A NEW APPROACH FOR BUCKLING AND VIBRATION ANALYSIS OF CRACKED COLUMN

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Abstract In this paper mathematical formulation for buckling analysis of a column and vibration analysis of a beam is presented. The beam and the column is assumed to be non-uniform and cracked. Using calculus of variations, the problem is expressed as an optimization problem. A technique of optimization is used for analysis of buckling load. Considering the similarity between the governing equation for buckling and free vibration, the fundamental frequency and mode shape of the beam is computed with the same method. Several examples are solved and the results are compared with other methods, as a result an excellent agreement was obtained.

Keywords Buckling, Vibration, Cracked, Non-Uniform, Beam, Column

چکیده در این مقاله برای بر رسی کمانش ستون و ارتعاش آزاد تیر با مقطع متغیر و ترک خورده فرمول سازی ریاضی انجام شده است. با استفاده از حساب تغییرات تحلیل به صورت یک مساله بهینه سازی در آمده است. در حل مساله از روش های شناخته شده بهینه سازی استفاده گردیده است. با توجه به تشابه بین معادلات حاکم بر کمانش ستون و ارتعاش آزاد تیر مد ارتعاشی تیر بر حسب مد کمانشی ستون گسترش یافته است. با حل چند مساله نمونه و مقایسه نتایج با نتایج حاصل از روش های دیگر صحت و دقت کار به تایید رسیده است.

1. INTRODUCTION

The presence of a crack in a structural member introduces a local flexibility that affects its dynamic response. The changes in dynamic characteristics can be measured and led to an identification of structural changes, which eventually might lead to the detection of a structural flaw. A wealth of analytical, numerical and experimental investigations now exists. References more related to the present study are cited here.

T.G. Chondros, et al [1] developed a continuous cracked beam vibration theory for the lateral vibration of cracked Euler-Bernoulli beams with single-edge or double-edge open cracks. They used the Hu-Washizu-Bar variational formulation to develop the differential equation and the boundary conditions of the cracked beam as a one-dimensional continuum. The displacement field about the crack was used to modify the stress and displacement field throughout the bar. A steel beam with a double-edge crack was investigated and results compared well with existing experimental data. They extended their theory for a beam with breathing crack [2].

E. I. Shifrin, et al [3] proposed a technique for calculating natural frequencies of a vibrating beam with arbitrary number of transverse open cracks. The main feature of their method is related to decreasing the dimension of the matrix involved in the calculation, so that reduced computation time is required for evaluating natural frequencies compared to alternative methods.

Vibration of beams with multiple open cracks subjected to axial force is studied by Baric Binici, et al [4]. He proposed a method to obtain the eigenfrequencies and mode shapes of beams containing multiple cracks and subjected to axial force. The method uses one set of end conditions at crack locations; mode shape functions of remaining parts are determined. Other set of boundary conditions yields a second-order determinant that needs to be solved for its roots. He considered both vibration and buckling load of the structure.

M. Behzad, et al [5] based on Hamilton principle developed the equation of motion and corresponding boundary conditions for bending vibration of a beam with an open edge crack. The natural frequencies of a uniform Euler-Bernoulli beam have been calculated using the new developed model in conjunction with the Galerkin projection method. Buckling of multi-step crack columns with shear deformation is studied by Q. S. Li, et al [6]. The governing differential equation for buckling of one-step cracked column with shear deformation is established and its solution is found first. Then a new approach that combines the exact buckling solution of a one-step column and the transfer matrix method (TMM) is presented for solving the entire and partial buckling of a multi-step column with various end conditions, with or without cracks and shear deformation, subjected to concentrated axial load. The main advantage of the proposed method is that the eigen-value equation for buckling of a multi-step column with an arbitrary number of cracks, any kind of two end supports and various spring supports at intermediate points can be conveniently determined from a system of two linear equations. He later [7], extended the method and proposed classes of exact solutions for buckling of multi-step non-uniform columns with an arbitrary number of cracks subjected to concentrated and distributed axial loads.

Ranjbaran,et al proposed a method for computation of fundamental eigen-pairs of shear building [8]. The fundamental mode was obtained as minimum of a quotient defined in terms of strain energy and moment of inertia as a function of lateral displacement. The mode shape is obtained as minimizer displacement function of the quotient.

In the present paper and in most references it is cited that, the crack is modeled as a rotational spring with specified flexibility [9]. Based on calculus of variations, the problem of computation of buckling load columns and vibration of beams, are defined as an optimization of a quotient. The minimizer function of the quotient is the mode shape and its minimum value is buckling load or fundamental frequency of the beam. As compared to others the proposed method is simple, more efficient and accurate.

2. THEORETICAL BASIS

2.1. Crack Model A crack in a column is modeled as a rotational spring with flexibility coefficient, C, defined as [9]:

$$C = 5.346Hf(\xi), \quad \xi = \frac{a}{H} \tag{1}$$

In which H is the column section height and a is the depth of the crack. The function f is defined as:

$$f(\xi) = 1.862\xi^2 - 3.95\xi^3 + 16.375\xi^4 - 37.22\xi^5 + 76.81\xi^6 - 126\xi^7 + 172\xi^8 - 143.97\xi^9 + 66.56\xi^{10} \tag{2}$$

The spring inserts a jump in the rotation of column centerline at the cracked point, i.e. as:

$$\frac{d\theta}{dx} = \frac{d\phi}{dx}\frac{d^2y}{dx^2}, \quad \text{OR} \quad \Delta\theta = C\frac{d^2y}{dx^2} = C\frac{M}{EI}, \quad C = d\phi \tag{3}$$

Where θ is rotation and φ is a potential function.

2.2. Buckling of A Column The equilibrium equation for a non-uniform cracked column in a displaced position is expressed as:

$$L^2y^{//} + qry = 0, \quad y(0) = y(L) = 0 \tag{4}$$

In which prime (/) denote differentiation with respect to x and:

$$q = \frac{PL^2}{EI_c}, \quad r = \left(1 + \frac{d\phi}{dx}\right)\frac{EI_c}{EI} \tag{5}$$

Where E is elastic modulus, I is second moment of area, L is length, q is a working parameter and c in the subscript denotes a specified section. The ratio r is selected for modeling non-uniform sections.

A functional in form of a quotient is defined as follows:

$$q = \frac{L^2 \int_0^L y'^2 dx}{\int_0^L r y^2 dx} = \frac{\int_0^L f_1 dx}{\int_0^L f_2 dx} \tag{6}$$

In which f_1 and f_2 are selected for better reference. According to calculus of variations, the Euler-Lasgrange equation corresponding to Equation 6 is written as:

$$F = -\left(f_{1y} - q_0 f_{2y}\right) + \frac{d}{dx}\left(f_{1y'} - q_0 f_{2y'}\right) = 0 \tag{7}$$

In which the letter in the subscript denotes differentiation and q_0 is minimum value of the quotient q. Substitution from Equation 6 into Equation 7 leads to the following equation:

$$F = -\left(0 - 2qry\right) + \frac{d}{dx}\left(2L^2 y' - 0\right) = 0 \rightarrow L^2 y'' + qry = 0 \tag{8}$$

Equation 8 clearly shows that the minimizer of quotient q is the displacement function y that satisfies the governing equation and corresponding boundary conditions. As a result the problem of computation of buckling load of the columns may be expressed as the following minimization problem:

$$\text{Minimize}: q = \frac{L^2 \int_0^L y'^2 dx}{\int_0^L r y^2 dx} \rightarrow y, q_0 \tag{9}$$

$$\text{Subject to}: y(0) = y(L) = 0$$

For numerical computation, the column is divided into n segments connected at n + 1 nodes. Finite difference scheme is used in place of derivatives. The quotient q may be expressed in terms of nodal values as follows:

$$q = \frac{L^2 \sum_{\alpha=1}^{n-1} y_\alpha \left(2y_\alpha - y_{\alpha-1} - y_{\alpha+1}\right)}{h^2 \sum_{\alpha=1}^{n-1} r_\alpha y_\alpha^2 dx}, \tag{10}$$

$$y_0 = y_n = 0, \quad h = \frac{L}{n}$$

The BFGS algorithm is used for carrying out the minimization process of quotient q in Equation 10 [10].

After finding the quotient q, the critical load P, ratio of critical load, P, to Euler load, Pe, i.e. R_P, and effective length factor, K, is computed as follows:

$$P = q\frac{EI_C}{L^2}, R_P = \frac{P}{P_e} = \frac{q}{\pi^2}, K = \frac{1}{\sqrt{R_P}} = \frac{\pi}{\sqrt{q}} \tag{11}$$

Example 1. In order to verify the formulation for non-uniform column a simply supported column with variable cross section is selected for analysis. The following parameters are used:

$$I_x = I_0 \left(\frac{x+a}{a}\right)^M$$

In which I_0, I_c, and I_x are second moment of area at sections o, c, and x respectively and M is a power number.

Solution

For this problem the ratio r_i may be shown as:

$$r_i = \left[\sqrt[M]{R_0} + 2\left(1 - \sqrt[M]{R_0}\right)\left(\frac{x_i}{L}\right)\right]^{-M}, \quad R_0 = \frac{I_0}{I_c}$$

Note that because of symmetry for half-length of the column r_i is computed as above and for the other half symmetry is used. The critical load of the column is denoted as:

$$P_{cr} = \frac{4mEI_c}{L^2}$$

This problem is solved in reference [11] for M = 2 and 4. The problem is solved for R0 = 0.1, 0.2,..., 1.0 and M = 4. The results of the present study and that of reference [11] are compared in Figure1. Excellent agreement of results obtained.

Example 2. A simply supported column of length L, crack depth a, and crack position βL with a rectangular cross section of 20 cm height and 15 cm width is considered. The elastic modulus is 200 Gpa. The buckling load is computed and its variation versus crack depth ratio for different length (slenderness ratio) is shown in Figure 2. The crack is at mid-height of the column. For L = 2 the results of Transformation Matrix Method (TMM) is also shown for comparison.

2.3. Vibration Analysis of A Beam The governing equation for free vibration of a beam is

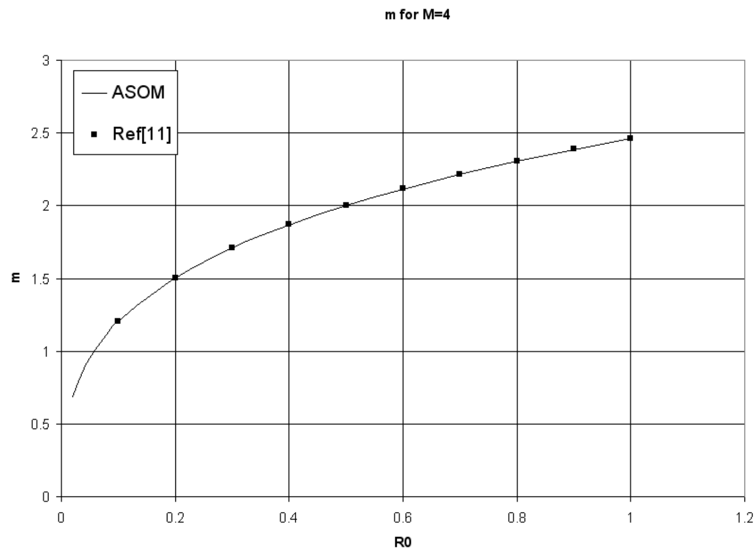

Figure 1. Variation of coefficient m versus R_0.

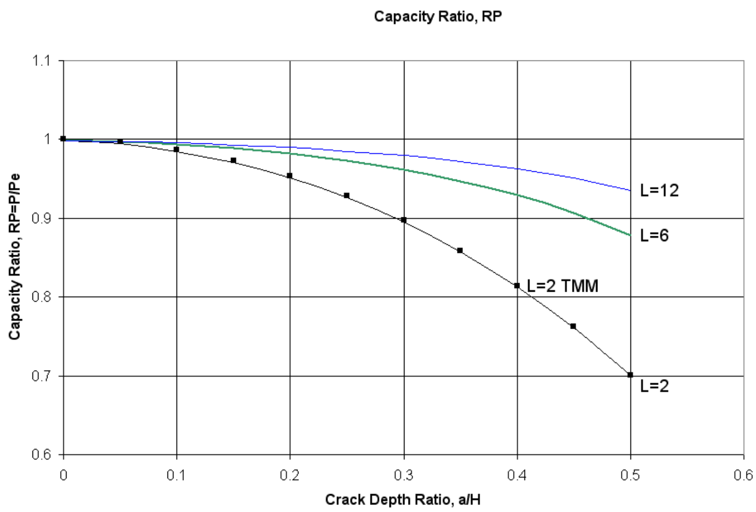

Figure 2. The capacity ratio of a cracked column.

similar to that of buckling load of a column except in the name of parameters. Based on this similarity the circular frequency may be obtained by using $P \approx \rho A(L/\pi)^2\omega^2$ where ρ is mass density, A is cross sectional area and ω is circular frequency. As a result the circular frequency of a beam is obtained as:

$$\omega^2 = \frac{q}{\pi^2}\left(\frac{\pi}{L}\right)^4 \frac{EI_C}{\rho A_c}, \quad \omega_c^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI_C}{\rho A_c} \qquad (12)$$

And the ratio of ω^2 to ω_c^2, i.e. R_ω is defined as:

$$R_\omega = \frac{\omega}{\omega_c} = \frac{\sqrt{q}}{\pi} = \sqrt{R_P} \qquad (13)$$

In case of uniform members no other consideration is needed. For the case of non-uniform column the ratio r should be defined as follows:

$$r = \left(1 + \frac{d\phi}{dx}\right)\frac{EI_c}{EI}\frac{A}{A_c} = \left(1 + \frac{d\phi}{dx}\right)\frac{Er_{gc}^2}{Er_g^2}, \quad r_g^2 = \frac{I}{A} \qquad (14)$$

Example 3. The same member as in example 2 is considered but now as a beam. The variation of frequency ratio versus crack depth ratio for different lengths is shown in Figure 3.

3. CONCLUSIONS

A new, efficient and accurate method for vibration analysis of a non-uniform cracked beam is presented. Based on the calculus of variations the theoretical basis of the method is developed. First computation method for analysis of buckling load of a non-uniform column is presented. Considering the similarity between the governing equations of buckling analysis of columns and free vibration analysis of beams the latter was formulated. To verify the proposed method several examples are presented. Results of this study are compared with others and an excellent agreement was obtained. Compared to others, the volume of formulations, the amount of computer algorithm needs, and also the time of computations, is considerably less. At the same time the presented method is quite general.

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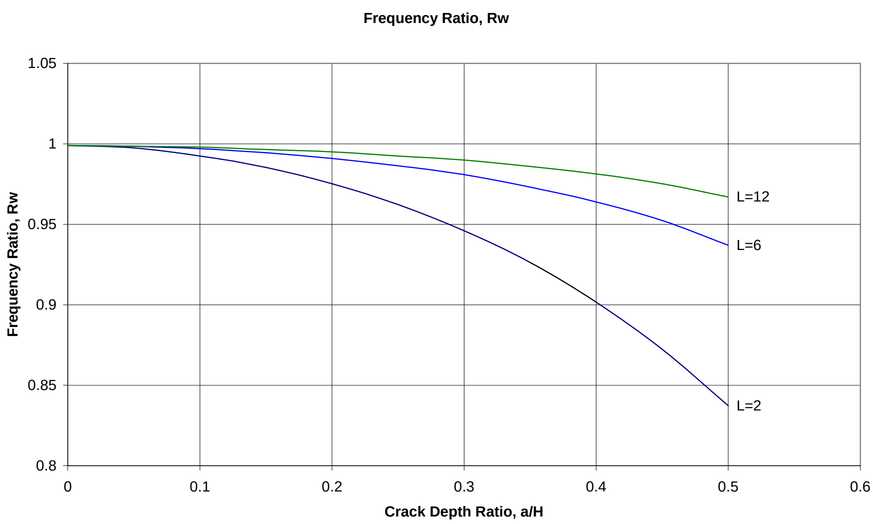

Figure 3. The capacity ratio of a cracked column.

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