A NOVEL HOMOTOPY PERTURBATION METHOD: KOROUSH METHOD FOR A THERMAL BOUNDARY LAYER IN A SATURATED POROUS MEDIUM

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Abstract In this paper a novel homotopy perturbation method has been presented for forced convection boundary layer problems in a porous medium. Considering the infinite condition, a homotopy form which is similar to the singular perturbation form has been considered. The inner and outer solutions have been achieved and the coincidence of the results has been investigated with a proper matching method. The results have been compared with those of other researchers in the field of mixed convection boundary layer problem in porous media. The comparisons show that this method can be used properly for analyzing similar problems. The results show that this method can be very powerful and efficient, and for this special problem the convergence is achieved rapidly. According to the presented matching condition, the Kourosh method can be used in various fields of science and engineering.

Keywords Boundary layer; Forced convection; Kourosh method; Perturbation

چکیده در این مقاله روش جدید هموتوپی پرتوبیشن در حل مسائل لایه مرزی جابجایی اجباری در محیط-های متخلخل معرفی شده است. با لحاظ نمودن شرایط حدی در حل مسائل، ساختار هموتوپی معادلات، مشابه با شکل سینگولار پرتوبیشن آنها در نظر گرفته شده است. حل معادلات در لایه داخلی (inner) و خارجی (Outer) به دست آمده است و تطابق نتایج با یک روش مناسب مورد ارزیابی قرار گرفته است. همچنین نتایج حاصل شده با نتایج تحقیقات دیگر محققان در زمینه مسائل لایه مرزی جابجایی اجباری در محیط‌های متخلخل نیز مقایسه شد که نشان می‌دهد این روش نیز می‌تواند به صورت مناسب و قابل قبولی در تحلیل مسائل مشابه مورد استفاده قرار گیرد. نتایج نشان می‌دهد که این روش می‌تواند بسیار قدرتمند و کارا عمل نماید و منجر به همگرایی سریع‌تری در حل این گونه مسائل ویژه خواهد گردید. با توجه به شرایط تطابق بیان شده، "روش کوروش" می‌تواند در زمینه‌های مختلف علمی و مهندسی مورد استفاده قرار گیرد.

1. INTRODUCTION

Numerous researches have studied mixed convection boundary layer flows. This subject is important due to its applications in fields such as geothermal energy extraction, oil reservoir modeling, food process industry, casting and welding in manufacturing processes [1].

Several published books on convection in porous media have lead the convective flow in porous media to become a classical subject [2].

The similarity of solutions for mixed convection boundary-layer flows was first considered by Sparrow et al. [3], who showed that the boundary-layer equations could be reduced to a pair of coupled ordinary differential equations and obtained some solutions. Furthermore, Cheng [4] investigated the mixed convection adjacent to inclined surfaces embedded in a porous medium using the boundary-layer approximation. Similar solutions have been obtained for the situation where the free stream velocity and the surface temperature distribution vary according to the same power function of the distance along the surface. The separation in mixed convection flow was first discussed by Merkin [5] who examined the effect of opposing buoyancy forces on the boundary-layer flow on a semi-infinite vertical flat plate at a constant temperature in a uniform free stream. Furthermore, this problem was studied by Wilks [6,7] and Hunt and Wilks [8] who also considered the case of uniform flow over a semi-infinite flat plate but heated at a constant heat flux rate. Merkin and Pop [9] obtained the similarity equations for mixed convection boundary-layer flow over a vertical semi-infinite flat plate in which the free stream velocity is uniform and the wall temperature is inversely proportional to the distance along the plate. Aly et al. [1] have investigated a vertically flowing fluid, in a fluid-saturated porous media maintained at a constant temperature, T∞, pass a thin vertical fin, which is modeled as a fixed and semi-infinite vertical surface. They showed that the problem depends on two parameters which are namely (the ratio of the Rayleigh to Péclet numbers) and λ (they was assumed T∞, to vary as x¯λ). In this paper a new addomian procedure which is based on homotopy perturbation method is presented in order to solve the Cheng and Minkowycz [10] problem with Aly et al assumptions [1]. The temperature of the fin, above the ambient temperature T∞, is assumed to vary as x¯λ, where x is the distance from the tip of the fin and λ is a pre-assigned constant. The Rayleigh number is assumed to be large enough to assume the boundary-layer approximation and the fluid velocity at the edge of the boundary-layer to vary as x¯λ, so that a similarity solution may be obtained. The convecting fluid and the porous media is assumed isotropic in thermodynamic equilibrium and have constant physical properties. The flow is assumed to be described by Darcy's law and the Boussinesq approximation is assumed valid. The governing equations are given by [1]:

$$\frac{1}{Pe}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{Ra}{Pe}\frac{\partial\theta}{\partial y} \tag{1}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\Theta}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial\Theta}{\partial y}=\frac{1}{Pe}\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2} \tag{2}$$

In the above equation, the following non-dimensional quantities are used:

$$x=\frac{\bar{x}}{L},\quad y=\frac{Pe^{1/2}}{L}\bar{y},\quad \psi=\frac{\bar{\psi}}{\alpha_m Pe^{1/2}},\quad \Theta=\frac{T-T_\infty}{T_0-T_\infty} \tag{3}$$

where x, y are vertical and horizontal co-ordinates respectively. L is an arbitrary length scale, and Pe is the Péclet number $Pe=U_0L/\alpha_m$, where $U_0=BL^{\lambda}$ is the reference velocity and B> 0, $\bar{\psi}$ is the stream function, α_m is effective thermal diffusivity of the porous medium. T_0 is the reference temperature which is defined as $T_0=T_\infty+|A|L^{\lambda}$ and Ra is the Raleigh number which is equal to $gKL(T_0-T_\infty)/(\nu\alpha_m)$, where g is the magnitude of the acceleration due to gravity, K is the permeability, β is the coefficient of thermal expansion, and ν is the kinematic viscosity of the fluid. Assuming that the Péclet number is very large, the resulting temperature boundary-layer is analogues to that in classical boundary-layer theory. Let Pe→∞ in Eq. (1) and (2) and also replacing Ra/Pe in Eq.(2) with ε and finally introducing the dimensionless similarity variables, $\psi(x,y)=(2x^{\lambda+1})^{1/2}$, $f(x,\eta)$, $\eta=y/(2x^{1-\lambda})^{1/2}$ and $\Theta(x,y)=x^{\lambda}\,\theta(x,\eta)$, the following similarity equations is obtained:

$$f'=1+\varepsilon\theta \tag{4}$$

$$\theta''+(1+\lambda)f\theta'-2\lambda f'\theta=0 \tag{5}$$

where f is the function to be determined. The similarity equations have to be solved subject to the boundary conditions:[1]

$$f=0,\quad \theta=1(A>0)\ or\ \theta=-1(A<0),\ on\ \eta=0 \tag{6}$$

$$f'\rightarrow 1,\quad \theta\rightarrow 0\ \ as\ \ \eta\rightarrow\infty \tag{7}$$

where $f=f(\eta)$ and $\theta=\theta(\eta)$, $\eta\geq 0$ are the similar stream function and the similar temperature field of a mixed convection boundary layer flow over a vertical plane surface adjacent to a saturated porous medium respectively. In the above equations $-\infty<\varepsilon<+\infty$ stands for the mixed convection parameter, and $\lambda>-1$ represents the power-law exponent of the surface temperature distribution. Other authors [11, 12] have reported some general features of the mixed convection solution. For $\varepsilon=0$ the equation will convert to the forced convection governing equation and the similar stream function can be considered as $f=f(\eta)$ and the equation of similar temperature can be expressed as: [13]

$$\theta''+(1+\lambda)\eta\theta'-2\lambda=0 \tag{8}$$

$$\theta(0)=1, \quad \theta(\infty)=0 \tag{9}$$

2. SUGGESTED HOMOTOPY STRUCTURE (KOUROSH METHOD)

Analytical solution which utilizes classical perturbation method needs to exert a small parameter in the equation. This limitation makes it difficult to develop this method for different applications. Recently, in order to overcome this problem, novel similar methods has been introduced. The homotopy perturbation method (HPM) is one of the most famous one. This method is introduced by He [14] for the first time and has been developed by his self and other authors [15–19].

In this study in order to solve the forced convection boundary layer problems in a porous medium, a new procedure are introduced which is based on Composite Asymptotic Expansion Method [20] and homotopy perturbation method. One technique of dealing with this method is to determine straight forward expansion (called outer expansion) using the original variables (in homotopy form) and to determine expansions (called inner expansions) describing the sharp change using magnitied scales.

In order to analyze Equation (4), while considering the infinite condition, the following homotopy form is suggested:

$$P(\theta'' - 2\lambda - (1+\lambda)\eta\theta) + (1+\lambda)\eta(\theta' + \theta) = 0 \tag{10}$$

The deduced equations for the inner and outer solutions of (10) can be considered as followed:

Outer solution

$$\theta^o = \theta_0^o + P\theta_1^o + P^2\theta_2^o + \ldots \tag{11}$$

$$P^0 \to \theta_0'^o + \theta_0^o = 0, \quad \theta_0^o(\infty) = 0 \tag{12}$$

$$\begin{aligned} P^1 \to \eta(\theta_1'^o + \theta_1^o) = -\frac{1}{1+\lambda}(\theta'' - 2\lambda\,{}_0^o \\ -(1+\lambda)\eta\theta_0^o), \quad \theta_1^o(\infty) = 0 \end{aligned} \tag{13}$$

For the inner solution under the condition $y = \eta/\sqrt{P}$, Equation (6) will become:

$$\begin{aligned} &P\left(\frac{\theta_y''}{P} - 2\lambda\theta - (1+\lambda)\sqrt{P}y\theta\right) + \\ &(1+\lambda)\sqrt{P}y\left(\frac{\theta_y'}{\sqrt{P}} + \theta\right) = 0 \end{aligned} \tag{14}$$

where θ_y' and θ_y'' are the first and second order derivative of θ with respect to y. Then the inner solution of (14) will be as follows:

Inner solution

$$\theta^i = \theta_0^i + P\theta_1^i + P^2\theta_2^i + \ldots \tag{15}$$

$$P^0 \to \theta_0''^i + (1+\lambda)\theta_0'^i = 0, \quad \theta_0^i(0) = 1, \quad \theta_0^i(\infty) = 0 \tag{16}$$

$$\begin{aligned} &P^1 \to \theta_1''^i - 2\lambda\,{}_1^i + (1+\lambda)y\theta_1'^i = 0, \\ &\theta_1^i(0) = 0, \quad \theta_1^i(\infty) = 0 \end{aligned} \tag{17}$$

3. SUGGESTED MATCHING METHOD

Noting the stretching variable [21] $y = \eta/\sqrt{P}$ for the inner solution and η for the outer solution, the $P^{1/4}$ region is considered for the coincidence of the results. Therefore:

$$(\theta^i)^o = \lim_{P\to 1} \theta^i(y = P^{-1/4}) \tag{18}$$

$$(\theta^o)^i = \lim_{P\to 1} \theta^o(\eta = P^{1/4}) \tag{19}$$

As a result, to determine the probable unspecified coefficients of the outer solution, the below relation can be used as the matching condition:

$$(y^i)^o = (y^o)^i \tag{20}$$

The finally:

$$\theta^i = \theta_0^i + \theta_1^i + \ldots, \quad \eta < 1 \tag{21}$$

$$\theta^o = \theta_0^o + \theta_1^o + \ldots, \quad \eta > 1 \tag{22}$$

$$\theta = \theta^i + \theta^o - (\theta^i)^o, \quad \eta \approx 1 \tag{23}$$

4. SOLUTION OF THE EQUATIONS

The answers of Equations (8) and (12) can be written as follows:

$$\theta_0^o = C_1 exp(-\eta) \tag{24}$$

$$\theta_0^i = 1 - erf(y\sqrt{\frac{1+\lambda}{2}}) \tag{25}$$

The C_1 coefficient can be achieved from the matching condition as follows:

$$C_1 = \left[1 - \sqrt{\frac{2(1+\lambda)}{\pi}}\int_0^1 exp(-\frac{(1+\lambda)y^2}{2})dy\right]exp(1) \tag{26}$$

Also, the answers of Equations (9) and (13) can be written as follows:

$$\theta_1^o = -\frac{C_1}{1+\lambda}\times\left[(1-2\lambda)exp(-\eta).Ei(\eta) + (1+\lambda)exp(-2\eta)\right] + C_2 exp(-\eta) \tag{27}$$

where $Ei(\eta) = \int_{-\eta}^{\infty}\frac{e^{-t}}{t}dt, \quad \eta > 0$.

$$\theta_1^i = (-\frac{2\sqrt{2}\lambda}{1+\lambda})erf(y\sqrt{1+\lambda}) - (\frac{2\lambda}{1+\lambda})\exp(-\frac{1+\lambda}{2}y^2) \times(1 - erf(y\sqrt{\frac{1+\lambda}{2}})) + (\frac{2\sqrt{2}-2}{1+\lambda})erf(y\sqrt{\frac{1+\lambda}{2}}) + \frac{2\lambda}{1+\lambda} \tag{28}$$

Using the matching condition, the C_2 coefficient can be written as:

$$C_2 = \exp(-1)\left\{(\frac{2\lambda}{1+\lambda})(1-\exp(-\frac{1+\lambda}{2})) + (\frac{2\sqrt{2}-2}{1+\lambda})(erf(\sqrt{\frac{1+\lambda}{2}})) - (\frac{C_1}{1+\lambda})\left[\frac{2\lambda-1}{\exp(1)}Ei(1) + (1+\lambda)\exp(-2)\right]\right\} \tag{29}$$

5. RESULTS AND DISCUSSIONS

According to the calculations, the final solution of forced convection boundary layer problem in a porous medium including two first terms can be deduced as:

$$\theta^i = 1 - erf(y\sqrt{\frac{1+\lambda}{2}}) - (\frac{2\sqrt{2}\lambda}{1+\lambda})erf(y\sqrt{1+\lambda}) - -(\frac{2\lambda}{1+\lambda})\exp(-\frac{1+\lambda}{2}y^2)\left[1 - erf(y\sqrt{\frac{1+\lambda}{2}})\right] + +(\frac{2\sqrt{2}-2}{1+\lambda})erf(y\sqrt{\frac{1+\lambda}{2}}) + \frac{2\lambda}{1+\lambda}, \quad y \prec 1 \tag{30}$$

$$\theta^o = (C_1 + C_2)exp(-\eta) - \frac{C_1}{1+\lambda}\left[(1-2\lambda)exp(-\eta).Ei(\eta) + (1+\lambda)exp(-2\eta)\right], \quad \eta \prec 1 \tag{31}$$

$$\theta = \theta^i + \theta^o - C_1 exp(-1) + \frac{C_1}{1+\lambda}\left[(1-2\lambda)exp(-1)Ei(-1) + (1+\lambda)exp(-2)\right] + C_2 exp(-1), \quad y \approx \eta \approx 1 \tag{32}$$

where C_1 and C_2 result from equations (26) and (27).

Figure (1) represents the θ values with respect to η for different values of λ

Also, the results of this study have been compared with the results of Brighi and Hoernel [11] in Figure (2). It is obvious that the results have proper coincidence.Considering of the result for $y<1$

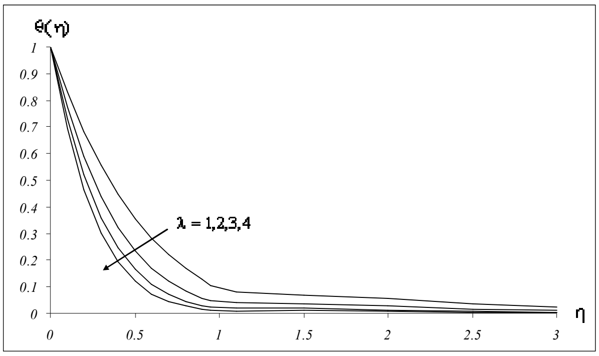

Figure 1. Unique temperature profiles for different positive values of λ

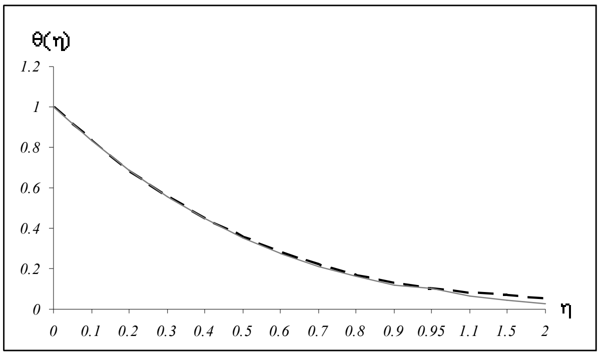

Figure 2. Comparison of the results with the Brighi and Hoernel's results ($\lambda = 1$)

(Eq. 30), the value of $\theta'(0)$ which is the temperature surface can be simply calculated as:

$$\theta'(0) = \frac{(4-5\sqrt{2})\lambda - \sqrt{2}}{\sqrt{\pi(1+\lambda)}} \tag{33}$$

For special cases $\lambda = 0, 1$:

$$\begin{aligned} &\theta'(0) = -\sqrt{\frac{2}{\pi}}, \quad \lambda = 0 \\ &\theta'(0) = \frac{4-6\sqrt{2}}{\sqrt{2\pi}} \approx -\sqrt{\pi}, \quad \lambda = 1 \end{aligned} \tag{34}$$

The similar wall heat flux $-\theta'(0)$ given by Equation (33) which is the most applicable case for engineers has been scaled with similarity exponent λ linearly for $0 < \lambda \le 1$ and with $\sqrt{\lambda}$ for $\lambda >> 1$:

$$-\theta'(0) = \begin{cases} \sqrt{\frac{2}{\pi}} + \frac{5\sqrt{2}-4}{\sqrt{\pi}}\lambda & 0 < \lambda \le 1 \\ \frac{5\sqrt{2}-4}{\sqrt{\pi}}\lambda^{1/2} & \lambda >> 1 \end{cases} \tag{35}$$

This shows a good agreement with Magyari & Aly's results [13].

6. CONCLUSION

In this paper a novel approach based on homotopy perturbation method has been used for problems with infinite conditions. The application of this method has been presented for the forced convection heat transfer in porous media. Comparison of the results shows that this method can be very powerful and efficient. Furthermore, the convergence of the special case which is studied is rapidly. According to the presented matching condition, the Kourosh method can be used in various fields of science and engineering.

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