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High Accuracy Relative Motion of Spacecraft Using Linearized Time-Varying J_2-Perturbed Terms

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A B S T R A C T

This paper presents a set of linearized equations which was derived for the motion, relative to an elliptical reference orbit, of an object influenced by J_2 perturbation terms. Approximate solution for simulations was used to compare these equations and the linearized keplerian equations to the exact equations. The inclusion of the linearized perturbations in the derived equations increased the high accuracy of the solution significantly in the out of orbit plane direction, while the accuracy within the orbit plane remained roughly unchanged. In fact, it will be determined whether the inclusion of this disturbance provides a significant increase in accuracy over Melton's problem. Becuase of replacing approximate terms (e, M) in this solution, for continues accuracy increase of time-varying parameters containing $\theta(t)$ and $R_O(t)$, this solution could be useful in the element-errors evaluation and analysis of orbital multiple rendezvous missions, that are involved to the short-period terms.

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NOMENCLATURE

a	Semi-major diameter (km)	n	Mean angular rate (rad/s)
e	Eccentricity of reference orbit	ω	Argument of perigee (rad/s)
f	True anomaly of reference orbit (rad)	θ	Argument of latitude at epoch (deg)
h	Specific angular momentum (m^2/s)	ECI	Earth Centered Inertial
i	Inclination angle (deg)	$R_\oplus$	Mean radius of the Earth (km)
M	Mean anomaly		

1. INTRODUCTION

The study of the dynamics of relative motion between spacecraft customarily begins with the Clohessy-Wiltshire equations [1]. They are a set of linearized equations for the motion, relative to a circular reference orbit, of an object in an inverse square gravity field. Tschauner and Hempel [2] first solved the problem for motion relative to an elliptical orbit, but they had to regularize the problem, resulting in a solution that does not explicitly include time. Other approaches by Lancaster [3] and Berreen and also Sved [4] include the time dependence, but they are limited to coplanar motion. Abrahamson and Stern [5] present a somewhat more general method that treats the problem for either elliptical or hyperbolic orbits. However, the solution is in terms of the eccentric anomalies, so that one must still solve the Kepler problem to obtain the explicit time dependence for each particular application. More recent efforts provide representations of the three-dimensional relative motion, but all of these motions are functions of either the true or eccentric anomaly of one satellite [6]. Solution of the Kepler problem is conceptually and numerically straight forward, but in instances where an onboard calculation of relative motion is required for estimation purposes, a direct, non-iterative algorithm would be preferred. In this regard, Melton [7] provides a method for generalizing the linear equations of motion

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to an elliptical orbit which enables the determination of a closed-form, time-explicit, approximate solution. Ross [8] gives a set of equations based on the C-W equations which incorporates the J_2 gravitational perturbations. He states in his paper introduction: "In principle, these equations can be developed for elliptical reference orbits as described by Melton" [8].

The objective of this paper is to develop a set of linearized expressions for the motion, relative to an elliptical reference orbit, of an object influenced by time-varying approximate perturbation terms. they will now be developed for the time-varying quantities containing $\theta(t)$ and $R_O(t)$ that had been referred by Melton [7]. The approximate solutions from numerical simulations have been used to compare basic equations given by Melton [7] and exact equations of motion. It will be determined whether the inclusion of the linear perturbations provides a high accuracy relative motion dynamics between two Spacecraft.

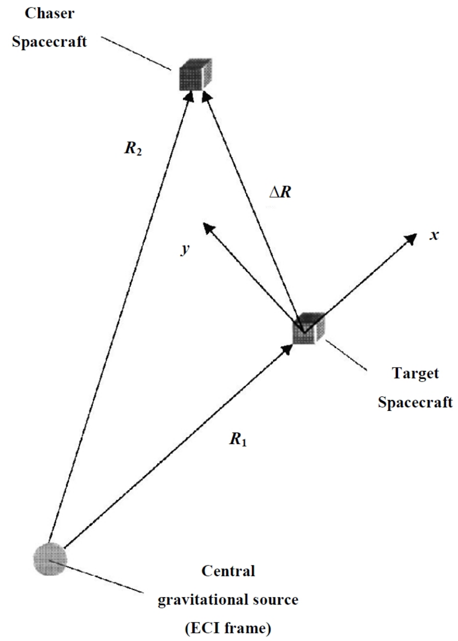


Figure 1. Position of chaser S/C relative to target S/C

2. BASIC EQUATIONS OF MOTION

The following subsections will review the linearized equations for the motion, relative to an elliptical reference orbit, to compare these equations and the equations given by Melton to the exact equations of motion.

2. 1. Nonlinearity of the Exact Equations The exact equations describing the motion of an object in an inverse square gravitational field follow directly from Newton's second law and his law of gravitation. It can be written in vector form as:

$$\ddot{\mathrm{R}} = \frac{\mu}{R^3}\mathrm{R} \tag{1}$$

In this equation, R is the position vector of the object relative to the inertially fixed of the gravitational field and μ is the gravitational coefficient. Equation (1) can be written in component form as:

$$\ddot{\mathrm{X}} = \mu\frac{X}{R^3}, \ddot{\mathrm{Y}} = \mu\frac{Y}{R^3}, \ddot{Z} = \mu\frac{Z}{R^3} \tag{2}$$

That R parameter is equal to $(X^2+Y^2+Z^2)^{1/2}$.

The relative motion between two spacecraft in orbit is determined simply by $\Delta\mathrm{R} = \mathrm{R}_2 - \mathrm{R}_1$, that is shown in Figure 1. Although the above equations may appear straightforward, they are nonlinear. Thus, they can be difficult to analyze. Linearized equations provide approximations to nonlinear equations. Their reduced accuracy is accepted as a trade-off for their relative mathematical simplicity. As an alternative to Equation (1), the motion of an object in orbit can be described relative to a reference orbit, and the equations for this relative motion can be linearized.

The linearization is made possible by the assumption that the distance of the object from the reference orbit is very small compared to the size of the reference orbit itself. The gravitational field about a perfectly uniform, perfectly spherical mass has an inverse square relation. However, the earth is neither perfectly uniform nor perfectly spherical, and the inverse square relation describes closely, but not exactly, the earth's actual gravitational field. This perturbation force is referred to as the J_2 perturbation acceleration and is given by Kaplan [9] as follow:

$$P = \frac{3J_2\mu R_\oplus^2}{2R^5}\left[\left(5\frac{Z^2}{R^2}-1\right)\left(X\hat{\mathrm{I}}+Y\hat{\mathrm{J}}\right)+Z\left(5\frac{Z^2}{R^2}-3\right)\hat{\mathrm{K}}\right] \tag{3}$$

This expression gives the J_2 perturbation acceleration in the Earth Centered Inertial (ECI) coordinate directions in terms of the ECI coordinates. Although this perturbation force is small, its effect can cause the motion of an orbiting object to depart significantly from the pure inverse square motion, or Keplerian motion. Incorporating this perturbation into the equations provides a more accurate description of the actual motion of an orbiting object [1].

2. 2. Linearized Keplerian Equations The analysis of the linearized equations of motion for a spacecraft subject to a pure inverse square gravitational field will be presented in this subsection. Although this derivation of equations was well known, its results are considered. Nonlinear equations for this case are as follows [10]:

$$\begin{cases} \ddot{x} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2 x - \dfrac{\mu}{R_O^2} = -\dfrac{\mu}{R^3}(R_O + x) \\ \ddot{y} + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2 y = -\dfrac{\mu}{R^3} y \\ \ddot{z} = -\dfrac{\mu}{R^3} z \end{cases} \tag{4}$$

Next, it will develop approximate linearized expressions for the terms on the right side of Equation (4). The position vector given by has the magnitude:

$$R = \left[(R_O + x)^2 + y^2 + z^2\right]^{1/2} \tag{5}$$

If it assumed that the distance of the object from the reference orbit is small compared to the reference orbit radial distance, $(x^2 + y^2 + z^2) << R_O^2$, then, linearized case is obtained as follows:

$$\begin{cases} \ddot{x} - \dfrac{2\mu}{R_O^3} - 2\dot{f}\dot{y} - \ddot{f}y - \dot{f}^2 x = 0 \\ \ddot{y} + \dfrac{\mu}{R_O^3} y + 2\dot{f}\dot{x} + \ddot{f}x - \dot{f}^2 y = 0 \\ \ddot{z} + \dfrac{\mu}{R_O^3} z = 0 \end{cases} \tag{6}$$

Equations (6) give the linearized equations of motion with respect to an elliptical reference orbit. These equations contain time-varying, periodic coefficients $\dot{f}$, $\ddot{f}$ and R_O.

2. 3. Ross's Equations to Circular Orbit

Ross [8] derived a set of linearized equations of motion for a spacecraft in a nearly circular orbit with the effects of J_2 perturbations included. The equations were derived by adding the effects as perturbation terms to the C-W equations. The primary work involved deriving, for the x, y and z directions, linearized expressions for the J_2 perturbation acceleration, in terms of x, y and z. Ross simply the derivation of this perturbation acceleration and gives the result [8]:

$$\begin{aligned} P_x \cong -\frac{3J_2 n^2 R_\otimes^2}{2R_O}\Bigg[&\left(1-\frac{7x}{R_O}\right)\left(1-3\sin^2 nt \sin^2 i\right)+\frac{x}{R_O}\left(9\cos^2 nt\right. \\ &\left.+9\cos^2 i \sin^2 nt - 6\right)+\frac{y}{R_O}\left(8\sin nt \cos nt \cos^2 i\right. \\ &\left.-8\sin nt \cos nt\right)-\frac{z}{R_O}\left(8\sin nt \sin i \cos i\right)\Bigg] \end{aligned} \tag{7a}$$

$$\begin{aligned} P_y \cong -\frac{3J_2 n^2 R_\otimes^2}{2R_O}\Bigg[&\left(1-\frac{7x}{R_O}\right)\sin 2nt \sin^2 i \\ &+\frac{x}{R_O}(6\sin nt \cos nt \sin^2 i)+\frac{y}{R_O}(7\cos^2 nt + 5\cos^2 i \\ &-4-7\cos^2 nt \cos^2 i)+\frac{z}{R_O}(2\cos nt \sin i \cos i)\Bigg] \end{aligned} \tag{7b}$$

$$\begin{aligned} P_z \cong -\frac{3J_2 n^2 R_\otimes^2}{2R_O}\Bigg[&\left(1-\frac{7x}{R_O}\right)\sin nt \sin 2i \\ &+\frac{x}{R_O}(6\cos i \sin i \sin nt)+\frac{y}{R_O}(2\cos nt \sin i \cos i) \\ &-\frac{z}{R_O}(7\cos^2 i + 5\cos^2 nt - 4 - 5\cos^2 nt \cos^2 i)\Bigg] \end{aligned} \tag{7c}$$

In the above equations, i is the inclination of the circular orbit and t is the time since passage of the ascending node and n is mean angular rate of circular reference orbit [8].

2. 4. Melton's Equations to Elliptical Orbit

Melton provides a time-explicit solution for the relative motion between elliptical orbits. The equations presented by Melton which were the basis for the development of his solution are equivalent to equations as follows [7]:

$$\begin{cases} \ddot{x} = \dfrac{2\mu}{R_O^3(t)} + 2\dot{f}(t)\dot{y} + \ddot{f}(t)y + \dot{f}^2(t)x + P_x(t) \\ \ddot{y} = -\dfrac{\mu}{R_O^3(t)} y - 2\dot{f}(t)\dot{x} - \ddot{f}(t)x + \dot{f}^2(t)y + P_y(t) \\ \ddot{z} = -\dfrac{\mu}{R_O^3(t)} z + P_z(t) \end{cases} \tag{8}$$

The portions of Melton's work that are pertinent to this paper are the expressions that he provides as approximations for the time-varying coefficients in Equation (8).

$$\dot{f}(t) = \frac{h}{a^2}\left[1 + 2e\cos M + \frac{e^2}{2}(1 + 5\cos 2M) + O(e^3)\right] \tag{9a}$$

$$\ddot{f}(t) = -\frac{2h}{a^2}\left[en\sin M + e^2(n\sin 2M + 3n\cos M \sin M) + O(e^3)\right] \tag{9b}$$

$$\dot{f}^2(t) = \frac{h^2}{a^4}\left[1 + 4e\cos M + \frac{e^2}{2}(3 + 7\cos 2M) + O(e^3)\right] \tag{9c}$$

where, M is the mean anomaly and is given by $n(t - t_p)$ and h is specific angular momentum, a is semi-major diameter, f is true anomaly and e is eccentricity of reference orbit [11].

The equations were generated using Lagrange's generalized expansion theorem. They are truncated series expansions that are functions of e and M, and are approximate to the order of e^2. Melton [7] provides as the equation:

$$\frac{R_O}{a} = 1 - e\cos M + \frac{e^2}{2}(1 - \cos 2M) + O(e^3) \tag{10}$$

Presumably for the purpose of providing an approximate expression for the quantity $1/R_0^3(t)$:

$$\frac{1}{R_O{}^3} \cong \frac{1}{a^3\left[1 - e\cos M + \dfrac{e^2}{2}(1 - \cos 2M)\right]^3} \tag{11}$$

This expression would be suitable for use in a computational algorithm. However, it would not be suitable for the purposes of creating a combined algebraic expression, since it would produce a lengthy expression in the denominator [7].

3. DEVELOPMENT OF MELTON'S PROBLEM

The primary task of this paper, the analysis of the linearized equations of motion for a J_2-perturbed object relative to an elliptical reference orbit, will be presented in this section. This derivation follows the same general procedure as outlined in Ross's paper for the case of a circular reference orbit. Whereas the basis equations used by Ross [8] were the C-W equations, the basis equations for this analysis are as described Equations (8) to (11).

The first step in this analysis is to find, in terms of the orbit frame coordinate variables *x, y, z*, linearized expressions for zonal perturbation acceleration in the orbit frame coordinate directions. These expressions will be linear in Cartesian frame and will have constant or time-varying periodic coefficients. The expression for zonal perturbation acceleration in the ECI frame was used in Ross's paper as given Equation (3). This equation gives perturbation acceleration as a function of position in space, and is independent of the motion of the object under consideration. This can also be written:

$$\begin{bmatrix} P_X \\ P_Y \\ P_Z \end{bmatrix} = \frac{3J_2\mu R_\oplus^2}{2}\frac{1}{R^7}\begin{bmatrix} 5Z^2-R^2X \\ 5Z^2-R^2Y \\ 5Z^2-3R^2Z \end{bmatrix} \tag{12}$$

Therefore, it is to linearize the quantity $1/R^7$ in Equation (12). This can be accomplished as follows:

$$\begin{aligned} R &= \sqrt{(X^2+Y^2+Z^2)} = \sqrt{(R_O+x)^2+y^2+z^2)} \\ &= R_O(t)\sqrt{\frac{x^2+y^2+z^2}{R_O(t)^2}+1+\frac{2x}{R_O(t)}}) \end{aligned} \tag{13}$$

By the considering of the assumption that $(x^2+y^2+z^2) << R_O^2$, allows can be written:

$$\frac{1}{R^7} \cong \frac{1}{R_O(t)^7}\left(1+\frac{2x}{R_O(t)}\right)^{-7/2} \tag{14}$$

This expression can be further simplified by using the binomial series approximation and neglecting terms containing $[x/R_O(t)]^n, n \geq 2$. Therefore:

$$\frac{1}{R^7} \cong \frac{1}{R_O(t)^7}\left(1-\frac{7x}{R_O(t)}\right) \tag{15}$$

The quantities X, Y and Z in Equation (12) can be transformed by:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = {}^N C^O \begin{bmatrix} x+R_O(t) \\ y \\ z \end{bmatrix} \tag{16}$$

That the direction cosine matrix $({}^N C^O = ({}^O C^N)^T)$ is defined by Ross [8]. When Equations (13) to (16) are substituted into Equation (12) and algebraically expanded, each of the resulting terms has an nth order product of *x*, *y*, and *z* in the numerator and a quantity $R_O{}^{4+n}$ in the denominator, where, for a particular term, *n* is equal 0, 1, 2, 3 or 4. Thus, the terms all have the form:

$$C\frac{1}{R_O^4}\frac{x^b y^c z^d}{R_O^n}, \quad b+c+d=n=0,1.,..,4 \tag{17}$$

where, *b*, *c* and *d* are zero or positive integers and *C* is some constant expression. The employing once again the assumption that the distance of the object from the reference orbit is small compared to the size of the reference orbit, can be written:

$$\frac{x^b y^c z^d}{R_O^n} \approx 0, \quad b+c+d=n=2,3,4 \tag{18}$$

Thus, the perturbation acceleration expressions can be approximated by retaining only the terms for which *n=0*, 1. These approximate expressions are given below by Equations (19). Hence, the symbolic manipulator, Maple was used for many of the steps to simplify the lengthy algebraic expressions. The Results are shown in Table 1. Equations (19) are the linearized expressions for the perturbation acceleration in the ECI coordinate directions in terms of the orbit frame coordinates *x*, *y*, and *z*. These equations can be transformed to give the perturbation accelerations in the orbit frame coordinate with the equation:

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = {}^O C^N \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{20}$$

Substituting Equations (19) into Equations (20) and simplifying gives (see Table 2). Expressions will now be developed for the time-varying quantities containing *θ(t)* and *R$_O$(t)*. These expressions will be in the form of approximate truncated series which are functions of *e* and *M*. Solutions for time-varying quantities related to the motion of an elliptical orbit can be expressed as Fourier-Bessel series expansions. Taff [12] gives as the following:

$$\cos f = -e + 2\frac{1-e^2}{e}\sum_{n=1}^{\infty} J_n(ne)\cos(nM) \tag{22}$$

$$\sin f = \sqrt{1-e^2}\sum_{n=1}^{\infty}\frac{2}{n}J_n'(ne)\sin(nM) \tag{23}$$

$$\frac{a}{r} = 1 + 2\sum_{n=1}^{\infty} J_n(ne)\cos(nM) \tag{24}$$

In the above equations, J_n are the Bessel functions of the first kind of order *n*, and $J_n'(ne) = \partial J_n(ne)/\partial e$. These Bessel functions can be expressed as series expansions, as given in Battin [13], by following expression.

TABLE 1. Linearized Perturbation Expressions in the ECI Coordinate

$$
\begin{aligned}
P_1 &= x\,\frac{3J_2\mu R_\otimes^2}{2R_O^5}\left(-1+5\cos 2i+10\cos 2\theta\sin^2 i\right)(\cos\theta\cos\Omega-\cos i\sin\theta\sin\Omega)+\\
&y\,\frac{3J_2\mu R_\otimes^2}{32R_O^5}\left(2\cos\Omega((3+5\cos 2i)\sin\theta+30\sin^2 i\sin 3\theta)+(15\cos 3i\cos\theta+\cos i(\cos\theta+60\cos 3\theta\sin^2 i))\sin\Omega\right.\\
&+z\,\frac{3J_2\mu R_\otimes^2}{16R_O^5}\left(20\cos\Omega\sin 2i\sin 2\theta-((3+5\cos 2\theta)\sin i+30\sin 3i\sin^2\theta)\sin\Omega\right)-\\
&\frac{3J_2\mu R_\otimes^2}{8R_O^4}\left(-1+5\cos 2i+10\cos 2\theta\sin^2 i\right)(\cos\theta\cos\Omega-\cos i\sin\theta\sin\Omega)
\end{aligned}
\tag{19a}
$$

$$
\begin{aligned}
P_2 &= x\,\frac{3J_2\mu R_\otimes^2}{2R_O^5}\left(-1+5\cos(2i)+10\cos(2\theta)\sin^2 i\right)(\cos i\cos\Omega\sin\theta+\cos\theta\sin\Omega)\\
&-y\,\frac{3J_2\mu R_\otimes^2}{8R_O^5}\left(\cos i\cos\theta\cos\Omega(-11+15\cos 2i+30\sin^2 i\cos 2\theta)+(-9+5\cos 2i-30\cos 2\theta\sin^2 i\right)\sin\theta\sin\Omega\\
&+z\,\frac{3J_2\mu R_\otimes^2}{16R_O^5}\sin i\left(2\cos\Omega(9-5\cos 2\theta+30\cos 2i\sin^2\theta)+80\cos i\cos\theta\sin\theta\sin\Omega\right)\\
&-\frac{3J_2\mu R_\otimes^2}{8R_O^4}\left(-1+5\cos 2i+10\cos 2\theta\sin^2 i\right)(\cos i\cos\Omega\sin\theta+\cos\theta\sin\Omega)
\end{aligned}
\tag{19b}
$$

$$
\begin{aligned}
P_3 &= x\,\frac{3J_2\mu R_\otimes^2}{8R_O^5}\left(3(\sin i+5\sin 3i)\sin\theta+20\sin^3 i\sin 3\theta\right)-y\,\frac{9J_2\mu R_\otimes^2}{32R_O^5}\left(15\cos 3\theta\sin i+\cos\theta(\sin i+20\sin 3i\sin^2\theta)\right)\\
&-z\,\frac{9J_2\mu R_\otimes^2}{16R_O^5}\left(5\cos 3i+\cos i(3+20\cos 2\theta\sin^2 i)\right)-\frac{3J_2\mu R_\otimes^2}{32R_O^4}\left(3(\sin i+5\sin 3i)\sin\theta+20\sin^3 i\sin 3\theta\right)
\end{aligned}
\tag{19c}
$$

TABLE 2. Linearized Perturbation Terms in the Orbit Coordinate

$$
\begin{aligned}
P_x &\cong J_2\mu R_\otimes^2\left\{\left[\frac{3}{2(R_O(t))^5}(1+3\cos 2i+6\cos 2\theta(t)\sin^2 i)\right]x+\left[\frac{6}{(R_O(t))^5}(\sin^2 i\sin 2\theta(t))\right]y\right.\\
&\left.+\left[\frac{6}{(R_O(t))^5}(\sin 2i\sin\theta(t))\right]z-\frac{3}{8(R_O(t))^4}(1+3\cos 2i+6\cos 2\theta(t)\sin^2 i)\right\}
\end{aligned}
\tag{21a}
$$

$$
\begin{aligned}
P_y &\cong J_2\mu R_\otimes^2\left\{\left[\frac{6}{(R_O(t))^5}(\sin^2 i\sin 2\theta(t))\right]x-\left[\frac{3}{8(R_O(t))^5}(1+3\cos 2i+14\cos 2\theta(t)\sin^2 i)\right]y\right.\\
&\left.-\left[\frac{3}{(R_O(t))^5}(\cos i\cos\theta(t)\sin i)\right]z-\frac{3}{2(R_O(t))^4}(\sin 2\theta(t)\sin^2 i)\right\}
\end{aligned}
\tag{21b}
$$

$$
\begin{aligned}
P_z &\cong J_2\mu R_\otimes^2\left\{\left[\frac{6}{(R_O(t))^5}(\sin 2i\sin\theta(t))\right]x-\left[\frac{3}{(R_O(t))^5}(\cos i\cos\theta(t)\sin i)\right]y\right.\\
&\left.-\left[\frac{3}{8(R_O(t))^5}(3+9\cos 2i+10\cos 2\theta(t)\sin^2 i)\right]z-\frac{3}{(R_O(t))^4}(\cos i\sin i\sin\theta(t))\right\}
\end{aligned}
\tag{21c}
$$

$$
J_n(ne)=\sum_{j=0}^{\infty}(-1)^j\,\frac{(0.5ne)^{n+2j}}{j!(n+j)!}
\tag{25}
$$

Note should be taken not to confuse J_2 (ne) with the constant J_2 associated with the gravitational field. Substituting Equation (25) into Equation (22) to (24) and carrying out the expansions for each gives:

$$
\begin{aligned}
\cos f(t) &= -e+\left(1-\frac{9}{8}e^2\right)\cos(M)+e\cos(2M)\\
&+\frac{9}{8}e^2\cos(3M)+O(e^3)
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\sin f(t) &= \frac{b}{a}\left[\left(1-\frac{3}{8}e^2\right)\sin(M)+e\sin(2M)\right.\\
&\left.+\frac{9}{8}e^2\sin(3M)+O(e^3)\right)
\end{aligned}
\tag{27}
$$

$$\frac{a}{r(t)} = 1 + e\cos(M) + e^2\cos(2M) + O(e^3) \tag{28}$$

With these expressions, it could develop the expressions which we require for the time-varying quantities in Equation (21). In this equation (developed Ross's Equations), $\theta(t)$ is the argument of latitude at epoch that has been replaced for nt in the Ross's results:

$$\theta(t) = \omega + f(t) \tag{29}$$

where, ω is the argument of perigee and $f(t)$ is the true anomaly. The trigonometric angle-addition formulae could be to write:

$$\cos\theta(t) = \cos(\omega + f(t)) = \cos\omega\cos f(t) - \sin\omega\sin f(t) \tag{30}$$

$$\sin\theta(t) = \sin(\omega + f(t)) = \sin\omega\cos f(t) + \cos\omega\sin f(t) \tag{31}$$

By substituting Equations (26) and (27) into Equations (30) and (31), results are obtained as follows:

$$\begin{aligned}\cos\theta(t) \cong & -e\cos\omega + \left(1 - \frac{9}{8}e^2\right)\cos\omega\cos M + e\cos\omega\cos 2M \\ & + \frac{9}{8}e^2\cos\omega\cos 3M + \frac{b}{a}\left(\frac{3}{8}e^2 - 1\right)\sin\omega\sin M \\ & - \frac{b}{a}e\sin\omega\sin 2M - \frac{9b}{8a}e^2\sin\omega\sin 3M\end{aligned} \tag{32}$$

$$\begin{aligned}\sin\theta(t) \cong & -e\sin\omega + \left(1 - \frac{9}{8}e^2\right)\sin\omega\cos M + e\sin\omega\cos 2M \\ & + \frac{9}{8}e^2\sin\omega\cos 3M + \frac{b}{a}\left(1 - \frac{3}{8}e^2\right)\cos\omega\sin M \\ & + \frac{b}{a}e\cos\omega\sin 2M + \frac{9b}{8a}e^2\cos\omega\sin 3M\end{aligned} \tag{33}$$

So, the trigonometric double-angle law could be written:

$$\sin 2\theta(t) = 2\sin\theta(t)\cos\theta(t) \tag{34}$$

$$\cos 2\theta(t) = 1 - 2\sin^2\theta(t) \tag{35}$$

As by Substituting Equations (32) and (33) into Equations (34) and (35), results are considered. For $R_O(t)$, Equation (28) could be written with approximate expression, as follows:

$$\frac{1}{R_O^3(t)} \cong \frac{1}{a^3}(1 + e\cos M + e^2\cos 2M)^3 \tag{36}$$

Algebraically expanding for Equation (36), it's resulted as:

$$\frac{1}{R_O^3(t)} \cong \frac{1}{a^3}\left(1 + \frac{3}{2}e^2 + 3e\cos M + \frac{9}{2}e^2\cos 2M\right) \tag{37}$$

Similarly, could be to derive the expressions:

$$\frac{1}{R_O^4(t)} \cong \frac{1}{a^4}\left(1 + 3e^2 + 4e\cos M + 7e^2\cos 2M\right) \tag{38}$$

$$\frac{1}{R_O^5(t)} \cong \frac{1}{a^5}\left(1 + 5e^2 + 5e\cos M + 10e^2\cos 2M\right) \tag{39}$$

Therefore, by substituting the all of above approximate terms, independent of time, in the Equation (21) or same developed Ross's equations, can be the approximate solution for development of Melton's problem about linearization of equations for the motion, relative to an elliptical reference orbit under linearized J_2 perturbation terms. In this development, obtained expansions are same shown that they will be functions of parameter e and M.

4. SIMULATION RESULTS AND DISCUSSION

For the errors evaluation and analysis, first it is required to define the initial conditions for presented problem. simulations are obtained and valided on the approximate solutions using initial conditions of a relative motion mission (see Table 3).

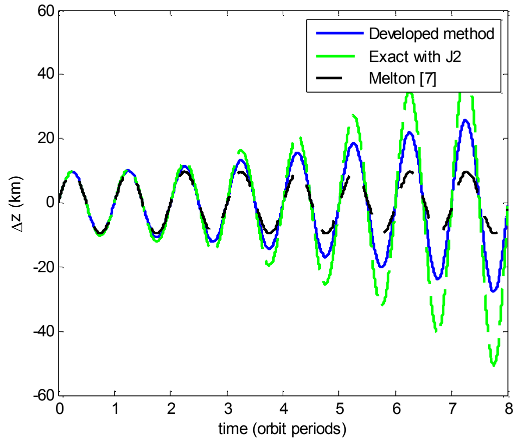
The plots for *x*, *y*, *z*, relative positions vs. time for 8 orbit periods are shown in Figures 2 and 3. In *x*-position, the exact keplerian and the exact perturbed curves have diverged noticeably by the end of the orbit period. The linear equations on the other hand nearly overlap each other over the entire period. Thus, high accuracy improvement can be discerned from this plot. For *y*-position, curves nearly overlap over most of the time range, until a divergence becomes apparent near the end of the orbit period. At this point, the two curves for the linearized equations continue to nearly overlap each other, as they diverge from the curves for the exact equations, which also continue to nearly overlap each other. This would seem to suggest that, at least in this position, the magnitude of perturbation is small relative to the error that exists between the exact and linearized keplerian equations. For *z*-position, the linear perturbed and exact perturbed results show very close correspondence over the entire orbit period.

TABLE 3. Initial Conditions Relative to Reference Orbit

Orbit Parameter	Value
Earth's oblateness (J_2)	1082.63×10^{-6}
Mean radius of the Earth ($R_\oplus$)	6.378136×10^3 km
Gravitational coefficient (μ)	3.986004×10^5 km^3/s^2
Semi-major diameter (a)	$R_\oplus + 1333.78$ km
Eccentricity (e)	0.1
Argument of perigee (ω)	90 deg
Ascending node (Ω)	116.55 deg
Inclination (i)	66.09 deg
Initial true anomaly (f_0)	0 deg
Relative position (Δx)	$[90\quad 90\quad 90]$m
Relative velocity (Δv)	$[10\quad 10\quad 10]$m/s

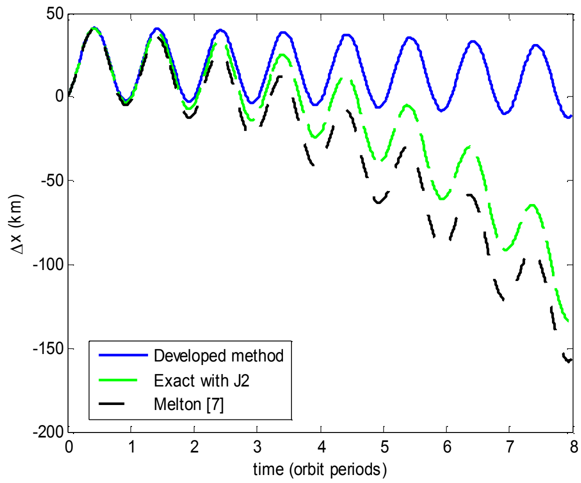
However, it should be noted that the keplerian solutions do not overlap as closely as the approximate perturbed solutions. The most useful aspect of the path plots is that they allow one to see the relative scale of the motion amongst the coordinate directions; axis scales for the plots were set equal (see Figure 4).

Figure 5 shows the error plots for the maximum errors of relative positions in the linearized solutions at 8 orbit periods. The error between each of the linear solutions and the developed method (time-varying perturbed solution) was determined at each time step as compared to the perturbed non-linear exact solution. The simulation results show that Range error (%) for developed method as compare to Melton's solution [7] is smoothly increased with the limited expansion of orbit periods due to applied approximate terms.

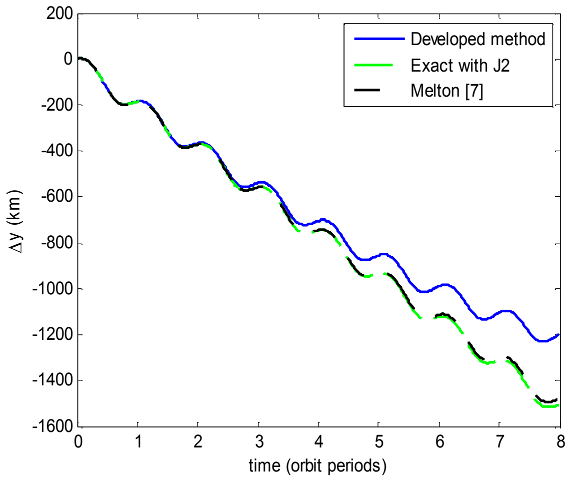


(c) Relative *z*-Position

Figure 2. Comparison of the relative positions vs. time for discussed solutions at 8 orbit periods



(a) Relative *x*-Position



(b) Relative *y*-Position

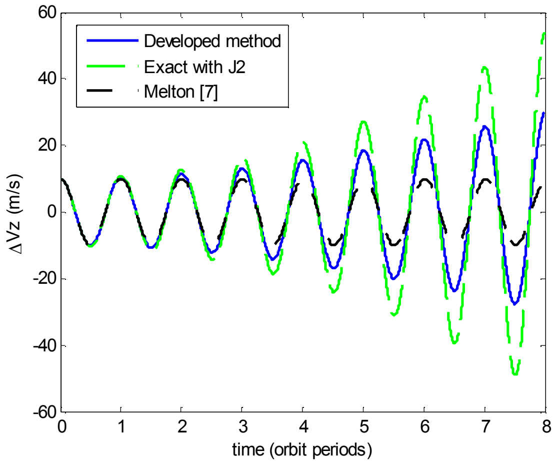


Figure 3. Relative *z*-velocity vs. time at 8 orbit periods

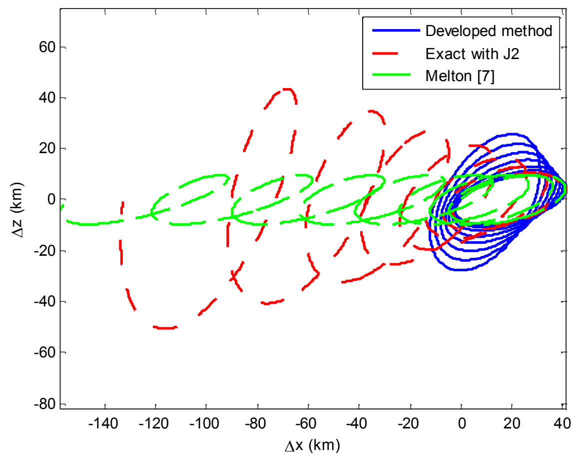


Figure 4. Relative *z*-position vs. *x*-position

The error results (Table 4) are shown the maximum errors in the linear solutions over one orbit period for a range of eccentricities. The error between each of the linear solutions and the approximate perturbed solution was determined at each time step as compared to the perturbed non-linear exact solution. These results seem to indicate that the equations developed in this paper do not significantly change the *x*- and *y*-position accuracy, regardless of eccentricity. So, results indicate a considerable increase in the *z*-position accuracy for eccentricities smaller than 0.3. For eccentricities greater than 0.3, the error is slightly greater. Therefore, the accuracy in the *x*- and *y*-directions, appears to be roughly unchanged

TABLE 4. Comparison of values of *z*-Position error versus eccentricity (one orbit period)

	Eccentricity(e)	0.1	0.2	0.3	0.4
	Solution Method				
Max. Error (%)	Exact with J_2	19.12	17.86	8.32	10.63
	Keplerian	17.23	13.65	6.47	7.47
	Melton [7]	19.87	11.23	6.24	6.82
	Developed method	8.26	3.14	2.12	2.84

5. CONCLUSION

A set of linearized equations was derived for the motion, relative to an elliptical reference orbit, of an object influenced by harmonic perturbations. Approximately, determined solutions were used to compare these equations and the linearized keplerian equations to the exact equations. The inclusion of the linear perturbations in the derived equations increased the accuracy of the solution significantly in the out of orbit plane direction, while the accuracy within the orbit plane remained roughly unchanged. Therefore, by reason of using approximate terms in the solution, for continues high accuracy increase of time-varying parameters, obtained results could be useful about the orbital element errors evaluation and analysis of orbital multiple rendezvous or formation flying missions.

In the future, the effective application of the linearized J_2-perturbed terms on relative motion dynamics could be studied, such as the multiple rendezvous or formation flying missions which are involved with the short-period times. In addition, the study on inclusion analysis of higher-level linearized zonal harmonic perturbation terms over Melton's problem could be done.

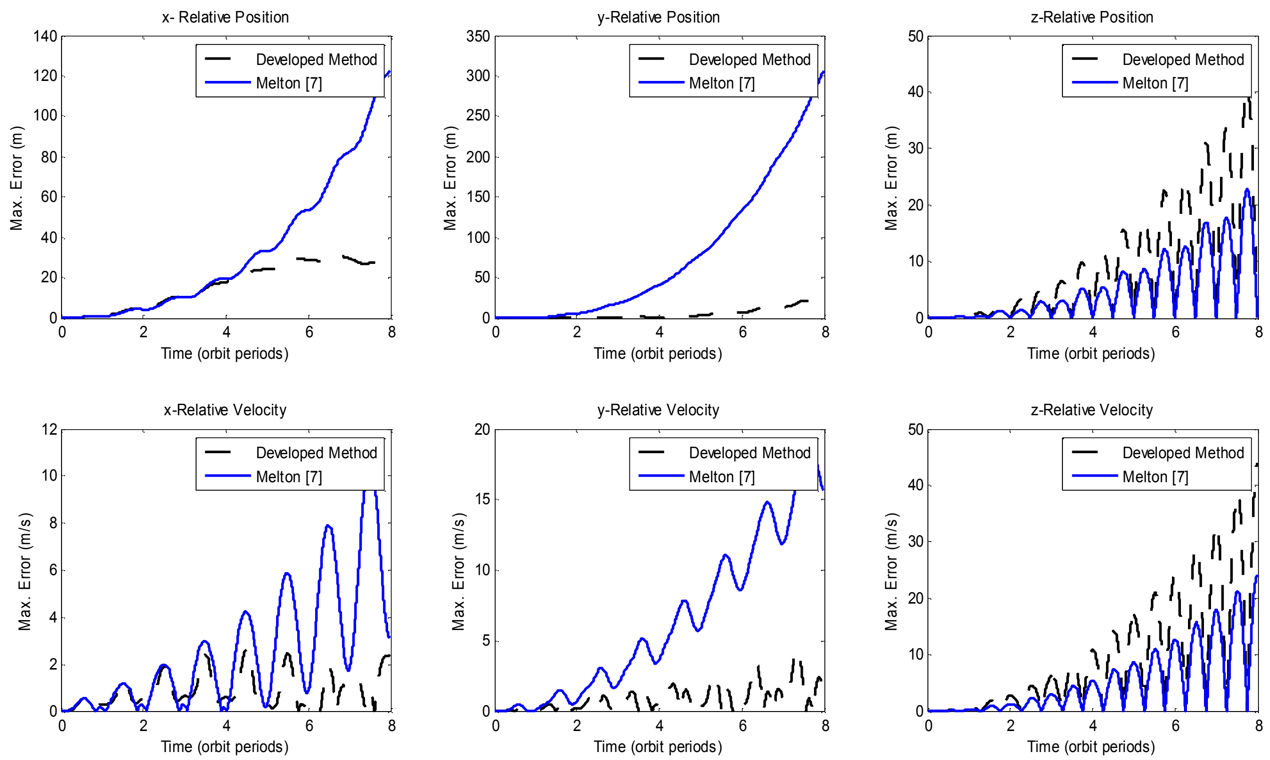


Figure 5. Comparison of max.error between Melton and developed method

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High Accuracy Relative Motion of Spacecraft Using Linearized Time-Varying J_2-Perturbed Terms

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چکیده

در این مقاله یک دسته معادلات خطی شده برای حرکت نسبی در مدار مرجع بیضوی تحت نفوذ اغتشاشات J_2 ارائه شده است. یک روش حل تقریبی نیز با رویکرد افزایش دقت و پایدارسازی معادلات حرکت اغتشاشی توسعه یافته است. در واقع هدف این تحقیق این است که نشان دهد، نفوذ اغتشاشات خطی شده در مسئله ملتون، سبب افزایش قابل ملاحظه دقت خطای موقعیت در جهت خارج از صفحه مدار شده، در حالیکه دقت در صفحه مداری تقریبا بدون تغییر باقی می‌ماند. به دلیل جایگزینی جمله‌های تقریبی بر حسب دو پارامتر مداری e و M در روش حل مذکور با رویکرد افزایش دقت پارامترهای زمان–متغیر شامل $\theta(t)$ و $R_O(t)$ ، نتایج حاصله می‌تواند جهت ارزیابی هر چه بهتر انحراف المان‌های مداری در فرآیندهایی همچون عملیات ملاقات چندگانه که هر یک با بازه زمانی کوتاه مدت مواجه هستند، بسیار مؤثر باشد.

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