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Numerical Solution of Optimal Control of Time-varying Singular Systems via Operational Matrices

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ABSTRACT

In this paper, a numerical method for solving the constrained optimal control of time-varying singular systems with quadratic performance index is presented. Presented method is based on Bernstein polynomials. Operational matrices of integration, differentiation and product are introduced and utilized to reduce the solution of optimal control problems with time-varying singular systems to the solution of algebraic equations set. The strength of the method is shown by exhibiting a numerical implementation using operational matrices that solves the determined control problem by solving an equation set. The method converges rapidly to the exact solution and gives very accurate results even by low value of m. Illustrative examples are included to demonstrate the validity and efficiency of the technique and convergence of method to the exact solution especially for unstable singular systems.

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1. INTRODUCTION

Problem of optimal control of singular systems is an immense interest; especially those researchers are investigating on the existing problems in the field of control theory and in numerical computation of the value of control vector which is controlling the state vector. These types of systems are encountered in many areas, such as network theory, economics, demography, neural systems, composite systems, etc. Chen and Hsiao [1] and Chen and Shih [2] have used Walsh series to study the problem of optimal control of time-invariant and time-varying linear systems. A review of the literature suggests that Cobb [3] and Pandolfi [4] were the first authors to consider the optimal regulator problem of continuous-time singular systems. Both the used state feedback and the results were derived with the aid of Ricatti-type matrix equations. Walsh functions have been widely used to study problem of optimal control of linear systems with quadratic performance index [1, 2]. Analysis of linear singular systems using orthogonal functions has been presented, among others, by Trzaska [5] and Marszalek [6, 7]. References [5, 7] used block-pulse and reference [6] applied Walsh operational matrices of integration to calculate the integral involved in the analysis of singular systems. Palanisamy [8] analyzed optimal control of linear systems using a single-term Walsh series (STWS) method. Balachandran and Murugesan [9] have applied the STWS method to optimal control of linear singular systems. Razzaghi and Marzban [10] introduced a piecewise linear polynomial function method for optimal control of singular systems. Observability and controllability of linear time-varying singular systems have been studied in [11, 12]. Murugesan et al. [13] and Park et al. [14] used RK-Butcher algorithm to compute numerical solution of the industrial robot arm control problem and optimal control of time-invariant linear singular systems. Optimal control of singular system has also been studied using genetic programming approach in [15]. In view of this situation, numerous research articles have been dedicated to singular systems in the past three decades. Liu and Sreeram [16] and Chang and

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Davison [17] constitute a representative collection of the other works. Recently, a numerical algorithm to obtain the consistent conditions satisfied by singular arcs for singular linear-quadratic(LQ) optimal control problems is presented in [18]. In [19] the existence of singular arcs for optimal control problems is studied by using a geometric recursive algorithm inspired in Dirac's theory of constraints. For more information on the mathematical modeling and solution of this model and some other similar models, we refer the interested reader to the papers [20-30]. [19, 20] [21-30]

In this paper, Bernstein polynomial basis is used for solving an optimal control of time-varying singular system with a quadratic cost function. In the following, major difficulties and challenges which are to be met in the paper are summarized. At first, state vector $\dot{x}(t)$ and control vector $u(t)$ are expanded in terms of Bernstein polynomial. Operational matrices of Bernstein polynomial are applied to estimate $x(t)$ using $\dot{x}(t)$. In section 6, when $k \leq n$ by a technic the system dynamics and cost function (performance index) are approximated, then a new minimization problem is attained. Approximated solution of problem is calculated using the Lagrange multipliers method. These unknown coefficients are determined in such a way that the necessary conditions for extremization are met. The presented method shows that a more accurate solution of the time-varying optimal control of linear singular systems with a quadratic performance index can be obtained. Numerical evidence of the stability of the algorithm will be presented by discussing various relevant numerical experiments.

This paper is structured as follows: in section 2, problem of time-varying singular system is described. Section 3 describes the basic formulation of Bernstein polynomials which is required for our subsequent development. Section 4 is devoted to the function approximation using Bernstein polynomial basis whilst the upper bound of approximation error is deduced. In section 5, we elaborate operational matrices of integration, differentiation, dual and product using Kronecker product [31]. In section 6, solution of time-varying singular system problem is approximated by Bernstein polynomial basis and an algebraic equation set is presented using Lagrange multipliers method. In section 7, numerical findings are presented which demonstrate the validity, accuracy and applicability of our new method. Section 8 consists of the remark and brief summary.

2. STATEMENT OF THE PROBLEM

Consider following time-varying singular system:

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0, \tag{1}$$

where the matrix $E(t)$ is singular, $x(t) \in \mathrm{R}^n$ is a generalized state vector, $u(t) \in \mathrm{R}^k$ is a control vector, $A(t)$ and $B(t)$ are known coefficient matrices associated with $x(t)$ and $u(t)$ with appropriate dimensions, respectively, and x_0 is the given initial state vector.

In order to minimize a cost function, considering both state and control signals of the feedback control system, a quadratic performance index is usually minimized:

$$J = \frac{1}{2}x^T(t_f)Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [x^T(t)F(t)x(t) + u^T(t)R(t)u(t)]dt, \tag{2}$$

where t_0 and t_f are prescribed times, $S \in \mathrm{R}^{n\times n}$ and $F(t)$ are weighting matrices for $x(t)$ and S is a symmetric and positive definite (or semi definite) matrix, $R(t)$ is a weighting matrix for $u(t)$, $R(t)$ and $F(t)$ are matrices with appropriate dimensions [32].

3. PROPERTIES OF BERNSTEIN POLYNOMIALS

Bernstein polynomials of m^{th}degree are defined on $[a, b]$ as [5, 6, 7, 23]:

$$B_{i,m}(x) = \binom{m}{i}\frac{(x-a)^i(b-x)^{m-i}}{(b-a)^m}, \quad 0 \leq i \leq m$$

where

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}.$$

These Bernstein polynomials form a basis on [a,b]. There are $m + 1$ number m^{th}degree polynomials. For convenience, $B_{i,m}(x) = 0$, if $i < 0$ or $i > m$. A recursive definition can also be used to generate the Bernstein polynomials over $[a, b]$ as:

$$B_{i,m}(x) = \frac{(b-x)}{b-a}B_{i,m-1}(x) + \frac{x-a}{b-a}B_{i-1,m-1}(x).$$

It can be shown that each of the Bernstein polynomials are positive and linear independent and the sum of all the Bernstein polynomials is unity for all real $x \in [a, b]$, i.e., $\sum_{i=0}^{m} B_{i,m}(x) = 1$. It is easy to show that any given polynomial of degree m can be expanded in terms of these basis functions.

4. APPROXIMATION OF FUNCTIONS

Suppose that $H = L^2[t_0, t_f]$ where $t_0, t_f \in \mathrm{R}$, let $\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\} \subset H$ is the set of Bernstein polynomials of m^{th} degree and

$$Y = Span\,\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\}$$

and f is an arbitrary element in H. Since Y is a finite dimensional vector space, f has a unique best approximation out of Y, say $y_0 \in Y$, that is:

$$\exists y_0 \in Y;\ \ \forall y \in Y\ \ ||f - y_0||_2 \leq ||f - y||_2,$$

where $||f||_2 = \sqrt{<f,f>}$ and $<f,g> = \int_{t_0}^{t_f} f_{(t)} g_{(t)} dt$.
In [39], it is shown that a unique coefficient vector $c^T = [c_0, c_1, \cdots, c_m]$ exists such that:

$$f \approx y_0 = \sum_{i=0}^{m} c_i B_{i,m} = c^T \phi, \tag{3}$$

where $\phi^T = [B_{0,m}, B_{1,m}, \cdots, B_{m,m}]$ and c^T can be obtained:

$$c^T = \left(\int_{t_0}^{t_f} f(x)\phi(x)^T dx\right) Q^{-1}, \tag{4}$$

which Q is said dual matrix of ϕ and introduced as:
$Q = <\phi\ ,\ \phi> = \int_{t_0}^{t_f} \phi(x)\phi(x)^T dx.$

Theorem 1. Suppose that H be a Hilbert space and Y be a closed subspace of H such that $dim\quad Y < \infty$ and $\{y_1, y_2, \cdots, y_n\}$ is any basis for Y. Let x be an arbitrary element in H and y_0 be the unique best approximation to x out of Y. Then
$||x - y_0||_2^2 = \frac{G(x, y_1, y_2, \cdots, y_n)}{G(y_1, y_2, \cdots, y_n)}$,
where
$$G(x, y_1, y_2, \cdots, y_n) = \begin{vmatrix} <x,x> & <x,y_1> & \cdots & <x,y_n> \\ <y_1,x> & <y_1,y_1> & \cdots & <y_1,y_n> \\ \vdots & \vdots & \vdots & \vdots \\ <y_n,x> & <y_n,y_1> & \cdots & <y_n,y_n> \end{vmatrix}.$$

Proof. [33].
Exact value of approximation error is presented by the Theorem 1. In the following lemma, an upper bound of approximation error is presented.

Lemma 1. Suppose that function $g: [t_0, t_f] \to R$ be $m+1$ times continuously differentiable, $g \in C^{m+1}[t_0, t_f]$, and $Y = Span\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\}$. If $c^T\phi$ be the best approximation g out of Y then the mean error bound is presented as follows:
$$||g - c^T\phi||_2 \le \frac{M(t_f - t_0)^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}} ,$$
where $M = \max_{x \in [t_0, t_f]} |g^{(m+1)}(x)|$.

Proof. [34].
Lemma 1 shows that the method of approximation converges to f when $m \to \infty$.
Now, let $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ where $x_i(t) \in H$ for $i = 1, 2, \ldots, n$. If we approximate $x_i(t)$ out of Y by (3), we have:
$x_i(t) \approx c_i^T \phi$
where $c_i^T = [c_{i,0}, c_{i,1}, \ldots, c_{i,m}]$ can be calculated by (4) for $i = 1, 2, \ldots, n$. Then

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \approx \begin{bmatrix} c_1^T\phi \\ c_2^T\phi \\ \vdots \\ c_n^T\phi \end{bmatrix}$$

$$= B_{0,m}\begin{bmatrix} c_{1,0} \\ c_{2,0} \\ \vdots \\ c_{n,0} \end{bmatrix} + B_{1,m}\begin{bmatrix} c_{1,1} \\ c_{2,1} \\ \vdots \\ c_{n,1} \end{bmatrix} + \cdots + B_{m,m}\begin{bmatrix} c_{1,m} \\ c_{2,m} \\ \vdots \\ c_{n,m} \end{bmatrix}$$

$$= [B_{0,m}I_n,\ B_{1,m}I_n,\ \ldots,\ B_{m,m}I_n] = \widehat{\phi}_n{}^T C,$$

where I_n is the identity matrix of order n ,
$C^T = [c_{1,0}, c_{2,0}, \ldots, c_{n,0}, c_{1,1}, c_{2,1}, \ldots, c_{n,1}, \ldots, c_{1,m}, c_{2,m}, \ldots, c_{n,m}]$
is a matrix $1 \times [n(m+1)]$ and

$$\widehat{\phi}_n = \begin{bmatrix} B_{0,m}I_n \\ B_{1,m}I_n \\ \vdots \\ B_{m,m}I_n \end{bmatrix} = \phi \otimes I_n, \tag{5}$$

where $\otimes$ denotes the Kronecker product [31], so

$$x^T = C^T\widehat{\phi}_n. \tag{6}$$

We can also approximate a matrix of functions using (6). Therefore, suppose that $A_{r\times s}(t) = [a_{i,j}(t)]_{r\times s} = \begin{bmatrix} A_1(t) \\ A_2(t) \\ \vdots \\ A_r(t) \end{bmatrix}$ where $a_{i,j}(t) \in H$ and $A_i(t)$ is i^{th} row of $A(t)$ for $i = 1, 2, \ldots, r$.
If we approximate $A_i(t)$ out of Y by (6), we will get $A_i(t) \approx \hat{A}_i{}^T(t)\widehat{\phi}_s(t)$ for $i = 1, 2, \ldots, r$, then

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{bmatrix} \approx \begin{bmatrix} \hat{A}_i{}^T\widehat{\phi}_s \\ \hat{A}_2{}^T\widehat{\phi}_s \\ \vdots \\ \hat{A}_r{}^T\widehat{\phi}_s \end{bmatrix} = \begin{bmatrix} \hat{A}_i{}^T \\ \hat{A}_2{}^T \\ \vdots \\ \hat{A}_r{}^T \end{bmatrix}\widehat{\phi}_s = \tilde{A}\widehat{\phi}_s \tag{7}$$

where $\tilde{A}_{r\times[s(m+1)]} = \begin{bmatrix} \hat{A}_i{}^T \\ \hat{A}_2{}^T \\ \vdots \\ \hat{A}_r{}^T \end{bmatrix}$.

5. OPERATIONAL MATRICES

Operational matrices of the integration P, differentiation D dual Q and product $\hat{C}$ of vector ϕ are respectively defined by:

$$\int_{t_0}^{x} \phi(t)dt \approx P\phi(x), \qquad t_0 \le x \le t_f$$

$$\frac{d\phi(x)}{dx} \approx D\phi(x),$$

$$Q = \int_{t_0}^{t_f} \phi(x)\phi^T(x)dx,$$

$$c^T\phi(x)\phi(x)^T \approx \phi(x)^T\hat{C},$$

which the details of obtaining these matrices are given in [34].

Analogously, we can define $[n(m+1)] \times [n(m+1)]$ operational matrices of $\widehat{\Phi}_n$ as:

$\int_{t_0}^{x} \widehat{\Phi}_n(t)dt \approx \widehat{\mathrm{P}}_{\mathrm{n}}\widehat{\Phi}_n(x), \qquad t_0 \le x \le t_f$,

$\frac{d\,\widehat{\Phi}_n(x)}{dx} \approx \widehat{\mathrm{D}}_{\mathrm{n}}\widehat{\Phi}_n(x)$,

$\widehat{\Phi}_n = \int_{t_0}^{t_f} \widehat{\Phi}_n(x)\widehat{\Phi}_n(x)^T dx$,

$C\widehat{\Phi}_n(x)\widehat{\Phi}_n(x)^T \approx \widehat{\Phi}_n(x)^T\hat{C}_n$,

which C is an arbitrary $n \times [n(m+1)]$ matrix.
It can be easily shown that the operational matrices of the integration, differentiation and dual of $\widehat{\Phi}_n$ are as:

$$\hat{P}_n = P \otimes I_n,$$

$$\hat{D}_n = D \otimes I_n, \tag{8}$$

$$\hat{Q}_n = Q \otimes I_n.$$

5. 1. Operational Matrix of Product of $\widehat{\Phi}_n$

It is aimed to derive an explicit formula for operational matrix of product of $\widehat{\Phi}_n$. Suppose that $C = [C_0, C_1, \ldots, C_m]$ is an arbitrary $n \times [n(m+1)]$ matrix where C_i is $n \times n$ matrix for $i = 0,1,2,\ldots,m$, then $\hat{C}_n$ is $[n(m+1)] \times [n(m+1)]$ operational matrix of product of $\widehat{\Phi}_n$ whenever
$C\widehat{\Phi}_n(x)\widehat{\Phi}_n(x)^T \approx \widehat{\Phi}_n(x)^T\hat{C}_n$.
Since $C\widehat{\Phi}_n(x) = \sum_{i=0}^{m} C_i B_{i,m}(x)$, we have
$C\widehat{\Phi}_n(x)\widehat{\Phi}_n(x)^T =$
$\left[\ \sum_{i=0}^{m} C_i B_{i,m}(x)B_{0,m}(x), \sum_{i=0}^{m} C_i B_{i,m}(x)B_{1,m}(x),\ \cdots,\ \sum_{i=0}^{m} C_i B_{i,m}(x)B_{m,m}(x)\right]$.
Now, we approximate all functions $B_{k,m}(x)B_{i,m}(x)$ in terms of $\{B_{i,m}(x)\}_{i=0}^{m}$ for $i,k = 0,1,\cdots,m$, i.e, we must find vector $e_{k,i}^T = [e_0^{k,i}, e_1^{k,i}, e_2^{k,i}, \ldots, e_m^{k,i}]$ by (4) such that $B_{k,m}(x)B_{i,m}(x) \approx e_{k,i}^T\Phi(x), \quad i,k = 0,1,\cdots,m$.
Therefore, for $k = 0,1,\cdots,m$

$$\sum_{i=0}^{m} C_i B_{k,m}(x)B_{i,m}(x) \approx \sum_{i=0}^{m} C_i\left(\sum_{j=0}^{m} e_j^{k,i} B_{j,m}(x)\right)$$

$$= \sum_{j=0}^{m} B_{j,m}(x)\left(\sum_{i=0}^{m} C_i e_j^{k,i}\right) = \widehat{\Phi}_n(x)^T \begin{bmatrix} \sum_{i=0}^{m} C_i e_0^{k,i} \\ \sum_{i=0}^{m} C_i e_1^{k,i} \\ \vdots \\ \sum_{i=0}^{m} C_i e_m^{k,i} \end{bmatrix}$$

$$= \widehat{\Phi}_n(x)^T\left[e_{k,0} \otimes I_n, e_{k,1} \otimes I_n,\ \cdots, e_{k,m} \otimes I_n\right] \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_m \end{bmatrix} = \widehat{\Phi}_n(x)^T\overline{C_{k+1}}$$

where

$$\overline{C_{k+1}} = \left[e_{k,0} \otimes I_n,\ e_{k,1} \otimes I_n,\ \cdots, e_{k,m} \otimes I_n\right] \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_m \end{bmatrix}.$$

If we define $[n(m+1)] \times [n(m+1)]$ matrix $\overline{C} = \left[\overline{C_1},\ \overline{C_2}, \cdots,\ \overline{C_{m+1}}\right]$, then

$$C\widehat{\Phi}_n(x)\widehat{\Phi}_n(x)^T$$

$$= \left[\ \sum_{i=0}^{m} C_i B_{i,m}(x)B_{0,m}(x), \sum_{i=0}^{m} C_i B_{i,m}(x)B_{1,m}(x),\ \cdots,\ \sum_{i=0}^{m} c_i B_{i,m}(x)B_{m,m}(x)\right]$$

$$\approx \widehat{\Phi}_n(x)^T\left[\overline{C_1},\ \overline{C_2},\ \cdots,\ \overline{C_{m+1}}\right] = \widehat{\Phi}_n(x)^T\overline{C},$$

therefore

$$C\widehat{\Phi}_n(x)\widehat{\Phi}_n(x)^T \approx \widehat{\Phi}_n(x)^T\overline{C}. \tag{9}$$

so $\overline{C}$ is the operational matrix of product of $\widehat{\Phi}_n$.

6. SOLUTION OF PROBLEM USING BERNSTEIN POLYNOMIALS BASIS

6. 1. Approximation of the System Dynamics

We approximate (1) as follows:
Let $k \le n$ (for $k \ge n$ follows quite same) and

$$\dot{x}(t) = [\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t)]^T, \tag{10}$$

$$u(t) = [u_1(t), u_2(t), \ldots, u_k(t)]^T. \tag{11}$$

Using (3), each of $\dot{x}_i(t)$ and each of $u_j(t)$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,k$, can be approximated in terms of basic functions as $\dot{x}_i(t) = c_i^T\Phi(t)$ and $u_j(t) = \widehat{U}_j^T\Phi(t)$ where $c_i^T = [c_{i,0}, c_{i,1}, \ldots, c_{i,m}]$ and $\widehat{U}_j^T = [u_{j,0}, u_{j,1}, \ldots, u_{j,m}]$ which can be calculated by (4).
Then we can write (10), (11) by (6) as:

$$\dot{x}(t) = \widehat{\Phi}_n(t)^T C \tag{12}$$

$$u(t) = \widehat{\Phi}_k(t)^T\widehat{U} \tag{13}$$

where
$C^T = [c_{1,0}, c_{2,0}, \ldots, c_{n,0},\ c_{1,1}, c_{2,1}, \ldots, c_{n,1}, \ldots, c_{1,m}, c_{2,m}, \ldots, c_{n,m}]$,
and

$$\widehat{U}^T = [u_{1,0}, u_{2,0}, \ldots, u_{k,0}, u_{1,1}, u_{2,1}, \ldots, u_{k,1}, \ldots, u_{1,m}, u_{2,m}, \ldots, u_{k,m}]. \tag{14}$$

From (10) and (8) we have:

$$x(t) = \widehat{\Phi}_n{}^T\hat{P}_n(t)^T C + x(t_0) = \widehat{\Phi}_n(t)^T\hat{X} \tag{15}$$

where $\hat{X} = \hat{P}_n{}^T C + d$ and $x(t_0) = \widehat{\Phi}_n(t)^T d$.
Now, we approximate matrices A and E by (7)

$$A \approx \tilde{A}\widehat{\Phi}_n, \tag{16}$$

$$E \approx \tilde{E}\widehat{\Phi}_n, \tag{17}$$

since $k \le n$,

$$Bu = [B_1, B_2, \ldots, B_k]\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} = [B_1, B_2, \ldots, B_n]\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = B^*U^*,$$

where$B^* = [B_1, B_2, \dots, B_n]$, $U^* = [u_1, u_2, \dots, u_n]^T$ in which $B_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $u_j = 0$ for $j = k+1, k+2, \dots, n$.

If we approximate B^* and U^* by (7) and (6), respectively, we get:

$$B^* \approx \tilde{B}^* \widehat{\Phi}_n,$$

$$U^* \approx \widehat{\Phi}_n{}^T \widehat{U}^*,$$

where
$\widehat{U}^* = [u_{1,0}, u_{2,0}, \dots, u_{n,0}, u_{1,1}, u_{2,1}, \dots, u_{n,1}, \dots, u_{1,m}, u_{2,m}, \dots, u_{n,m}]^T$,
therefore

$$Bu = B^* U^* \approx \tilde{B}^* \widehat{\Phi}_n \widehat{\Phi}_n{}^T \widehat{U}^*. \tag{18}$$

Substituting (12), (15), (16), (17) and (18) in (1) we obtain:

$$\tilde{\mathrm{E}} \widehat{\Phi}_n \widehat{\Phi}_n{}^T C = \tilde{A} \widehat{\Phi}_n \widehat{\Phi}_n{}^T \widehat{X} + \tilde{B}^* \widehat{\Phi}_n \widehat{\Phi}_n{}^T \widehat{U}^*, \tag{19}$$

using (9) we have:

$$\tilde{A} \widehat{\Phi}_n \widehat{\Phi}_n{}^T \approx \widehat{\Phi}_n{}^T \widehat{A}_n , \tag{20}$$

$$\tilde{B}^* \widehat{\Phi}_n \widehat{\Phi}_n{}^T \approx \widehat{\Phi}_n{}^T \widehat{B}_n^* , \tag{21}$$

$$\tilde{E} \widehat{\Phi}_n \widehat{\Phi}_n{}^T \approx \widehat{\Phi}_n{}^T \widehat{E}_n, \tag{22}$$

where $\widehat{A}_n$, $\widehat{B}_n^*$ and $\widehat{E}_n$ are operational matrices of product, by replacing (20), (21) and (22) in (19) we get
$\widehat{\Phi}_n{}^T \widehat{E}_n C = \widehat{\Phi}_n{}^T \widehat{A}_n \widehat{\mathrm{X}} + \widehat{\Phi}_n{}^T \widehat{B}_n^* \widehat{U}^*$,
so

$$\widehat{A}_n \widehat{\mathrm{X}} + \widehat{B}_n^* \widehat{U}^* - \widehat{E}_n C = 0. \tag{23}$$

6.2. The Performance Index Approximation

Now, we approximate (2) as follows:
At first, we approximate matrices F and R by (7), i.e.

$$F \approx \tilde{F} \widehat{\Phi}_n, \tag{24}$$

$$R \approx \tilde{R} \widehat{\Phi}_k. \tag{25}$$

Substituting (13), (15), (24) and (25) in (2) we obtain:

$$J = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x^T(t) F(t) x(t) + u^T(t) R(t) u(t)] dt \quad \approx \frac{1}{2} \widehat{X}^{\ T} \widehat{\Phi}_n(t_f)\ S\ \widehat{\Phi}_n(t_f)^{\ T} \widehat{X} + \frac{1}{2} \int_{t_0}^{t_f} [\widehat{X}^{\ T} \widehat{\Phi}_n(t) \tilde{F}(t) \widehat{\Phi}_n(t) \widehat{\Phi}_n(t)^{\ T} \widehat{X} + \widehat{U}^{\ T} \widehat{\Phi}_k(t) \tilde{R}(t) \widehat{\Phi}_k(t) \widehat{\Phi}_k(t)^{\ T} \widehat{U}] dt, \tag{26}$$

by using (9) we have:

$$\tilde{F}(t) \widehat{\Phi}_n(t)\ \widehat{\Phi}_n(t)^{\ T} \approx \widehat{\Phi}_n(t)^{\ T} \widehat{F}_n, \tag{27}$$

$$\tilde{R}(t) \widehat{\Phi}_k(t)\ \widehat{\Phi}_k(t)^{\ T} \approx \widehat{\Phi}_k(t)^{\ T} \widehat{R}_k, \tag{28}$$

which $\widehat{F}_n$ and $\widehat{R}_k$ are operational matrices of product. By replacing (27) and (28) in (26), and using (8) we get:

$$J \approx \frac{1}{2} \widehat{X}^{\ T} \widehat{\Phi}_n(t_f)\ S\ \widehat{\Phi}_n(t_f)^{\ T} \widehat{X} + \frac{1}{2} [\widehat{X}^{\ T} \widehat{Q}_n \widehat{F}_n \widehat{X} + \widehat{U}^{\ T} \widehat{Q}_k \widehat{R}_k \widehat{U}]. \tag{29}$$

6. 3. Solution of the Optimization Problem From (23) and (29) in sections (6.1) and (6.2), respectively, the main problem is reduced to:

Min
$\frac{1}{2} \widehat{X}^{\ T} \widehat{\Phi}(t_f) S\, \widehat{\Phi}(t_f)^{\ T} \widehat{X} + \frac{1}{2} [\widehat{X}^{\ T} \widehat{Q}_n \widehat{F}_n \widehat{X} + \widehat{U}^{\ T} \widehat{Q}_k \widehat{R}_k \widehat{U}]$

Subjected to:
$\widehat{A}_n \widehat{X} + \widehat{B}_n^* \widehat{U} - \widehat{E}_n C = 0.$
Using the Lagrange multipliers method, Lagrangian equation for this problem is:
$J^* = \frac{1}{2} \widehat{X}^{\ T} \widehat{\Phi}_n(t_f)\ S\ \widehat{\Phi}_n(t_f)^{\ T} \widehat{X} + \frac{1}{2} [\widehat{X}^{\ T} \widehat{Q}_n \widehat{F}_n \widehat{X} + \widehat{U}^{\ T} \widehat{Q}_k \widehat{R}_k \widehat{U}] + \lambda^T [\widehat{A}_n \widehat{X} + \widehat{B}_n^* \widehat{U} - \widehat{E}_n C]$,
therefore, unknown coefficients cab be calculated by solving the following system of algebraic equations:

$$\frac{\partial J^*}{\partial C} = 0, \qquad \frac{\partial J^*}{\partial \lambda} = 0, \qquad \frac{\partial J^*}{\partial \widehat{U}} = 0.$$

Unknown coefficients can be found by simultaneously solving the above system of algebraic equations (e.g. using $Mathematica^{TM}$).

7. ILLUSTRATIVE EXAMPLES

In order to show the performance of the presented method in this paper we applied it to solve some examples. The following case studies are given to show the merit of the proposed method. This method differs from other methods presented in [9, 10, 23, 35] and thus could be used as a basis for comparison.

Example 1. Consider the optimal control of time-varying singular system presented in [10]:

$$\begin{bmatrix} 1 & 0 \\ -t & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1+t & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$0 \le t \le 2$,

$$x(0) = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix},$$

with the optimal control minimizing the performance index

$$J = \frac{1}{2} \int_0^2 (x^T x + u^2) dt,$$

whose exact solutions under the above-mentioned constraints is:
$x_1(t) = e^{-t}, \qquad x_2(t) = \frac{e^{-t}}{2}, \qquad u(t) = -\frac{e^{-t}}{2}.$

Here, we solve the problem using Bernstein polynomials of degree $m = 5$, 10 and present the absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ on some points in Table 1 and Table 2, respectively. In Figures 1 and 2 images of exact and estimated solutions of $x_1(t)$, $x_2(t)$ and $u(t)$ for $m = 10$ are ploted, respectively.
The exact value of optimal control and estimated values of optimal control by $m = 5$, 10 are equal to $J_{exact} = 0.36813163541672467$ and $J_{est.} = 0.3681316353957$ and $J_{est.} = 0.368131635789$, respectively. This example shows that absolute error decreases rapidly when degree of Bernstein polynomials are doubled.
Note that, the given exact solutions in [10]

$$x_1(t) = e^{-t}, \qquad x_2(t) = e^{-t} + \sin t, \qquad u(t) = \sin t$$

with initial points $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is failed to meet the optimality requirement because the value of optimum control is $J_{exact} = 2.14667246697579$ by the presented exact solutions in [30] whereas there are some solutions which satisfy on given constraints with lower optimal control value, for example:

$$x_1(t) = e^{-t}, \qquad x_2(t) = e^{-t} + \frac{t}{2}, \qquad u(t) = \frac{t}{2}$$

provide the optimum control value $J_{est.} = 1.4545059223673806$. Likewise

$$x_1(t) = e^{-t}, \qquad x_2(t) = e^{-t} + \frac{t}{100}, \qquad u(t) = \frac{t}{100}$$

provide the optimum control value $J_{est.} = 0.49704878872520114$ or

$$x_1(t) = e^{-t}, \qquad x_2(t) = e^{-t} + \frac{\sin t}{900}, \qquad u(t) = \frac{\sin t}{900}$$

the optimum control value $J_{est.} = 0.4913621261074325$ and etc. In fact, using Euler-Lagrange equation it can be demonstrated that this problem does not have an optimum solution with initial points $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so we have to change the initial points to $x(0) = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ and then solve it analytically using Euler-Lagrange equation to obtain exact solutions.

TABLE 1. The absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ with m=5 for example 1.

T	$x_1(t)$	$x_2(t)$	$u(t)$
0	0.0000370274	0.0000185137	0.0000185137
0.25	9.25254×10^{-6}	4.62627×10^{-6}	4.62627×10^{-6}
0.5	0.000012124	6.062×10^{-6}	6.062×10^{-6}
0.75	8.2268×10^{-7}	4.11341×10^{-7}	4.1134×10^{-7}
1	0.0000112909	5.64543×10^{-6}	5.64543×10^{-6}
1.25	2.6185×10^{-6}	1.30925×10^{-6}	1.30925×10^{-6}
1.5	0.0000113565	5.67823×10^{-6}	5.67823×10^{-6}
1.75	0.0000113912	5.69559×10^{-6}	5.69559×10^{-6}
2	0.0000370272	0.0000185136	0.0000185136

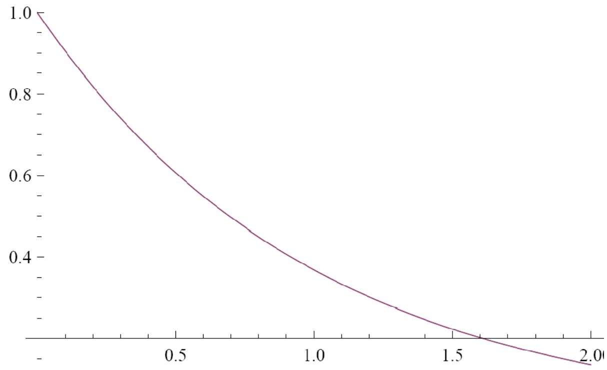

Figure 1. Exact and estimated solutions for $x_1(t)$ with $m = 10$ in example 1.

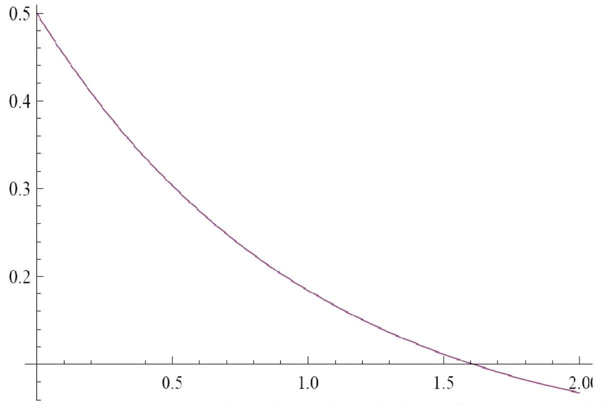

Figure 2. Exact and estimated solutions for $x_2(t)$ with $m = 10$ in example 1.

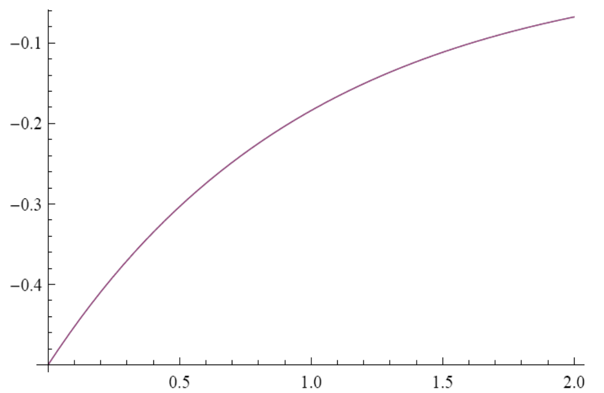

Figure 3. Exact and estimated solutions for $u(t)$ with $m = 10$ in example 1.

Example 2. Consider the optimal control of unstable time-varying singular system

$$\begin{bmatrix} 2 & 0 \\ t & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -t & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t),$$

$0 \le t \le 2$,
where $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

and the performance index

$J=\frac{1}{2}\int_0^{t_f}(x^Tx+u^2)dt.$

The objective is to determine an optimal control u(t) that will drive from an admissible initial state $x(0-)=x_0$ to some desired final state x_f in a given time t_f and minimize the above cost function (performance index). The exact solution for $t_f=2$ is:

$$x_1(t)=e^{2t}, \qquad x_2(t)=\frac{3}{5}\,t\,e^{2t}, \qquad u(t)=\frac{6}{5}\,t\,e^{2t}.$$

The absolute error of $x_1(t)$, $x_2(t)$ and corresponding optimal control $u(t)$ are calculated in some points using Bernstein polynomials of degree $m=5,\ 10$ and the results are presented in Table 3 and Table 4, respectively. In Figures 3, 4 and 5 images of exact and estimated solutions of $x_1(t)$, $x_2(t)$ and $u(t)$ for $m=10$ are ploted, respectively. The exact value of optimal control and estimated values of optimal control by $m=5,\ 10$ are equal $J_{exact}=2468.4527080$ and $J_{est.}=2468.36, J_{est.}=2468.452638$, respectively. From numerical results, it can be found that the method provides high efficiency and uniformly converges to the exact solution and gives accurate solutions especially for unstable singular systems. Also, this example shows that absolute error decreases rapidly when degree of Bernstein polynomials are doubled.

Example 3. Consider the optimal control of singular system presented in [9, 23, 35]:

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u(t),$$

$0\le t\le 2,$

$$x(0)=\begin{bmatrix}1\\ -\frac{\sqrt{2}}{2}\end{bmatrix},$$

by minimizing the performance index

$J=\frac{1}{2}\int_0^{2}(x^Tx+u^2)dt,$

whose exact solutions under the above-mentioned constrains is:

$$x_1(t)=e^{-\sqrt{2}t}, \qquad x_2(t)=-\frac{\sqrt{2}}{2}\,e^{-\sqrt{2}t}, \qquad u(t)=\frac{\sqrt{2}}{2}e^{-\sqrt{2}t}.$$

We approximate $x_1(t)$, $x_2(t)$ and $u(t)$ by Bernstein polynomials of degree $m=3,\ 5,\ 7$ on interval $[0,2]$ and present the absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ for some points in Table 5, Table 6 and Table 7, respectively. The exact value of optimal control and estimated values of optimal control by $m=3,\ 5,\ 7$ are equal $J_{exa\square t}=0.35231825561$ $J_{est.}=0.351089$, $J_{est.}=0.351092$ and $J_{est.}=0.351092$, respectively. Numerical results demonstrate the feasibility of the method.

TABLE 2. The absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ with m=10 for example 1.

T	$x_1(t)$	$x_2(t)$	$u(t)$
0	2.74002×10^{-11}	1.15576×10^{-11}	1.49232×10^{-11}
0.25	2.20091×10^{-12}	9.27175×10^{-12}	1.10639×10^{-12}
0.5	2.23621×10^{-12}	2.67986×10^{-11}	7.24032×10^{-13}
0.75	2.03171×10^{-12}	3.66686×10^{-10}	8.26469×10^{-12}
1	5.57443×10^{-13}	1.37799×10^{-9}	2.19486×10^{-11}
1.25	9.66172×10^{-13}	2.55455×10^{-9}	2.9886×10^{-11}
1.5	1.23643×10^{-12}	3.33117×10^{-9}	2.32852×10^{-11}
1.75	3.0749×10^{-12}	3.5851×10^{-9}	6.61682×10^{-12}
2	2.74002×10^{-11}	3.59851×10^{-9}	1.371×10^{-11}

TABLE 3. The absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ of example 2 for $m=5$.

T	$x_1(t)$	$x_2(t)$	$u(t)$
0	0.0543755	0.097876	0.195752
0.25	0.0177606	0.0301454	0.0602908
0.5	0.0152232	0.0362046	0.0724092
0.75	0.00607243	0.0051087	0.0102174
1	0.0153619	0.0398034	0.0796068
1.25	0.00322383	0.0000417667	0.0000835332
1.5	0.0177799	0.0476396	0.0952791
1.75	0.0113037	0.0393502	0.0787004
2	0.0552443	0.164169	0.328338

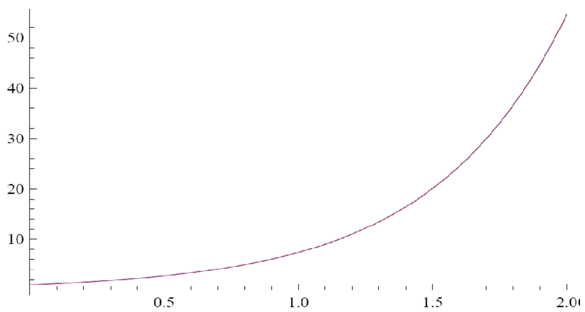

Figure 4. Exact and estimated solutions for $x_1(t)$ with $m=10$ in example 2.

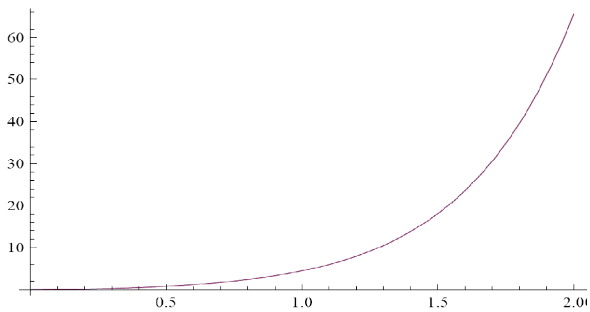

Figure 5. Exact and estimated solutions for $x_2(t)$ with $m=10$ in example 2.

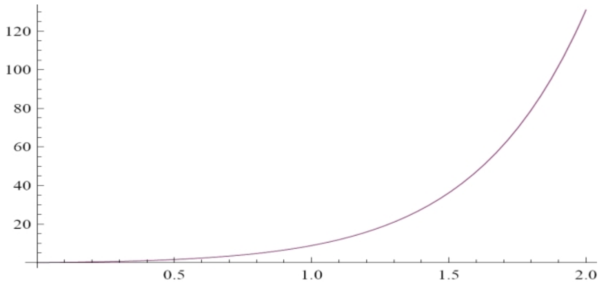

Figure 6. Exact and estimated solutions for $u(t)$ with $m = 10$ in example 2.

TABLE 4. The absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ of example 2 for $m = 10$.

T	$x_1(t)$	$x_2(t)$	$u(t)$
0	1.21003×10^{-6}	3.99602×10^{-6}	7.9868×10^{-6}
0.25	1.54127×10^{-7}	1.93468×10^{-7}	8.00346×10^{-7}
0.5	3.1485×10^{-8}	8.06245×10^{-7}	4.75277×10^{-7}
0.75	1.85167×10^{-8}	2.71612×10^{-6}	5.2744×10^{-7}
1	4.81083×10^{-8}	3.63834×10^{-6}	9.99076×10^{-8}
1.25	1.10634×10^{-7}	3.27331×10^{-6}	6.04075×10^{-7}
1.5	1.18365×10^{-7}	3.36985×10^{-6}	7.09836×10^{-7}
1.75	7.77893×10^{-8}	3.20602×10^{-6}	9.26404×10^{-7}
2	1.21003×10^{-6}	8.01143×10^{-6}	0.00001089

TABLE 5. The absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ of example 3 for $m = 3$.

T	$x_1(t)$	$x_2(t)$	$u(t)$
0	0.0169508	0.00794424	0.00794424
0.25	0.00350157	0.00478137	0.00478137
0.5	0.00689598	0.00542015	0.00542015
0.75	0.0137541	0.0044623	0.0044623
1	0.0185624	0.0068358	0.0068358
1.25	0.0213305	0.0135629	0.0135629
1.5	0.0259471	0.0229231	0.0229231
1.75	0.0390242	0.0312704	0.0312704
2	0.0690859	0.0336063	0.0336063

TABLE 6. The absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ of example 3 for $m = 5$.

T	$x_1(t)$	$x_2(t)$	$u(t)$
0	0.00715461	0.000152787	0.000152787
0.25	0.00735793	0.00181398	0.00181398
0.5	0.00883999	0.00372923	0.00372923
0.75	0.0112511	0.00626271	0.00626271
1	0.0151088	0.00957249	0.00957249
1.25	0.0210056	0.0139872	0.0139872
1.5	0.0295156	0.020195	0.020195
1.75	0.0415902	0.0290857	0.0290857
2	0.0592977	0.0413503	0.0413503

TABLE 7. The absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ of example 3 for $m = 7$.

T	$x_1(t)$	$x_2(t)$	$u(t)$
0	0.00696458	1.53036×10^{-6}	1.53036×10^{-6}
0.25	0.00740281	0.00177682	0.00177682
0.5	0.00877684	0.00377894	0.00377894
0.75	0.0112602	0.00625787	0.00625787
1	0.0151665	0.00952659	0.00952659
1.25	0.0209867	0.0139999	0.0139999
1.5	0.029459	0.0202406	0.0202406
1.75	0.0416522	0.0290377	0.0290377
2	0.0591077	0.0415015	0.0415015

8. CONCLUSION

In the present work, a technique has been developed for obtaining an optimal control of time-varying singular systems with a quadratic cost function using Bernstein polynomials. The operational matrices of integration, differentiation and product of Bernstein polynomials basis are introduced and are utilized to reduce the optimal control of time-varying singular system to the solution of algebraic equations. The proposed method is general, easy to implement, and yields accurate results. Absolute error reduced quickly when degree of Bernstein polynomials is increased, but when $m \geq 15$, volume of computations increases and obtained algebraic equation set is difficultly solved which is one of the limitation of this method. Simulation results give a satisfactory solution and demonstrate good performance of the proposed methods for solving optimal control of time-varying singular systems. Numerical tests also show that the method converges to the exact solution and can be used for unstable singular systems.

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Numerical Solution of Optimal Control of Time-varying Singular Systems via Operational Matrices

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چکیده

در این مقاله، یک روش عددی برای حل مسئله ی کنترل بهینه ی سیستم های تکین وابسته به زمان با شاخص عملکرد درجه ی دوم ارائه شده است. این روش بر اساس چندجمله ای های برنشتاین پایه ریزی شده است. ماتریس های عملیاتی انتگرال، مشتق و ضرب معرفی شده اند و از آنها برای حل مسایل مذکور به صورت حل دستگاه معادلات جبری بکار گرفته شده است. قدرت این روش در ارائه ی یک تکنیک عددی بوسیله ی ماتریس های عملیاتی برای حل مسئله ی کنترل بهینه با کمک یک دستگاه معادلات می باشد. این روش به سرعت به جواب دقیق همگراست ونتایج بسیار دقیقی حتی برای مقادیر کم m ارائه می دهد. مثال های گویایی برای اثبات عملکرد و تاثیرگذاری این تکنیک و همگرایی این روش به جواب های دقیق بخصوص برای سیستم های تکین ناپایدار بیان شده است.

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