RESEARCH NOTE

OSCILLATORY MHD FLOW OF BLOOD THROUGH AN ARTERY WITH MILD STENOSIS

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Abstract The purpose of this work is to study the effect of oscillatory MHD blood flow in stenosed artery. The analytical and numerical results are obtained for oscillatory MHD blood flow, which is assumed to be a Newtonian fluid. It was also assumed that the surface roughness is of cosine shaped and the maximum height of roughness is negligible, compared with the radius of un-constricted tube. The fluid mechanics of MHD blood flow in a stenosed artery is studied through a mathematical analysis, and the impact of magnetic effect on the instantaneous flow rate is discussed, which reduces, if we increase the Hartman number.

Keywords Oscillatory MHD Blood Flow, Stenosed Artery

چکیده هدف از این تحقیق بررسی نوسانات اثر جریان خون MHD در شریان آئورتی است و آنالیز عددی نتایج حاصل برای نوسانات جریان خون که سیال نیوتونی است، بررسی شد. زبری و ناهمواری سطح دیوارهٔ شریان آئورتی کوسینوسی فرض شده است و حداکثر ناهمواری نسبت به لوله‌های غیرمتمرکز قابل اغماض است. مکانیک سیالات MDH خون در شریان‌های آئورتی با معادلات ریاضی بررسی شد. اثر مغناطیسی شدت جریان لحظه‌ای مورد بحث قرار گرفت که این شدت جریان با افزایش عدد هارتمن کاهش یافته است.

1. INTRODUCTION

There are many investigations for blood flow in a stenosed artery but only a few have considered oscillatory MHD flow, yet never gone through the computational approximation. In this experiment the effort is made to approximate an analysis for such problem.

Many works are available but Womersley [1] discussed the oscillatory motion of a viscous fluid in a rigid tube under a simple harmonic pressure gradient. They observed the influence of frequency on the instantaneous flow rate, while Newmann [2] discussed the oscillatory flow in a rigid tube with stenosis. Imacda, et al [3] made an analysis of non linear pulsatile blood flow in arteries, and Mishra, et al [4] investigated the flow in arteries in the presence of stenosis. Mann, et al [5] discussed the flow of non-Newtonian blood analog fluids in rigid, curved and straight artery models, while Sud [6] worked on the simulation of steady cardiovascular flow in the presence of stenosis using a finite element method. Taylor, et al [7] also discussed the finite element modeling of three-dimensional pulsatile flow in the abdominal aorta: Relevance to atherosclerosis, while Waters, et al [8] discussed the oscillatory flow in a tube of time dependent curvature part-1 perturbation to flow in a stationary curved tube.

Qiu, et al [9] investigated the numerical simulation of pulsatile flow in a compliant tube model of a coronary artery, and Zhang, et al [10] studied the blood constitutive parameters in different blood constitutive equations, although Secomb, et al [11] discussed the blood flow and red blood cell deformation in non uniform capillaries effects of the endothelial surface layer. Anand, et al [12] worked on a shear thinning visco-elastic fluid model for describing the flow of blood. Steinman, et al [13] studied the flow imaging and computing the large artery hemodynamics. Kumar, et al [14] numerically worked on the study of the axi-symmetric blood flow in a constricted rigid tube, while Bali, et al [15] observed the effect of a magnetic field on the resistance of blood flow through stenotic artery. Therefore on the basis of above information, an effort was made to study the oscillatory MHD flow of blood through an artery with mild stenosis. Here, the impedance is a complex quantity and its absolute value normalized to the study value which is the inverse value of the maximum flow rate Q_{max} relative to the study flow rate.

2. MATHEMATICAL MODEL

In the present analysis, the artery is considered to be a circular, cylindrical and rigid tube. Let (r,z,t) be the co-ordinate system as z-axis is taken along the axis of the artery and r -axis is taken along the radius of the artery. A laminar MHD flow of blood, which is assumed to be Newtonian in character, is considered, through an artery with mild stenosis. We also consider the density and viscosity of blood to be constant, and the cylindrical geometry of stenosis, in the arterial segment is given by:

$$\frac{R(z)}{R_0} = 1 - \frac{\in}{2R_0}\left(1 + \cos\frac{\pi z}{d}\right) \quad (1)$$

Where R(z) is the artery radius in stenosis region, R_0 is the radius of the normal artery, d is the semi length of the stenosis, and $\in$ is the maximum height of the stenosis, such that $\frac{\in}{R_0} << 1$ (Figure 1).

The governing equation for analyzing this model, in which we are also introducing magnetic field, is:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\upsilon}{{R_0}^2}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\} + \frac{\upsilon}{{R_0}^2}\beta R_0^2 \frac{\sigma}{\mu} u$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \lambda\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\} + \lambda M^2 u \quad (2)$$

Where u is the velocity axial direction, υ is the kinamatics viscosity, ρ is the density, μ is the viscosity, p is the fluid pressure, $\lambda = \upsilon / {R_0}^2$, $\beta^2 = \frac{\rho R_0^2 \omega}{\mu}$ and $M^2 = \beta R_0^2 \frac{\sigma}{\mu}$,

While the boundary conditions are as:

$$\left.\begin{array}{lll} u = 0 & \text{on} & r = \frac{R}{R_0} \\ \frac{\partial u}{\partial r} = 0 & \text{on} & r = 0 \end{array}\right| \quad (3)$$

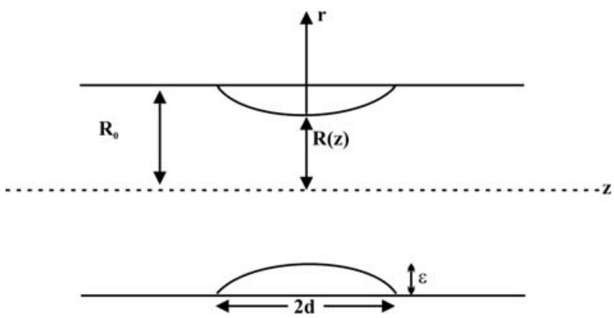

Figure 1. Cylindrical geometry of stenosis in an artery.

Now let's assume the following expression for the solution purpose of the problem:

$$u(r,t)=\bar{u}(r)e^{i\omega t}, \qquad -\frac{\partial p}{\partial z}=Pe^{i\omega t} \tag{4}$$

$$\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}-i\left(\frac{\rho R_0^2\omega}{\mu}+i\beta R_0^2\frac{\sigma}{\mu}\right)\bar{u}=-\frac{R_0^2}{\mu}P \tag{5}$$

Now Equation 5 may also be written as:

$$\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}-i\left(\beta^2+iM^2\right)\bar{u}=-\frac{R_0^2}{\mu}P$$

$$\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}-i\alpha^2\bar{u}=-\frac{R_0^2}{\mu}P \tag{6}$$

Where

$$\alpha^2=\beta^2+iM^2$$

and the corresponding boundary conditions along with the expression (4) are:

$$\bar{u}=0 \quad \text{at} \quad r=\frac{R}{R_0}$$

$$\frac{d\bar{u}}{dr}=0 \quad \text{at} \quad r=0 \tag{7}$$

The solution of Equation 6 subject to boundary conditions (7):

$$\bar{u}(r)=\frac{PR_0^2}{i\mu\alpha^2}\left[1-\frac{J_0\left(\frac{\alpha r}{R_0}i^{3/2}\right)}{J_0\left(\frac{\beta R}{R_0}i^{3/2}\right)}\right] \tag{8}$$

Where J_0 is the Bessel function of order zero with complex argument.

Then the resulting expression for the axial velocity:

$$u(r,t)=\frac{PR_0^2}{i\mu\alpha^2}\left[1-\frac{J_0\left(\frac{\alpha r}{R_0}i^{3/2}\right)}{J_0\left(\frac{\beta R}{R_0}i^{3/2}\right)}\right]e^{i\omega t} \tag{9}$$

Following the notation of McLachlan [16] given by

$$J_0\left(z i^{3/2}\right)=M_0(z)e^{i\theta_0(z)}$$

This can also be written as:

$$u(r,t)=\frac{PR_0^2M_0}{\mu\alpha^2}\left[\sin\left(\omega t+\epsilon_0\right)-i\cos\left(\omega t+\epsilon_0\right)\right] \tag{10}$$

Where

$$\epsilon_0=\tan^{-1}\left[\frac{h_0\sin\phi}{1-h_0\cos\phi}\right],$$

$$\phi=\theta_0\left(\alpha R/R_0\right)-\theta_0\left(\alpha r/R_0\right),$$

$$M_0=\left[1+h_0^2-2h_0\cos\phi\right]^{1/2},$$

$$h_0=M_0\left(\alpha r/R_0\right)/M_0\left(\alpha R/R_0\right)$$

Now the expression for axial velocity (if the real part of simple harmonic pressure gradient is $P\cos\omega t$) is:

$$u(r,t)=\frac{PR_0^2M_0}{\mu\alpha^2}\sin\left(\omega t+\epsilon_0\right) \tag{11}$$

and the volumetric flow rate:

$$Q=\frac{\pi R_0^4P}{i\mu\alpha^2}\left(\frac{R}{R_0}\right)\left[\frac{R}{R_0}-\frac{2J_1\left(\frac{\alpha R}{R_0}i^{3/2}\right)}{i^{3/2}J_0\left(\frac{\alpha R}{R_0}i^{3/2}\right)}\right]e^{i\omega t} \tag{12}$$

Flow rate for pressure gradient $P\cos\omega t$ is:

$$Q=\frac{nPR_0^4M_1}{\mu\alpha^2}\left(\frac{R}{R_0}\right)\sin\left(\omega t+\epsilon_1\right) \tag{13}$$

Where

$$\epsilon_1 = \tan^{-1}\left[\frac{h_1 \sin\theta}{\left(R/R_0 - h_1 \cos\psi\right)}\right],$$

$$M_1 = \left[\left(\frac{R}{R_0}\right)^2 + h_1^2 - 2\left(\frac{R}{R_0}\right)h_1 \cos\psi\right]^{1/2},$$

$$h_1 = \frac{2M_1\left(\frac{\alpha R}{R_0}\right)}{\alpha M_1\left(\frac{\alpha R}{R_0}\right)},$$

$$\psi = \frac{3\pi}{4} - \theta_1\left(\frac{\alpha R}{R_0}\right) + \theta_0\left(\frac{\alpha R}{R_0}\right)$$

Wall shear stress:

$$\tau_R = \mu\left(\frac{\partial u}{\partial r}\right)_{r=R} \tag{14}$$

3. RESULTS AND DISCUSSION

Let's consider $\frac{2d}{1} = 1$ and $\frac{R}{R_0} = 1 - \frac{\epsilon}{R_0}$ for the numerical solution of the problem. Now since the frequency parameter α plays an important role in the flow pattern, therefore we are explaining walls' shear stress and flow rate with the help of parameter α.

Figure 2 shows the variation of walls' shear stress with frequency for different values of stenosis height. It's observed that for a constant value of frequency parameter α the wall shear stress $|\tau|$ increases with increasing the stenosis height (ε/R_0), in other words shear stress increases along with the stenosis height.

Figure 3 shows the variation of instantaneous flow rate with frequency for different values of stenosis height. It was also observed that for a particular value of the frequency parameter $|\alpha|$, the flow rate decreases with increasing stenosis height ϵ/R_0.

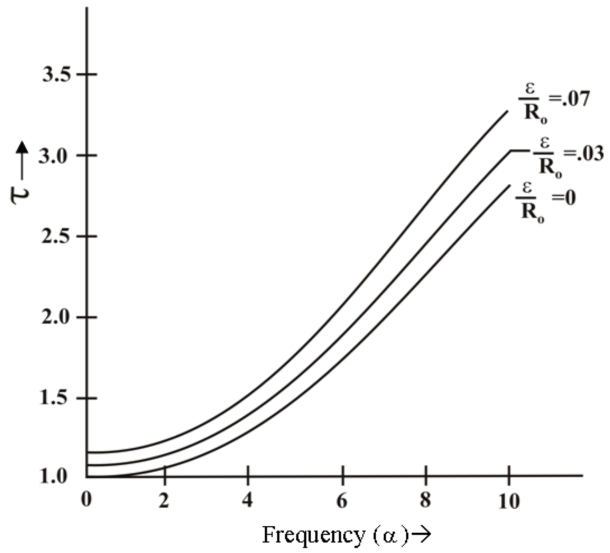

Figure 2. Variation of wall shear stress with frequency parameter α for different values of stenosis height.

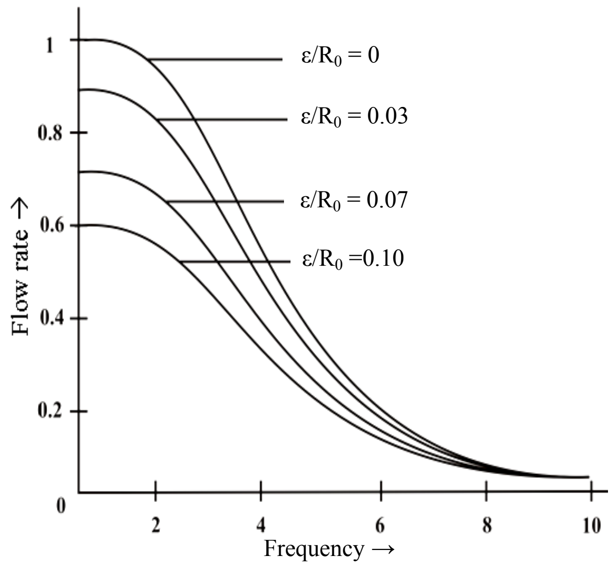

Figure 3. Variation of instantaneous flow rate with frequency for different value of stenosis height.

The deviation between any two consecutive curves is approximately constant in the range $0 \le |\alpha| \le 1$ and beyond this range it decreases remarkably for all values of $|\alpha|$ which fall on the steep falling parts of the curve.

Figure 4 shows the variation of instantaneous flow rate in the absence of Hartmann number i.e. magnetic field and in the presence of Hartmann number. In case of no magnetic field the result is the same, which is given by Haldar [17] in the oscillatory flow work of blood in a stenosed artery and in the presence of magnetic field, the increase of Hartmann numbers decreases the variation of instantaneous flow rate and vice versa.

4. CONCLUSIONS

In order to understand the abnormal flow conditions of blood in a locally constricted blood vessel, the analytical and numerical results are obtained for oscillatory MHD blood flow, which is assumed to be a Newtonian fluid. Here assumed that the surface roughness is cosine-shaped and the maximum height of the roughness is very small compared with the radius of the un-constricted tube. Numerical solutions are presented for the instantaneous flow rate, walls' shear stress and instantaneous flow rate with frequency in the absence and also the presence of Hartmann number for different value of stenosis height.

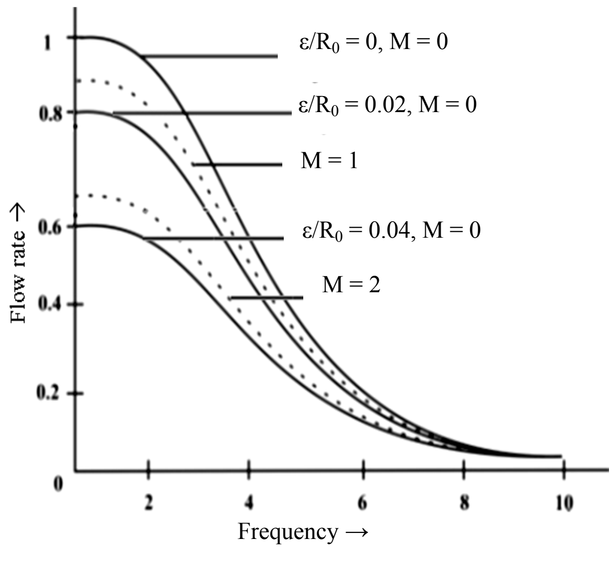

Figure 4. Variation of instantaneous flow rate with frequency in the absence and in the presence of hartmann number.

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