we found that each interaction involving γ is significant. Evaluating the effect of measurement errors on the VSSI $\bar{X}$ control chart, we concluded that higher measurement errors ratio (γ) would result in larger out-of-control ATSs and ARLs, meaning worsened conditions. Later, we showed that, in order to decease the negative effect of measurement errors, one may consider multiple measurements of each item. The results also showed that increasing the value of constant *B*, decreases the negative effect of measurement errors as well. We also found out that the multiple measurements is effective up to 4 measurements, and also if $B \geq 4$, then multiple measurements has no effect on the chart's performance. Finally, we illustrated this scheme using a real case.

Investigating the effect of measurement errors on the performance of VSSI $\bar{X}$ control chart under the linearly increasing error variance can be considered as a future research.

Moreover, since our model is based on the assumption that the process parameters (μ_0 and σ_0) are already known, future studies may include estimating them. Researchers may also consider the effect of measurement errors on the other univariate or multivariate adaptive control charts.

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The Effect of Measurement Errors on the Performance of Variable Sample Size and Sampling $\bar{X}$ Interval Control Chart

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چکیده

در طول سالهای گذشته، اثر خطاهای اندازه گیری برروی نمودارهای کنترلی تطبیقی و غیرتطبیقی به ندرت بررسی شده است. با این وجود، آن اثر بر روی نمودارهای کنترلی با فاصله نمونه گیری و اندازه نمونه متغیر (VSSI) تاکنون بررسی نشده است. در این مقاله، اثر خطاهای اندازه گیری بر روی نمودارهای کنترلی $\bar{X}$ VSSI ارزیابی می شود. پس از توسعه یک مدل، اثر خطاهای اندازه گیری و چندبار اندازه گیری نیز بر روی کارایی مدل $\bar{X}$ VSSI، با استفاده از معیار ATS که توسط یک روش زنجیره مارکوف محاسبه می شود، ارزیابی می شود. در آخر نیز یک مثال واقعی برای نشان دادن کاربرد الگوی پیشنهادی ارائه می شود.

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