RESEARCH NOTE

MATHEMATICAL ANALYSIS OF MHD FLOW OF BLOOD IN VERY NARROW CAPILLARIES

Madhu Jain, G.C. Sharma and Atar Singh*

Department of Mathematics, Institute of Basic Science, Dr. B.R. Ambedkar University
Agra 28002, Khandari, India
drmadhujain@sancharnet.com – gokulchand@sancharnet.com - atarsingh1968@rediffmail.com

*Corresponding Author

Abstract   A mathematical model for blood flow in narrow capillaries under the effect of transverse magnetic field has been investigated. It is assumed that there is a lubricating layer between red blood cells and tube wall. The transient flow of the fit red blood cell surrounded by plasma annulus in the narrow capillary is considered. The analysis of fluid flow between red cell and tube wall, when the cell appears to be at rest and the capillary wall moving backwards, is made. The effect of transverse magnetic field on the flow is examined. Computational results are shown through graphs to describe the effect of various parameters on velocity profile, leak-back flow rate and skin friction.

Keywords   MHD Flow, Single File Flow, Narrow Capillaries, Lubricating Zone, Magnetic Field

چكیده   یک مدل ریاضی برای جریان خون در مویرگ‌های نازک تحت تأثیر میدان مغناطیسی عرضی مورد بررسی قرارگرفته است. فرض شده است که یک لایه روان‌ساز بین گلبول‌های قرمز و دیواره رگ وجود دارد. جریان‌گذرای گلبول‌های قرمز سالم (مناسب) که با انیول‌های پلاسما در مویرگ‌های نازک احاطه‌شده، مدنظر قرارگرفته است. تحلیل جریان سیال بین گلبول‌های قرمز و دیواره رگ در حالتی انجام‌گرفته که گلبول درحال استراحت است و دیواره مویرگ به سمت عقب حرکت می‌کند. تأثیر میدان مغناطیسی عرضی روی جریان بررسی‌شده است. نتایج محاسبات در نمودارهایی نشان‌داده‌شده‌اند تا تأثیر پارامترهای مختلف روی پروفیل سرعت، نرخ جریان‌نشتی رو به عقب (برگشتی) و اصطکاک پوستی را نشان‌دهند.

1. INTRODUCTION

Recently Bio-magnetic Fluid Dynamics has emerged as a new field for the study of the fluid dynamical behavior of biological fluids in the presence of magnetic field. The control of the blood pressure has been possible by using magnetic field in cases of hypertension and related diseases. In human physiology, we come across mainly two types of circulations, macro circulation and microcirculation. The macrocirculation system consists of large arteries and vessels while microcirculation system comprises of the smallest arteries, veins, and capillaries. Conditions are very different for the circulations in narrow capillaries than large vessels. For describing the mechanics of red blood cell motion in narrow capillaries, we

distinguish two situations according to the convenience with which the cells fit into the vessels. In the first case when the capillary has diameter larger than that of the cell, the cell can fit into the tube without distortion; this flow situation is called positive clearance. In the second situation called negative clearance, when the diameter of the cell is larger than that of capillary as such the cell will be deformed in order to fit into the capillary. In this case pressure must be generated in thin layer of fluid round the edge of the cell in order to deform it and depends on elastic properties of the cell.

When red cell is severely deformed then in blood flow the red cell seems to plug the capillary of blood vessel and the motion of the plasma in capillary between successive red cells is called

bolus flow. The significance of bolus flow was pointed out by Prothero, et al [1]. Pressure within capillaries shows periodic fluctuations; such variation can result both from local functions and from changes in central arterial and venous pressure. Systematic measurement of pressure distribution in small blood vessels was usually done by Wiedertielm, et al [2]. Lighthill [3] discussed the behavior of tightly fitting solid pellets, which may be deformable and may be forced by pressure difference to move. Lighthill model was extended by Fitz-Gerald [4] to examine the effects of axisymmetry and tube porosity. Zweifach, et al [5] made extensive measurements of pressure distribution in micro vessels. Caro, et al [6] described that the measurement of representative flow rates and velocity profiles in the very narrow and small vessels has proved even more difficult than the measurement of pressure. This has been mainly due to the difficulty of observing the movement of individual red cell in vessels larger than capillaries. But in addition any one who has observed a micro vascular bed in vivo with a microscope would have noted the variability of flow in a given vessel; flow can be steady for a period of time and then suddenly slow down or stop altogether. Such changes in flow rate have not been closely correlated to vascular pressure; this is perhaps to be expected because in such a network pressures and flow rates in adjacent vessels are related. In capillaries, where individual red cell can be observed in the flow, flow rates can be estimated on the basis red cell velocity.

Tozern, et al [7,8] analyzed the steady flow of elastic spheres in a circular cylindrical tube, using a series expansion for the practical displacement and lubrication theory for the fluid motion. The results revealed that the apparent viscosity depends on the shear modulus of elasticity of the particles. Ozkayan [9] formulated a mathematical model for the flow of red blood cells in narrow capillaries and used lubrication theory to describe the plasma flow in narrow gap between red cells and tube wall. Relative apparent blood viscosity as a function of tube diameter and hematocrit was described by Pries, et al [10]. Bishop, et al [11] recognized that the rheological behavior of blood flow needed to understand the function of circulatory system. Sharan, et al [12] developed a two phase model for the flow of blood in narrow

tubes and assumed that the fluid is more viscous near the wall. Haik, et al [13] investigated the magnetic effect on apparent additive viscosity of the human body. Pries [14] investigated rheology of the microcirculation and noted that the apparent viscosity declines as diameter of vessel decrease. Baumgartnes, et al [15] experimented on 26 individuals that decreasing distance between red cells increases in hematocrit. The effect of transverse magnetic field on blood flow in arteries was analyzed by Sharma, et al [16]. Mittal, et al [17] analyzed coronary blood flow distribution through a network of vessels. Huo, et al [18] developed a mathematical model to analyze pulsatile blood flow in diastole in the absence of vessel tone in the entire coronary arterial tree. Aviles, et al [19] introduced a new implant assisted magnetic drug targeting approach for radiation capillaries beds.

Various studies have been performed on rheological behavior of blood flow in narrow capillaries to establish relationship among resistance, viscosity, clearance (both positive and negative) of cell and other parameters. From earlier works, we observe that the flow resistance for plasma in narrow capillaries is greater than in large capillaries. The study of flow under the influence of a magnetic field is also important so far as flow and resistance are concerned. In the present investigation, we study the motion of the blood through a very narrow capillary under the action of transverse magnetic field. Our study is confined to microcirculation i.e. single file flow of red blood cell through narrow capillary. The rest of paper is organized in the following manner. Section 2 is devoted for model description and governing equations. Section 3 provides volumetric flow rate of the blood in narrow capillaries. In Section 4, the numerical results are given. Finally, Section 5 is intended for the concluding remarks.

2. MATHEMATICAL MODEL AND GOVERNING EQUATIONS

In this model we consider the axially symmetric and Newtonian flow of blood in a tube of uniform radius. The blood is assumed to be homogeneous fluid, while the red blood cells are assumed to be

elastic and incompressible. The single cell is fitted in the tube so as to generate a single file flow. In this investigation, we study fluid flow in lubricating zone i. e. fluid flow between red blood cell and tube wall. The effect of transverse magnetic field on the flow of narrow capillary is taken into account. The induced magnetic field has been neglected. The viscous forces are pre-dominant in the flow of such tubes. The inertial terms are considered negligible. During passing down single red cell in narrow capillary, it deforms due to its elastic property. The shape of red cell is bi-concave disk. The axial velocity is taken zero at the surface of red blood cell and-W at the tube wall. To obtain the axial velocity of the fluid relative to the tube, we add W velocity in the direction of the flow of fluid (cf. 5).

Let us assume the coordinate z in the direction of the axis of the tube. r is transverse distance from the highest point of the surface of the RBC (Figure 1).

Notations used for mathematical formulation are as follows:

R   Tube radius
h   Thickness of gap between the red cell and tube wall
z   Axial distance
v(w)   Radial (Axial) velocity of fluid
w   Velocity relative to the tube wall

w_m Mean velocity flow
W   Velocity of red blood cell
P   Pressure
C_f Skin friction coefficient
i   Imaginary number
n   Frequency of pulse
ρ Density of fluid
μ Viscosity of the fluid
μ_e Magnetic permeability
σ Coefficient of conductivity
τ_w Shear stress at tube wall
B_0 Transverse magnetic field intensity
Q_{lb} Leak-back flow rate
MHD   Magneto hydrodynamic

The equations governing the motion fluid flow are:

$$\frac{\partial(rv)}{\partial r} + \frac{\partial(rw)}{\partial z} = 0 \tag{1}$$

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right) - \sigma\mu_e^2 B_0^2 w \tag{2}$$

Boundary conditions are:

$$w = -We^{-int}, \quad v = 0, \quad \text{at} \ \ r = h \quad t > 0 \tag{3}$$
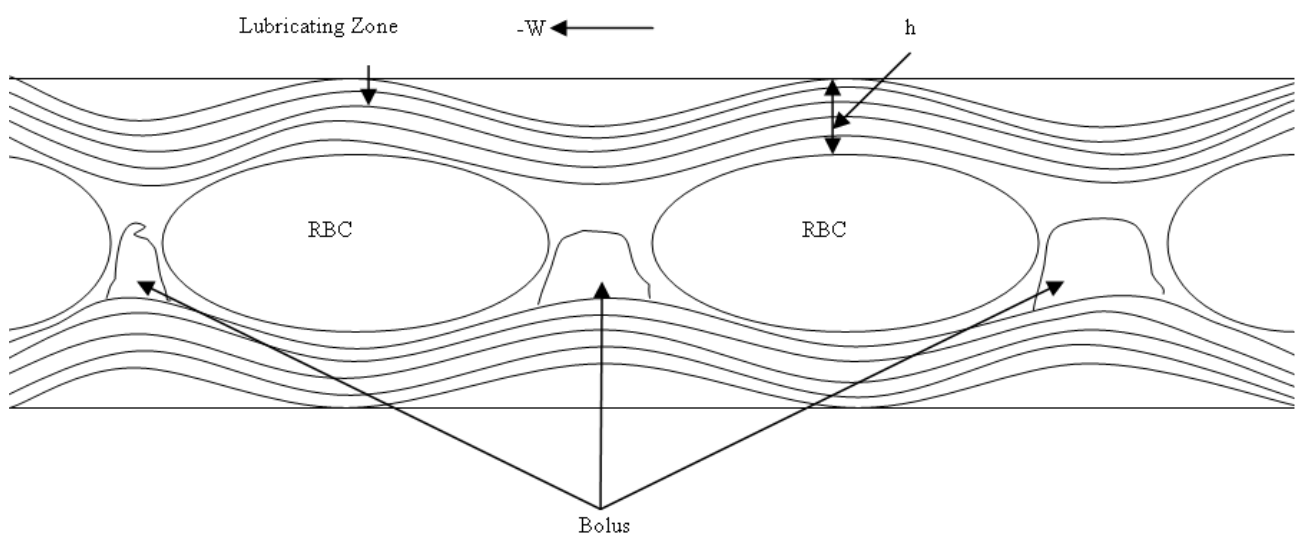$$w = 0, \quad\quad\quad v = 0, \quad \text{at} \ \ r = 0 \quad t > 0$$

Figure 1. Schematic diagram of fluid flow in lubricating zone.

From Equations 1 and 3, we have:

$$\int_0^h rwdr = -Q_{lb} \quad \text{(Constant)} \tag{4}$$

The pressure gradient of blood flow can be taken as:

$$\frac{\partial p}{\partial z} = -Pe^{-int}$$

Also we may take the velocity as $w(r,t) = f(r)e^{-int}$ without loss of generality.

From Equation 2 using these values of $\frac{\partial p}{\partial z}$ and $w(r,t)$, we get:

$$f''(r) + \frac{1}{r}f'(r) - \frac{A}{\mu}f(r) = -\frac{P}{\mu} \tag{5}$$

Where

$$A = -in\rho + \sigma\mu_e^2 B_0^2$$

The solution of (5) is given by

$$f(r) = A_0 J_0(ir\sqrt{A/\mu}) + B_0 Y_0(ir\sqrt{A/\mu}) + \frac{P}{A} \tag{6}$$

Where J_0 and Y_0 are Bessel's functions of first kind and second kind respectively. Now $B_0 = 0$ as otherwise at $r = 0$, $f(r)$ is not finite.

Then the solution of (6) becomes:

$$f(r) = A_0 J_0(ir\sqrt{A/\mu}) + \frac{P}{A} \tag{7}$$

Now transformed boundary conditions are:

$$
\begin{aligned}
f(r) &= 0, & \text{at} \quad r &= 0, & t &> 0 \\
f(r) &= -W, & \text{at} \quad r &= h, & t &> 0
\end{aligned} \tag{8}
$$

Using second condition of Equation 8 in (7), we have:

$$A_0 = -\frac{P}{A}$$

Thus (7) becomes

$$f(r) = \left[1 - J_0(ir\sqrt{A/\mu})\right]\frac{P}{A} \tag{9}$$

By using first condition of Equation 8 in (7), we get:

$$P = -\frac{AW}{\left(1 - J_0(ih\sqrt{A/\mu})\right)}$$

Now Equation 7 yields:

$$f(r) = -W\left[\frac{\left(1 - J_0(ir\sqrt{A/\mu})\right)}{\left(1 - J_0(ih\sqrt{A/\mu})\right)}\right] \tag{10}$$

Thus

$$w = -W\left[\frac{\left(1 - J_0(ir\sqrt{A/\mu})\right)}{\left(1 - J_0(ih\sqrt{A/\mu})\right)}\right]e^{-int}$$

Expanding the Bessel function in a series and retaining only up to bi-quadratic terms,

$$w = -W\frac{r^2\left(16\mu + Ar^2\right)}{h^2\left(16\mu + Ah^2\right)}e^{-int} \tag{11}$$

The real part of w is given by:

$$\text{Re}(w) = -W$$

$$\frac{r^2\left[\begin{array}{l}\left((16\mu + \sigma\mu_e^2 B_0^2 r^2)(16\mu + \sigma\mu_e^2 B_0^2 h^2) + (nh\rho r)^2\right) \\ \cos nt - \left(\begin{array}{l}n\rho h^2(16\mu + \sigma\mu_e^2 B_0^2 r^2) - \\ n\rho r^2(16\mu + \sigma\mu_e^2 B_0^2 h^2)\end{array}\right)\sin nt\end{array}\right]}{h^2\left(16\mu + \sigma\mu_e^2 B_0^2 h^2)^2 + n^2\rho^2 h^4\right)} \tag{12}$$

The velocity relative to tube wall is given by:

$$w_1 = w + W\cos nt \tag{13}$$

3. THE LEAK-BACK FLOW RATE AND SKIN FRICTION

With the help of Equation 4, the leak-back flow

rate Q_{lb} is given by:

$$Q_{lb} = -\int_0^h rwdr = W$$

$$\frac{h^2 \left[\left(64\mu + \frac{20\sigma\mu_e^2 B_0^2 \mu h^2}{3} + \frac{h^4}{6}(\sigma^2\mu_e^4 B_0^4 + n^2\rho^2) \right) \left[\cos nt - \frac{4}{3}n\rho\mu h^2 \sin nt \right] \right]}{\left((16\mu + \sigma\mu_e^2 B_0^2 h^2)^2 + n^2\rho^2 h^4 \right)} \tag{14}$$

The skin friction at the RBC surface is given by:

$$\tau_{rbc} = -\mu\left(\frac{1}{r}\frac{\partial w}{\partial r} \right)_{r=0} =$$

$$32\mu^2 W\left((16\mu + \sigma\mu_e^2 B_0^2 h^2)\cos nt - n\rho h^2 \sin nt \right) / \tag{15}$$

$$h^2\left((16\mu + \sigma\mu_e^2 B_0^2 h^2)^2 + n^2\rho^2 h^4 \right)$$

4. NUMERICAL RESULTS AND DISCUSSION

In this section, we validate the analytical results for blood flow in very narrow capillaries by setting the default parameters as $h = 0.5$, $\mu = 3$, $\sigma = 2$, $\mu_e = 1$, $\rho = 1.05$, $n = 8$, $w = 1.1$, $r = 0.05$, $\mu = 2.5$ and $B_0 = 3$.

Figures 2a-c depicts the relative velocity profiles for different values of B_0, μ and r in lubricating zone. It is noticed that the relative flow velocity along the capillary between RBC and tube wall is almost uniform and the effect of magnetic intensity on flow velocity is negligible (see Figure 2a). It is also clear from these figures that the flow is pulsatile where velocity changes periodically. It is seen in Figure 2b that the effect of viscosity (μ) on the flow velocity relative to the tube wall is almost negligible. From Figure 2c, we see that the velocity relative to the tube (remember that the tube velocity relative to RBC is-W) increases as transverse distance from highest point of RBC surface decreases and at the wall it is zero. From this we infer that the velocity relative to the tube wall declines towards wall and the velocity along the transverse distance is independent of viscosity.
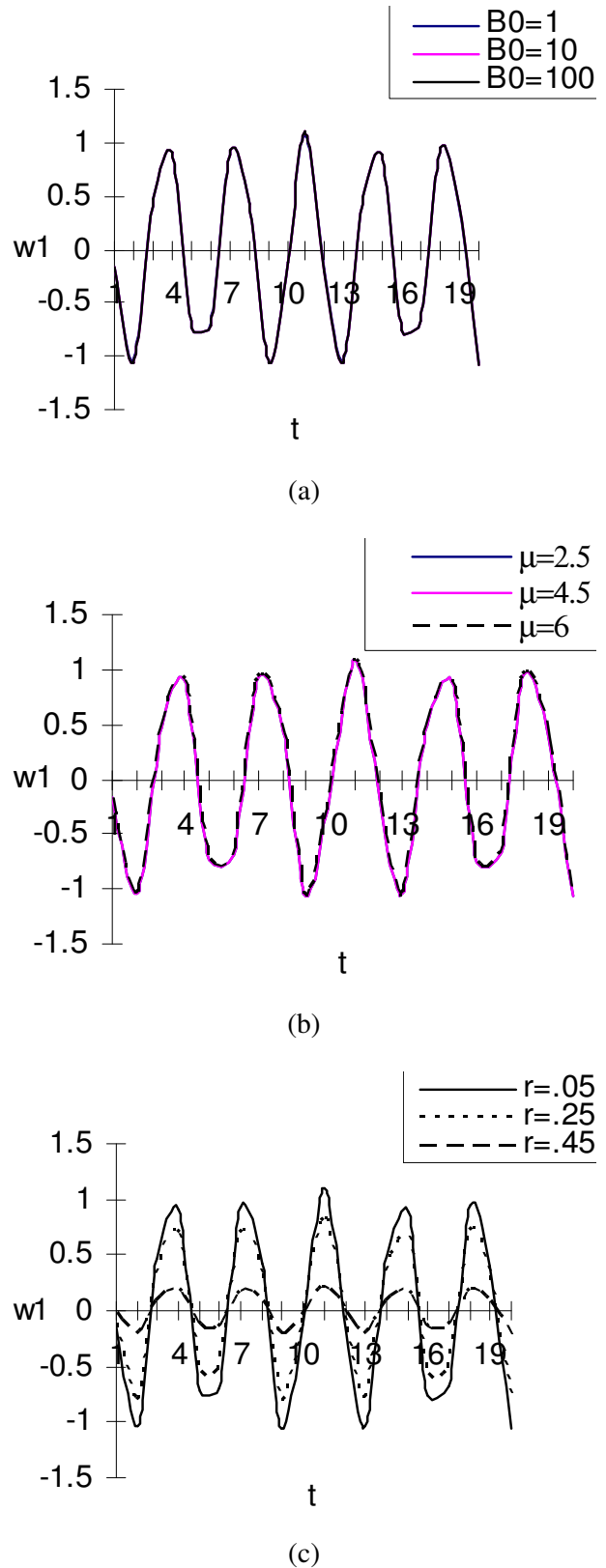

(a)

(b)

(c)

Figure 2. Velocity profiles for different values of (a) B_0, (b) viscosity (μ) and (c) r.

Figures 3a-c display leak-back flow rate for different values of μ and B_0. It is noted that the leak-back flow rate increases as B_0 increases. For the values of B_0 from 10 to 100, the flow rate increases slightly but beyond these values, the flow rate is independent of B_0. But on decreasing the value of B_0 (<10), the leak-back flow rate suddenly declines. For the increasing viscosity the leak-back flow rate retards significantly.

Figures 4a-c displays the effect of thickness (gap between RBC and tube wall) of fluid on leak-back for different values of μ. The leak-back flow rate increases as h increases whereas on increasing viscosity, the flow rate decreases. This demonstrates that the effect of thickness and viscosity on flow rate is remarkable. Figures 5a-c exhibit skin friction vs. time for different values of B_0 and μ the skin friction decreases (increases) as B_0 (μ). At $B_0 = 100$ and beyond this value the skin friction is zero. Thus the effects of B_0 and viscosity on skin friction can not be ignored.

Figures 6a-c displays the profiles for skin friction for different values of h and μ. The skin friction at the surface of RBC decreases as the thickness (gap between RBC and tube wall) of lubricating zone increases. However the increasing viscosity enhances the skin friction.

Overall from numerical simulation performed, we conclude:

- The velocity flow of lubricating zone relative to the tube wall in narrow capillaries is periodic and almost independent of time, viscosity and B_0.
- The velocity flow relative to the tube wall increases towards RBC surface.
- The effect of B_0 on leak-back flow rate is worth mentioning as higher values for B_0 enhance the leak-back flow rate.
- The viscosity effect within the lubricating zone is worth noting with respect to the increasing thickness of lubricating zone.
- The skin friction at RBC surface decreases as B_0 and h increase.

5. CONCLUDING REMARKS

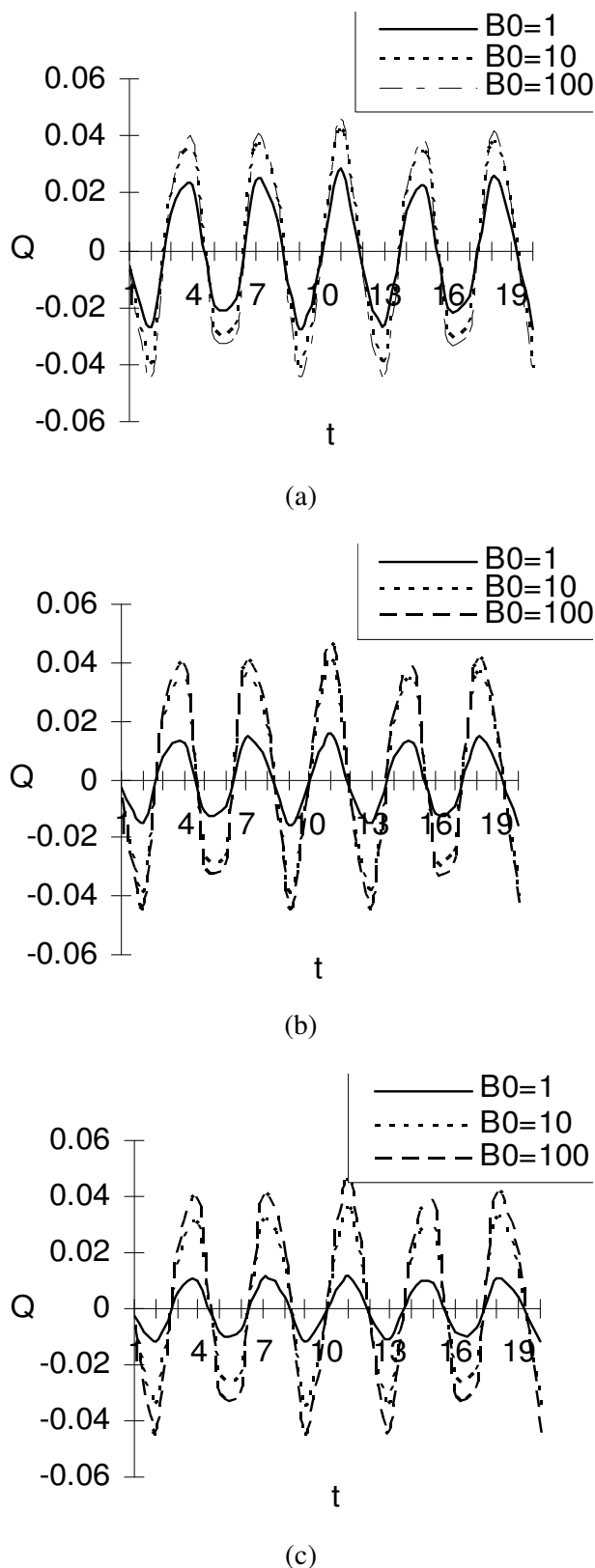In this investigation, we have developed a transient
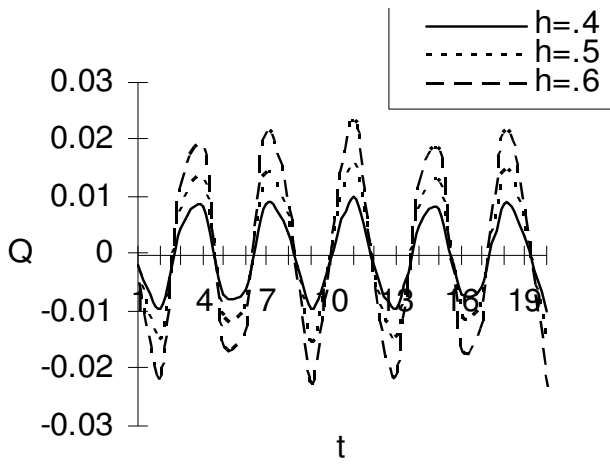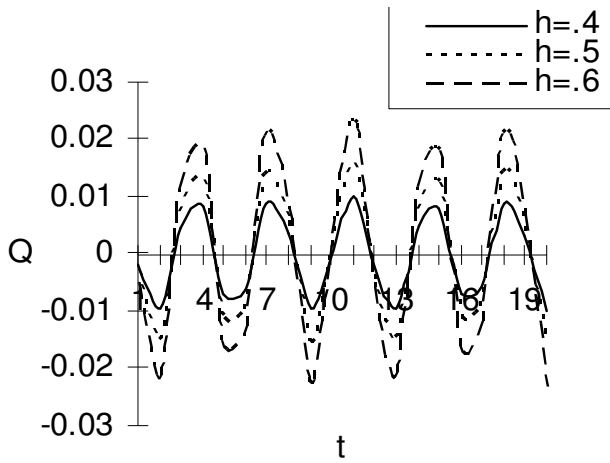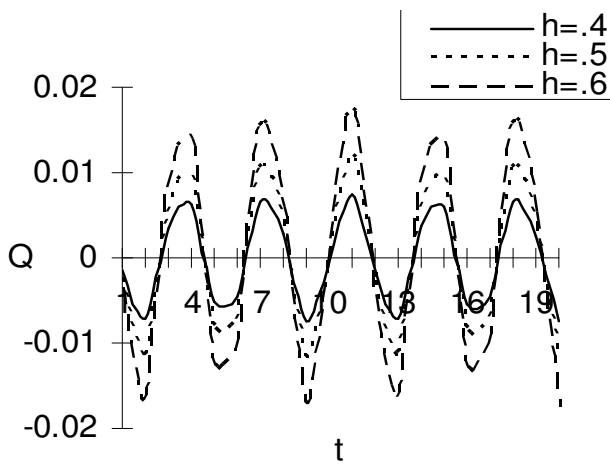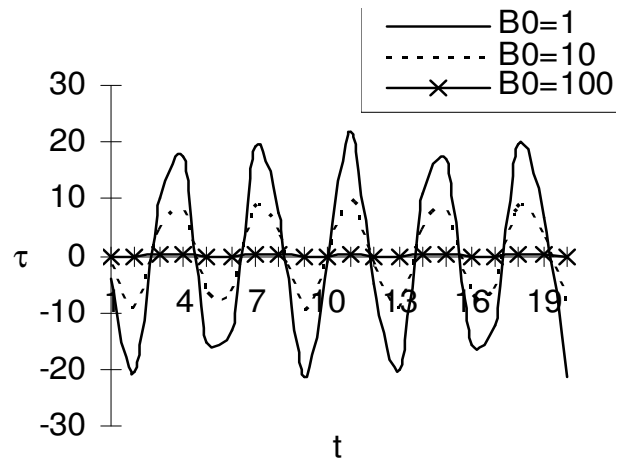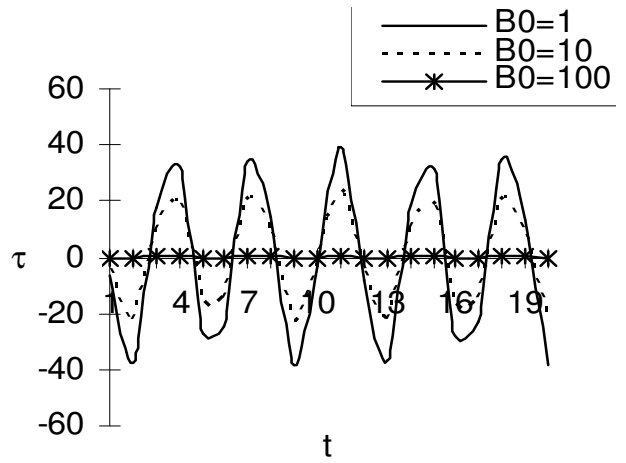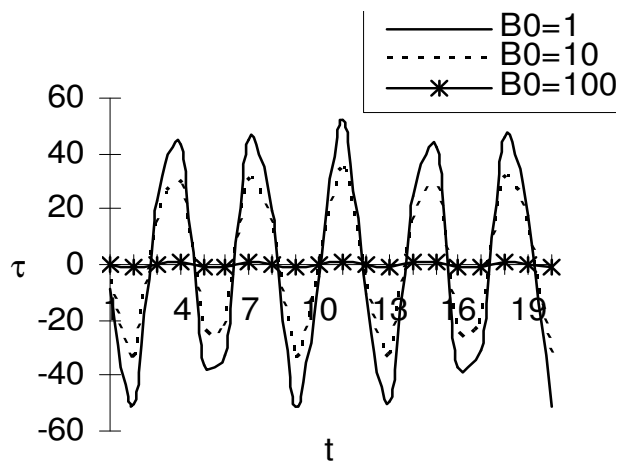

(a)

(b)

(c)

Figure 3. Leak-back flow rate for different values of B_0, and for (a) $\mu = 2.5$, (b) $\mu = 4.5$ and (c) $\mu = 6$.

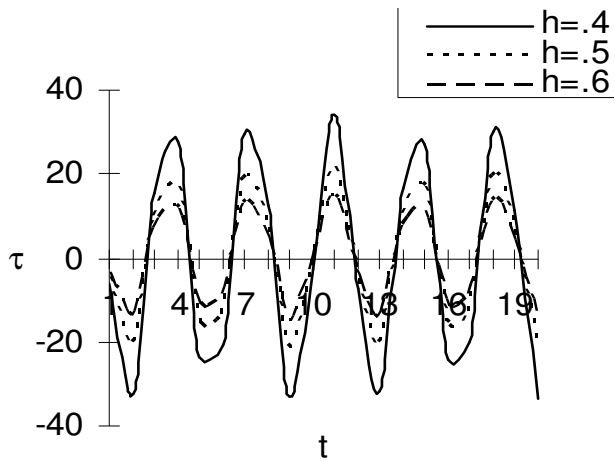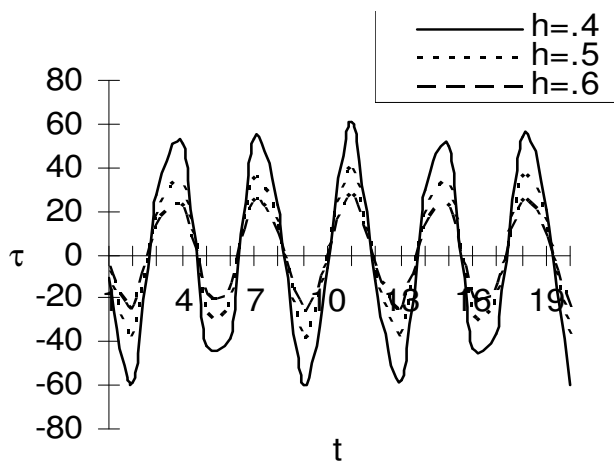**Figure 4**. Leak-back flow rate for different values of h, and for (a) μ = 2.5, (b) μ = 4.5 and (c) μ = 6.

Figure 5. Skin friction vs time for different values of B_0, and for (a) μ = 2.5, (b) μ = 4.5 and (c) μ = 6.

mathematical model for blood flow in very narrow capillaries. The velocity and volumetric flow rate are calculated and validated by numerical results. We make the sensitivity analysis to examine the effect of various parameters. It is found that the magnetic field has no effect on velocity flow in lubricating zone relative to the tube wall and it is also independent viscosity. The effect of magnetic field on leak-back flow rate and skin friction seems to be significant. The viscosity effect is very much dependent on the thickness of the lubricating zone (thickness between RBC and tube wall). The magnetic field and thickness of lubricating zone affect the skin friction at RBC surface remarkably.

(a)

(b)

(c)

Figure 6. Skin friction vs time for different values of h, and for (a) μ = 2.5, (b) μ = 4.5 and (c) μ = 6.

6. REFERENCES

1. Prothero, J. and Burton, A.C., "The Physics of Blood in Capillaries", the Nature of Motion, *Biophys. J*., Vol. 1, (1961), 567-579.
2. Weiderhielm, C.A., Woodbury, J.W., Kirk, S. and Rushmer, R.F., "Pulsatile Pressure in Microcirculation of the Frog's Mesentery", *Amer. J. Physi*., Vol. 207, (1964), 173-176.
3. Lighthill, M.J., "Pressure Forcing of Tightly Fitting Pellets Along Fluid Filled Elastic Tubes, *J. Fluid Mech*., Vol. 34, (1968), 113-143.
4. Fitz-Gerald, J.M., "Mechanics of Red Cell Motion through Very Narrow Capillaries", *Proc. Roy. Soc*. London, U.K., Vol. 174, No. 1035, (1969), 193-227.
5. Zweifach, B.W. and Lipowsky, H.H., "Quantitative Studies of Microcirculatory Structure and Function, Mcrovascular Hemodynamics of Cat Mesentery and Rabbit Omentum", *Circul. Res*., Vol. 41, No. 3, (1977), 380-390.
6. Caro, C., Pedley, G., Schroter, T.J. and Seed, W.A., "The Mechanics of the Circulation", Oxford University Press, New York, U.S.A., (1978).
7. Tozern, H. and Skalak, R., "The Steady Flow of Closely Fitting in Incompressible Elastic Spheres in a Tube", *J. Fluid Mech*., Vol. 87, (1978), 1-16.
8. Tozern, H. and Skalak, R., "Flow of Elastic Compressible Spheres in Tubes", *J. Fluid Mech*., Vol. 95, No. 4, (1979), 743-760.
9. Ozkayan, A.N., "Viscous Flow of Particles in Tubes: Lubrication Theory and Finite Element Model", *Disser. Abs. Int. Part B: Scie. Eng*., Vol. 47, No. 7, (1987), 270.
10. Pries, A.R., Neuhaus, D. and Gaehtgens, P., "Blood Viscosity in Tube Flow: Dependence on Diameter and Hematocrit", *Amer. J. Physiol. Heart Circ. Physiol*., Vol. 263, No. 6, (1992), 1770-1778.
11. Bishop, J.J., Popel, A.S., Intaglietta, M. and Johnson, P.C., "Rheological Effects of Red Blood Cell Aggregation in the Venous Network: A Review of Recent Studies", *Biorheol*., Vol. 38, No. 2-3, (2001), 263-274.

12. Sharan, M. and Popel, A.S., "A Two Phase Model for Flow of Blood in Narrow Tubes with Increased Effective Viscosity Near The Wall", *Biorheol.*, Vol. 38, No. 5-6, (2001), 415-428.

13. Haik, Y., Pai, V. and Chen, C.J., "Apparent Viscosity of Human Blood in a High Static Magnetic Field", *J. Magnetism Magn. Mater.*, Vol. 225, (2001), 180-186.

14. Pries, A.R. and Secomb, T.W., "Rheology of the Microcirculation", *Clin. Hemorheol. Microcir.*, Vol. 29 (3-4), (2003), 143-148.

15. Baumgartner, A.W., Peterson, A.J., Presson, R.G., Tanabe, N., Jaryszak, E.M. and Wagner, W.W., "Blood Flow Switching Among Pulmonary Capillaries is Decreased During High Hematocrit", *J. Appl. Physiol.*, Vol. 97, (2004), 522-526.

16. Sharma, G.C., Jain, M. and Kumar, A., "Performance Modeling and Analysis of Blood Flow in Elastic Artery", *Math. Comp. Modelling*, Vol. 39, (2004), 1491-1499

17. Mittal, N., Zhou, Y., Linares, C., Ung, S., Kaimovitz, B., Molloi, S. and Kassab, G.S., "Analysis of Blood Flow in the Entire Coronary Arterial Tree", *Am. J. Physiol. Heart Cir. Physiol.*, Vol. 289, (2005), H439-H446.

18. Huo, Y. and Kassab, G. S., "Pulsatile Blood Flow in the Entire Coronary Arterial Tree: Theory and Experiment", *Am. J. Physiol. Heart Cir. Physiol.*, Vol. 291, (2006), H1074-H1087.

19. Aviles, M.O., Ebnes, A.D. and Ritter, J.A., "Ferromagnetic Seeding for the Magnetic Targeting of Drugs and Radiation in Capillaries Beds", *J. Magnetism Magn. Mater.*, Vol. 310, (2007), 131-144.