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A Novel Similarity Solution of Turbulent Boundary Layer Flow over a Flat Plate

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A B S T R A C T

In this paper, the similarity solution of turbulent boundary layer flow on the flat plate with zero pressure gradients is presented. By employing similarity variables the governing partial differential equations are transformed to ordinary ones with inconsistent coefficients and solved numerically with the use of Runge–Kutta and shooting methods in conjunction with trial and error procedure. For a large domain of Reynolds number, the distribution of velocity, friction coefficient and thickness of boundary layer are calculated and compared with the experimental results extracted from the literature, where a good agreement between them are observed. The novelty of this study is to propose two new relations for the friction coefficient and thickness of the boundary layer.

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NOMENCLATURE

C_f	Friction coefficient	**Greek Symbols**	
l	Mixing length	δ	Boundary layer thickness
L	Length of the plate	η	Dimensionless variable
Re_x	Reynolds number	μ	Dynamic viscosity
$\bar{u}$	Velocity component in x direction	ρ	Density
$\overline{u'v'}$	Time-averaged value of $u'v'$	τ_0	Wall shear stress
U_∞	Free stream velocity	υ	Kinematic viscosity
$\bar{v}$	Velocity component in y direction	υ_t	Eddy viscosity
		ψ	Stream function

1. INTRODUCTION

Facing turbulent flows is inevitable in daily life and there is a certain need to study this kind of flows in details to understand its characteristics [1, 2]. Turbulent boundary layer flow over a flat plate is one of the most common phenomena which occur in turbo machine blades, rotary compressors and calculating the friction force on lifting surfaces and fuselage [3]. Since da Vinci's time, many scientists and researchers have been concerning about finding different aspects of turbulent flow. Blasius [4] presented a technique called "similarity solution" to reduce the partial differential boundary layer equation to nonlinear ordinary differential. Blasius seminal study became a base to simplify complex turbulent equations. A turbulent flow is called self-similar when all or some of its statistical properties are dependent to particular combination of independent variables [5]. Thus, self-similar flow depends on fewer variables and obviously dealing with this kind of flows is much easier. Turbulence is a very complicated phenomenon and its analysis and accurate identification is not routine, so many scientists have

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tried to find a similarity solution which can simplify the solution process [6-9]. Experimental activities and various theories about the self-similarity boundary layer, demonstrated that the free-stream velocity as a function of power law is required for self-similarity. Self-similar boundary layer is a useful phenomenon that simplifies the solution and helps us for better understanding of the boundary layer; organized category laboratory and experimental results. Principle of self-similarity solution in turbulent flow was introduced by Townsend [10]as the symbol of a dynamic equilibrium. A dimensionless variable that is written as a function of dimensionless coordinates is called self-similar if the stream line at the downstream of the flow does not change. Wolfshtein[1, 2, 11] performed a feasibility study about the existence of self-similar solution for the 2-D incompressible turbulent boundary layer. Clauser [12] carried out some experiments in which a desirable pressure gradient generates a self-similarity turbulent boundary layer. He introduced a constant dimensionless pressure gradient as a condition for self-similarity of boundary layer. Mellor and Gibson [13] showed that self-similarity may be achieved when free stream velocity can be declared as a function of longitude coordinates. Townsend [10]with the use of length and velocity scales analyzed outer layer equation turbulent boundary layer, which led to an addition condition that the length scale must vary linearly with the downstream coordinate. Shome [14] studied numerically the oscillating boundary layer flow over a flat plate. The $k-k_L-\omega$ turbulence model was used for the Reynolds number ranging from fully laminar flow to fully turbulent flow.

Despite of frequently mentioned applications in the industry, to the best knowledge of authors, no analytical solution has been presented for turbulent boundary layer flow over a flat plate yet. The only accomplished research in this field is based on direct numerical simulation. This simulation is based on four common turbulence models: algebraic $K-\varepsilon$, $K-\omega$ and Reynolds stress modeling [3, 15]. Still there is much debate about similarity solutions of turbulent boundary layer on the flat plate or pressure gradient. In addition, knowing these solutions is useful for better understanding of turbulence concepts; it helps us to guess an accurate initial scale for the experimental studies using wind tunnel [7]. It should be mentioned that turbulent boundary layer flow is more complicated than shear flow and turbulent jet flow because of the presence of a solid wall that imposes an additional force to the problem. It is obvious that fluid viscosity exerts no-slip condition to boundary layer conditions i.e. fluid velocity on a solid surface must be equal to the surface velocity [16, 17]. Ganji et al. [18] investigated the problem of forced convection over a horizontal flat plate under condition of variable plate temperature. In order to compute an approximation to the solution the homotopy perturbation method (HPM) is used. Moreover, recently some researches have worked on similarity solution [19, 20].

In the current study, a similarity solution for calculating the distribution of velocity, friction coefficient and thickness of the boundary layer are presented for a turbulent flow over the flat plate. Using Reynolds decomposition, turbulence viscosity is appeared in the equations. Employing Prandtl mixing length reported in the literature [21] and similarity variables, the PDE equations collapse into an ODE one. Finally the obtained equation is solved by Runge–Kutta and shooting methods in conjunction with trial and error procedure. The main advantage of this method is the independency of the solution from upstream and downstream characteristics of the flow. Furthermore, based on the results, two new relations for the friction coefficient and boundary layer thickness are presented.

2. GOVERNING EQUATION

An incompressible flow over a flat plate with no pressure gradient is considered. It is assumed that the free stream velocity is U_∞. Thus, the Reynolds averaged Navier-Stokes equations for a two-dimensional turbulence flow can be expressed as follows [3, 5]:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0 \tag{1-a}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=\frac{\partial}{\partial y}(\upsilon\frac{\partial \bar{u}}{\partial y}-\overline{u'v'}) \tag{1-b}$$

where $\bar{u}$ and $\bar{v}$ are components of the velocity in x and y direction, respectively and υ is the kinematic viscosity. Also, the boundary conditions are defined as:

$$\begin{cases} y=0 & : \ u=v=0 \\ y\to\infty & : \ u=U_\infty \end{cases} \tag{2}$$

In Equation (1-b), Reynolds stress can be expressed by the following relation [3, 5]:

$$-\overline{u'v'}=\upsilon_t(y)\frac{\partial \bar{u}}{\partial y} \tag{3}$$

in which υ_t is the Eddy viscosity. By substituting Equation (3) into Equation (1-b), it can be written:

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=\upsilon\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial}{\partial y}(\upsilon_t\frac{\partial \bar{u}}{\partial y}) \tag{4}$$

It is obvious that υ_t does not appear in the laminar boundary layer equations.

By employing Prandtl mixing length [3, 5], the turbulent viscosity can be assumed to be in the following form:

$$\upsilon_t = l^2 \frac{\partial \overline{u}}{\partial y} \tag{5}$$

The stream function is defined as bellow:

$$\psi = U_\infty g(x) f(\eta) \tag{6}$$

where:

$$\eta = y/g \tag{7}$$

It should be noted that $g(x)$ and $f(\eta)$ are exclusive functions of x and η, respectively.

According to definition of the stream function, the terms of Equation (4) can be obtained as:

$$\begin{aligned}
&\overline{u} = \frac{\partial \psi}{\partial y} = U_\infty f' \\
&\overline{v} = -\frac{\partial \psi}{\partial x} = U_\infty g' \left[\eta f' - f\right] \\
&\frac{\partial \overline{u}}{\partial x} = -U_\infty \frac{g'}{g} \eta f'' \\
&\frac{\partial \overline{u}}{\partial y} = \frac{U_\infty}{g} f'' \\
&\frac{\partial^2 \overline{u}}{\partial y^2} = \frac{U_\infty}{g^2} f''' \\
&\frac{\partial}{\partial y}(\upsilon_t \frac{\partial \overline{u}}{\partial y}) = \frac{\partial}{\partial y}(l^2 (\frac{\partial \overline{u}}{\partial y})^2) = \frac{U_\infty^2}{g^3} \frac{\partial}{\partial \eta}(l^2 f''^2)
\end{aligned} \tag{8}$$

Hence, Equation (4) can be formulated as:

$$-U_\infty^2 \frac{g'}{g} f\, f'' = \upsilon \frac{U_\infty}{g^2} f''' + \frac{U_\infty^2}{g^3} \frac{\partial}{\partial \eta}(l^2 f''^2) \tag{9}$$

The boundary layer over a plate can be divided into two layers: inner and outer layers with their own specific scaling [3, 5]. In this research, based on the experimental mixing length curve, presented by Anderosn and Kays [20], the following equation is proposed:

$$\frac{l}{\delta} = \alpha (\frac{y}{\delta})^n \tag{10}$$

where δ is the thickness of the boundary layer and α is a constant. Comparing Equation (10) with the mentioned curve [20], it can be concluded that:

$$l \propto \begin{cases} y & if \quad 0 \le y \le 0.1\,\delta \\ y^{0.5} & if \quad 0.1\delta \le y \le \delta \end{cases} \tag{11}$$

Regarding Equation (10):

$$l = \alpha \delta (\frac{y}{\delta})^n \quad \Rightarrow \quad l^2 = \alpha^2 \delta^{2-2n} y^{2n} \tag{12}$$

Substituting Equation (7) into Equation (12), the mixing length can be reformed as:

$$l^2 = \alpha^2 \delta^{2-2n} g^{2n} \eta^{2n} \tag{13}$$

By using Equation (13), Equation (9) can be reformulated as:

$$\begin{aligned}
&f''' + \frac{U_\infty g g'}{\upsilon} f\, f'' + \\
&g^{2n-1} \frac{U_\infty}{\upsilon} \alpha^2 \delta^{2-2n} \frac{\partial}{\partial \eta}(\eta^{2n} f''^2) = 0
\end{aligned} \tag{14}$$

In order to obtain a similarity solution, the coefficients have to be independent of x. Thus, it is assumed that:

$$\frac{U_\infty g g'}{\upsilon} = 1 \quad \Rightarrow \quad g = \sqrt{\frac{2\upsilon x}{U_\infty}} \tag{15}$$

Therefore, based on Equation (11), the solution domain can be divided into two sections:

A) At the vicinity of the wall ($n = 1 \Rightarrow l = \chi\, y$ where $\chi = 0.41$), Equation (14) will be reduced to:

$$f''' + f\, f'' + g \frac{U_\infty}{\upsilon} \chi^2 \frac{\partial}{\partial \eta}(\eta^2 f''^2) = 0 \tag{16}$$

According to Equation (7), it can be concluded:

$$\eta_{99} = \frac{\delta}{g} \tag{17}$$

It should be noted that $f'(\eta_{99}) = 0.99$.

Substituting Equation (17) into Equation (16) and based on the definition of $\mathrm{Re}_\delta = U_\infty \delta / \upsilon$, Equation (16) can be simplified as:

$$f''' + f\, f'' + (\chi^2 \frac{\mathrm{Re}_\delta}{\eta_{99}}) \frac{\partial}{\partial \eta}(\eta^2 f''^2) = 0 \tag{18}$$

and the boundary conditions are defined as:

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1\left(f'(\eta_{99}) = 0.99\right) \tag{19}$$

B) Farther from the wall ($y \ge 0.1\delta,\ n = 1/2$). In this case, Equation(14) will be changed into:

$$f''' + f\, f'' + \frac{U_\infty \delta}{\upsilon} \alpha^2 \frac{\partial}{\partial \eta}(\eta f''^2) = 0 \tag{20}$$

where the numerical value of $\alpha = 0.13$ is obtained from a simple curve fitting over the one reported by Andreson and Kays [20].

According to the nonlinear Equations (18, 20), a similarity solution for the turbulent boundary layer can be obtained. It should be mentioned that the relation between Re_δ and Re_x is as:

$$\mathrm{Re}_\delta = \frac{U_\infty \delta}{\upsilon} = \frac{U_\infty \eta_{99} g}{\upsilon} = \frac{U_\infty \eta_{99}}{\upsilon}\sqrt{\frac{2\upsilon x}{U_\infty}} = \eta_{99}\sqrt{2\mathrm{Re}_x} \quad (21)$$

3. SOLUTION ALGORITHM

As already mentioned, the governing equations for a turbulent boundary layer flow Equations (18, 20) can be presented as follows:

$$f''' = \frac{-ff'' - 2\chi^2\sqrt{2\mathrm{Re}_x}\eta\, f''^2}{1 + 2\chi^2\sqrt{2\mathrm{Re}_x}\eta^2 f''} \quad if \quad 0 \le \frac{y}{\delta} \le 0.1$$

$$f''' = \frac{-ff'' - \alpha^2\eta_{99}\sqrt{2\mathrm{Re}_x}\, f''^2}{1 + 2\alpha^2\eta_{99}\sqrt{2\mathrm{Re}_x}\eta\, f''} \quad if \quad 0.1 < \frac{y}{\delta} \le 1 \quad (22)$$

with the boundary conditions defined in Equation (19).

In order to solve the system of Equations (22), with the use of shooting method $f''(0)$ is obtained, then an iterative procedure is employed to calculate η_{99} at a specific Reynolds number. In other words, for a specific Reynolds number, $f''(0)$ and η_{99} are obtained simultaneously by shouting method and trial and error procedure, respectively. It should be stated that $f''(0)$ represents friction coefficient and can be expressed as follows:

$$\tau_0 = \mu\frac{\partial \bar{u}}{\partial y} = \mu\frac{U_\infty}{g} f''(0) = \mu\frac{U_\infty}{\sqrt{\frac{2\upsilon x}{U_\infty}}} f''(0) = \sqrt{\frac{\rho\mu U_\infty^3}{2x}}\, f''(0)$$

$$\frac{C_f}{2} = \frac{\tau_0}{\rho U_\infty^2} = \frac{f''(0)}{\sqrt{2\mathrm{Re}_x}} \quad (23)$$

where μ and ρ are dynamic viscosity and density, respectively. Furthermore, according to Equation (17) the boundary layer thickness can be written as follows:

$$\frac{\delta}{x} = \eta_{99}\sqrt{\frac{2}{\mathrm{Re}_x}} \quad (24)$$

4. RESULT AND DISCUSSION

Equation (22) with the boundary conditions of Equation (19), has been solved numerically via the shooting method based on the Runge-Kutta scheme. A FORTRAN code has been prepared to find the numerical solution of the present boundary value problem (BVP). The Runge-Kutta scheme as a standard integration scheme is used to determine the distributions of the velocity. In order to avoid the grid dependency, the integration step was chosen from 10^{-5} to 10^{-6} and no dependency was observed. It should be mentioned that the major difficulty of the solution is existansce of the unknown upper limit of integration (η_{99}) in Equation (22). Thus, to calculate η_{99} which is dependent on the Reynolds number, the trial and error procedure must be employed. The variation of this parameter is shown in Figure 1.

Once Equation (22) is solved, the velocities, friction coefficient and boundary layer thickness would be obtained from Equations (8), (23) and (24), respectively.

The comparision of dimensionless velocity profie ($\bar{u}/U_\infty$) versus y/δ with the 1/7 power law [5] at $\mathrm{Re} = 1.125\times10^8$ is illustrated in Figure 2 and an acceptable agreement beween them is observed. It should be stated that from Equations (7) and (24): $\eta/\eta_{99} = y/\delta$.

The influence of Reynolds number on dimensionless velocity profiles versus y/δ is demonstrated in Figure 3. According to this figure, it is clear that as Re_x increases, the momentum transferred between the core of the flow and the wall has to be increased, thus the velocity gradient at the vicinity of the wall increases.

Figure 4 depicts the variation of friction coefficient (C_f) versus Reynolds number. Also, the results of the well-known relation of $C_f/2 = 0.455/\left[\ln(0.06\mathrm{Re}_x)\right]^2$ [5, 22] are plotted too and an excellent agreement between them is observed. According to this figure, by increasing Reynolds number, C_f decreases. Furthermore, by using curve fitting method it is possible to obtain a novel formula for C_f as:

$$\frac{C_f}{2} = \frac{0.009397}{\mathrm{Re}_x^{0.1181}} \quad (25)$$

The comparison of Equation (25) with the relation of $C_f/2 = 0.455/\left[\ln(0.06\mathrm{Re}_x)\right]^2$ is shown in Figure 5. From Equation (24) and obtaining η_{99} from Figure 1, the variation of δ/x versus Reynolds number is demonstrated in Figure 6. Based on these data, another novel formula can be presented by curve fitting method for thickness of boundary layer as follows:

$$\frac{\delta}{x} = \frac{0.13}{\mathrm{Re}_x^{0.082}} \quad (26)$$

This formula satisfies the fact that the thickness of boundary layer should increase when Reynolds number growths (here $\delta \propto \mathrm{Re}^{0.918}$).

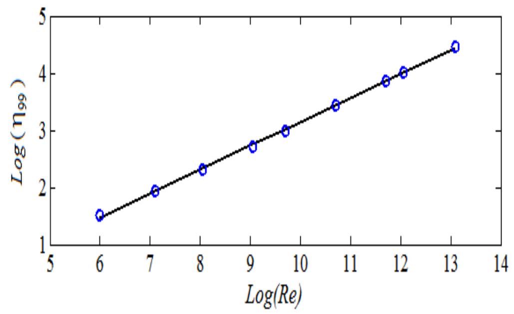

Figure 1. Variation of η_{99} versus Reynolds number

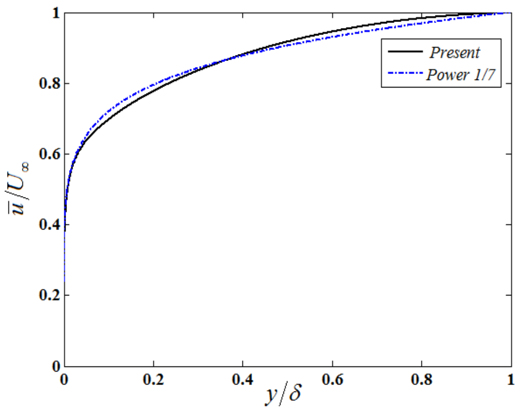

Figure 2. Comparision of dimensionless velocity profie ($\bar{u}/U_{\infty}$) versus y/δ with the 1/7 power law at $\mathrm{Re} = 1.125\times10^{8}$

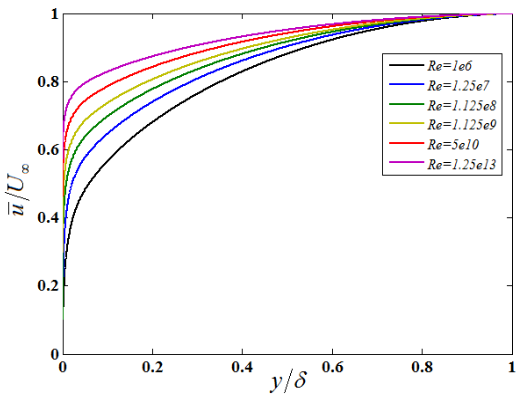

Figure 3. Variation of of dimensionless velocity profie ($\bar{u}/U_{\infty}$) versus y/δ for different Reynolds numbers

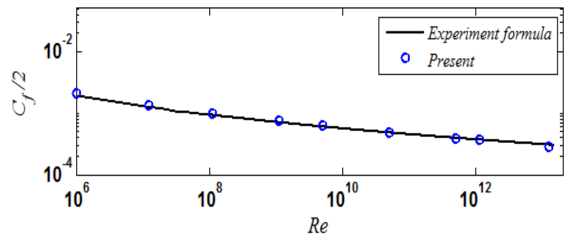

Figure 4. Comparison of the friction coefficient obtained by the present work with the experimental formula [5, 21]

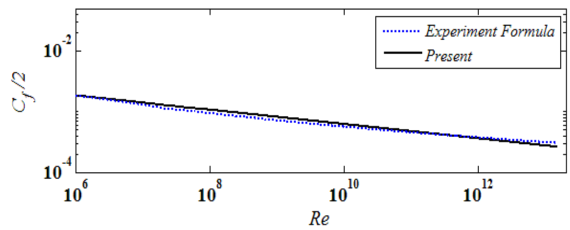

Figure 5. Comparison of the proposed relation for friction coefficient (Equation (25)) with the experimental formula [5, 21]

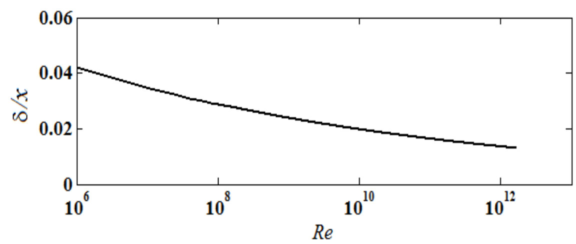

Figure 6. Variation of δ/x versus Reynolds number

5. CONCLUSION

This research studied the turbulent boundary layer flow over a flat plate in a large domain of Reynolds number. Employing similarity transformation, the basic partial differential equations (PDE) were reduced to an ordinary differential equation (ODE). The solution of this equation is independent of upstream and downstream characteristics of the flow. It should be noted that unlike the laminar boundary layer equation (Blasius equation), the Reynolds number was appeared in the obtained ODE. Therefore, a semi-similarity solution was used. Accordingly, a trial and error procedure in conjunction with Runge-Kutta and shooting methods were employed to solve the equation. The results were compared with the experimental data and a good agreement between them was observed. For a large domain of Reynolds number, the distribution of velocity, friction coefficient and thickness of boundary layer were obtained and discussed. Moreover, two new relations for friction coefficient and boundary layer thickness as exclusive functions of Reynolds number were proposed.

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چکیده

مقاله حاضر به بررسی حل تشابهی جریان درهم بر روی صفحه تخت با گرادیان فشار صفر می‌پردازد. در ابتدا معادلات پیوستگی و حرکت با حضور ترم تنش رینولدز و رابطه‌ی آن با طول اختلاط پرانتل در نظر گرفته شده و سپس با تغییر متغیرهای حل تشابهی، معادلات فوق به یک معادله دیفرانسیل معمولی غیرخطی تبدیل شده و در این بین شرایط تشابهی منجر به یافتن توابع مجهول به خدمت گرفته شده در تغییر متغیرها می‌گردد. معادله دیفرانسیل بدست آمده یک متغیره بوده اما یکی از ضرایب این معادله بر حسب عدد رینولدز بدست می‌آید. این معادله با استفاده از روش رانگ-کوتا و پرتابی به ازای اعداد رینولدز مختلف حل شده و نهایتا تنش برشی دیواره و ضریب اصطکاک محاسبه و با نتایج تجربی مقایسه شده که تطابق بسیار عالی مشاهده گردید. همچنین با استفاده از برازش منحنی‌ها برای اصطکاک پوسته‌ای و ضخامت لایه مرزی دو رابطه‌ی مستقل پیشنهاد شده‌است.

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