ANALYSIS OF NATURAL FREQUENCIES FOR A LAMINATED COMPOSITE PLATE WITH PIEZOELECTRIC PATCHES USING THE FIRST AND SECOND EIGENVALUE DERIVATIVES

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Abstract In this paper, the first and second order approximations of Taylor expansion are used for calculating the change of each natural frequency by modifying an arbitrary parameter of a system with a known amount and based on this approximation, the inverse eigenvalue problem is transformed to a solvable algebraic equation. The finite element formulation, based on the classical laminated plate theory (CLPT) is presented for laminated composite plates with piezoelectric patches. Using the proposed FE model, sensitivity analysis is carried out, to find the effects of the changes made in the design parameters such as the piezoelectric patch thickness and the fiber angles in each layer on the natural frequencies of the structure. The inverse eigenvalue problem is solved in order to find the thickness of piezoelectric patches and stacking sequence for relocating the natural frequencies.

Keywords Composite, Piezoelectric, Natural Frequency, Sensitivity Analysis, Inverse Eigenvalue Problem

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چکیده   در این مقاله تقریب های مرتبۀ اول و دوم از بسط تیلور برای محاسبۀ تغییر هر یک از فرکانس های طبیعی به واسطۀ ایجاد تغییر دلخواهی در هر یک از پارامترهای سازه استفاده شده اند و با استفاده از این بسط ها مسئلۀ مقدار ویژۀ معکوس به حل یک دسته معادلات جبری تبدیل شده است. روابط المان محدود برای تحلیل یک ورق کامپوزیت که دارای لایه یا قطعاتی از پیزوالکتریک است بر پایۀ تئوری کلاسیک ورق استخراج شده اند. با استفاده از روابط المان محدود ارائه شده، برای هر یک از فرکانس های طبیعی آنالیز حساسیت برای یافتن مؤثرترین پارامترهای سازه (مانند زاویه الیاف هر لایه یا ضخامت هر قطعۀ پیزوالکتریک) بر روی فرکانس انجام شده است. در نهایت مسئلۀ مقدار ویژۀ معکوس برای یافتن چیدمان مناسب لایه ها و ضخامت هر یک از قطعات پیزوالکتریک برای تعیین تغییرات دلخواه در فرکانس های طبیعی حل شده است.

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1. INTRODUCTION

Dynamic behavior modification is an active area of research due to its several applications in structural design/optimization. In contrast with the direct eigenvalue problem (or finding the modal characteristics of a known structure), the inverse eigenvalue problem (finding the necessary changes in structural parameters for achieving predefined

modal behavior) deals with several difficulties. Traditional methods for solving the inverse eigenvalue problem are restricted to trial and error iterative methods. Fox, et al [1] proposed exact expressions for derivatives of eigenvalues and eigenvectors with respect to any design variables for simple un-damped vibratory systems. This was rapidly developed to more complicated cases such as damped systems with complex eigenvalue and

eigenvectors [2,3]. Based on these derivatives many researches were carried out on formulating the inverse eigenvalue problem for different classes of structures. The earliest works in this field were restricted to modification of stiffness and mass matrixes in order to achieve desired shifts in natural frequencies for simple systems approximated with mass-spring systems [4]. Studying the inverse eigenvalue problem for continuous systems needs the mathematical or physical discretization. The finite element formulation is the most common method in modeling structures as continuous systems. So, formulating the inverse eigenvalue problem based on FE may be very efficient. The inverse eigenvalue problem in conjunction with FE was first performed for beam and bar elements [5,6] and consequently was developed for two dimensional elements [7,8] and more complicated structures like composite laminated plates [9] or functionally graded material (FGM) plates covered with piezoelectric layers [10]. In this paper, the inverse eigenvalue problem or finding the necessary changes in geometrical or physical properties of a structure in order to achieve desired changes in natural frequencies and relocating them in a favorable manner, is investigated and presented for a laminated composite plate with piezoelectric patches using the first and second order approximations of Taylor expansion.

2. CONSTITUTIVE EQUATIONS FOR PIEZOELECTRIC LAMINATES

For a composite plate consists of arbitrary layers (including the piezoelectric layers), the constitutive equation for the k_{th} layer of the laminate is expressed as

$$
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} -
$$

$$
\begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}
$$

(1a)

$$
\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{31} & e_{32} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} -
$$

$$
\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}
$$

(1b)

Which relates the stress vector $\{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}\}^T$ to the strain vector $\{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}\}^T$ through the elastic constants matrix, [C]. In these equations, E_i is the component of the electric field vector, D_i refers to the electric displacement vector components, the quantities C_{ij} and e_{ij} represent the elastic and the piezoelectric constants, respectively and k_{ij} is the electric permittivity.

When the poling direction of the piezoelectric layer coincides with the thickness direction, the components of the electric field vector can be expressed as gradient of electric potential φ in the thickness direction as

$$
E_i = \frac{\partial \varphi}{\partial x_i}
$$

(2)

By assuming the electric potential applying and varying linearly only in the thickness direction Equation 2 is written as

$$
\begin{Bmatrix} 0 \\ 0 \\ E_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \varphi/t \end{Bmatrix}
$$

(3)

Where t is the piezoelectric layer thickness.

3. DISPLACEMENT AND STRAIN FIELDS

By defining vector $\{\bar{u}\}$ as

$$
\{\bar{u}\} = \{u_0 \quad v_0 \quad w_0 \quad \partial w_0/\partial x \quad \partial w_0/\partial y\}^T
$$

(4)

Where, u_0, v_0 and w_0 are the mid-plane displacements, the displacement components based on the classical laminated plate theory (CLPT) may be expressed as [11]

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -z & 0 \\ 0 & 1 & 0 & 0 & -z \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \{\bar{u}\} = [H]\{\bar{u}\} \qquad (5)$$

The strains associated with these displacements are

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \qquad (6a)$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} \qquad (6b)$$

$$\gamma_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - 2z\frac{\partial^2 w_0}{\partial x \partial y} \qquad (6c)$$

4. FINITE ELEMENT FORMULATION

A four node plate element with five degrees of freedom in each node containing u_0, v_0, w_0, $\partial w_0/\partial x$, $\partial w_0/\partial y$ is chosen. The terms u_0 and v_0 are interpolated in terms of nodal values inside the element by means of bilinear Lagrangian interpolation functions as

$$u_0(x,y,t) \cong \sum_{j=1}^{nn} u_j^e(t)\,\psi_j^e(x,y) \qquad (7)$$

$$v_0(x,y,t) \cong \sum_{j=1}^{nn} v_j^e(t)\,\psi_j^e(x,y) \qquad (8)$$

and for approximating the term w_0 inside the element according to nodal values, the Hermit interpolation is used as

$$w_0(x,y,t) \cong \sum_{k=1}^{n} \Delta_k^e(t)\,\varphi_k^e(x,y) \qquad (9)$$

Where u_j, v_j, w_j are the nodal values, nn is the

number of nodes per element, Δ_k^e is $\{w_0, \partial w_0/\partial x, \partial w_0/\partial y\}$ vector in each node, ψ and φ are the Lagrange and Hermit interpolation functions respectively [11].

In addition to five degrees of freedom in each node, the elements that contain piezoelectric layers on the top and bottom surfaces have two electric potential degrees of freedom. Now the continuous displacement field in the element can be defined in terms of nodal variables as

$$\{\hat{u}\} = [H][N_u]\{u^e\} \qquad (10)$$

Where $\{u^e\}$ is a vector that contains all nodal displacements. $[N_u]$ is the structural shape function that for a four nodded element is expressed as

$$[N_u] = [[N_{u1}] \vdots [N_{u2}] \vdots [N_{u3}] \vdots [N_{u4}]] \qquad (11)$$

and each of the sub-matrices is

$$[N_{ui}] = \begin{bmatrix} \psi_i & 0 & 0 & 0 & 0 \\ 0 & \psi_i & 0 & 0 & 0 \\ 0 & 0 & g_{i1} & g_{i2} & g_{i3} \\ 0 & 0 & \dfrac{\partial g_{i1}}{\partial x} & \dfrac{\partial g_{i2}}{\partial x} & \dfrac{\partial g_{i3}}{\partial x} \\ 0 & 0 & \dfrac{\partial g_{i1}}{\partial y} & \dfrac{\partial g_{i2}}{\partial y} & \dfrac{\partial g_{i3}}{\partial y} \end{bmatrix} \quad i=1,2,3,4$$

$$(12)$$

Where the terms ψ_i are the linear interpolation functions and g_{ij} are the non-conforming Hermit cubic interpolation functions [11].

The electrical potential is defined in terms of the layer variables as

$$\{\varphi\} = [N_\varphi]\{\varphi^e\} \qquad (13)$$

Where $\{\varphi^e\}$ is a vector that contains the electrical potentials in piezoelectric layers and $[N_\varphi]$ is the electrical shape function.

Using the interpolation expressed in Equation 10, strain components can be obtained through the derivatives of Equation 6 as:

$$\{\varepsilon\varepsilon = [B_u]\{u^e\}$$ (14)

The matrix $[B_u]$ for a four nodded element is expressed as

$$[B_u] = [[B_{u1}] \vdots [B_{u2}] \vdots [B_{u3}] \vdots [B_{u4}]]$$ (15)

Where

$$[B_{ui}] =$$

$$\begin{bmatrix} \dfrac{\partial \psi_i}{\partial x} & 0 & -z\dfrac{\partial^2 g_{i1}}{\partial x^2} & -z\dfrac{\partial^2 g_{i2}}{\partial x^2} & -z\dfrac{\partial^2 g_{i3}}{\partial x^2} \\[3mm] 0 & \dfrac{\partial \psi_i}{\partial y} & -z\dfrac{\partial^2 g_{i1}}{\partial y^2} & -z\dfrac{\partial^2 g_{i2}}{\partial y^2} & -z\dfrac{\partial^2 g_{i3}}{\partial y^2} \\[3mm] \dfrac{\partial \psi_i}{\partial y} & \dfrac{\partial \psi_i}{\partial x} & -2z\dfrac{\partial^2 g_{i1}}{\partial x \partial y} & -2z\dfrac{\partial^2 g_{i2}}{\partial x \partial y} & -2z\dfrac{\partial^2 g_{i3}}{\partial x \partial y} \end{bmatrix}$$ (16)

and using the interpolation of Equation 13 with the expression of Equation 2 gives

$$\{E\} = [B_\varphi]\{\varphi^e\}$$ (17)

Where

$$[B_\varphi] = -\nabla[N_\varphi].$$

The dynamic equation of motion for a laminated composite plate with piezoelectric layers can be derived from the Hamilton's principle

$$\delta \int_{t_0}^{t_1} (T - U - W)dt = 0$$ (18)

Where the term T denotes the kinetic energy, U represents the strain energy and W is the external work. The term T for a laminated composite plate is expressed as

$$T = \frac{1}{2}\iint\left[\sum_{k=1}^{nl} \int_{h_k}^{h_{k+1}} \rho_k \{\dot{u}^k \quad \dot{v}^k \quad \dot{w}^k\} \times \{\dot{u}^k \quad \dot{v}^k \quad \dot{w}^k\}^T dz\right]dxdy$$ (19)

Where nl represents the number of layers, ρ_k is the density, $(h_{k+1} - h_k)$ is the thickness and $\{\dot{u}^k \quad \dot{v}^k \quad \dot{w}^k\}$ is the velocity vector for the k_{th} layer. The strain energy is expressed as

$$U = \frac{1}{2}\iint\left[\sum_{k=1}^{nl} \int_{h_k}^{h_{k+1}} \{\sigma\}^T \{\varepsilon\}dz\right]dxdy - \sum_{n=1}^{np} \int_{V_p} \{E\}^T \{D\}dv$$ (20)

Where np represents the number of piezoelectric layers. The external work done by the prescribed surface traction f_s, body forces f_b and the applied electrical charge density Q on the piezoelectric layers may be expressed as

$$W = \int_{S_f} \{u\}^T f_s dA + \int_V \{u\}^T f_b dv + \int_{S_p} \{\varphi\}^T QdA$$ (21)

Where S_f and S_p are the surfaces in which the mechanical and electrical loads are applied, respectively. Substituting Equations 19-21 into Equation 18 and using Equations 10, 13, 14 and 17 the governing equations of motion for a laminated composite plate with piezoelectric layers are obtained as

$$[M_{uu}]\{\ddot{u}\} + [K_{uu}]\{u\} + [K_{u\varphi}]\{\varphi\} = \{F_m\}$$ (22a)

$$[K_{\varphi u}]\{u\} - [K_{\varphi\varphi}]\{\varphi\} = \{F_q\}$$ (22b)

Where $\left[M_{uu}\right]$, $\left[K_{uu}\right]$, $\left[K_{u\varphi}\right]$ and $\left[K_{\varphi\varphi}\right]$ are the mass, structural stiffness, coupled structural-electrical stiffness and electric stiffness matrices, respectively. $\{F_m\}$ and $\{F_q\}$ are the applied mechanical and electrical loads.

The matrices and vectors in Equations 22a and 22b are given by

$$[M_{uu}] = \int_V [N]^T [H]^T [\rho][H][N]dv$$ (23a)

$$[K_{uu}] = \int_V [B_u]^T [C][B_u]dv$$ (23b)

$$[K_{u\varphi}] = \int_v [B_u]^T [e]^T [B_\varphi] dv \qquad (23c)$$

$$[K_{\varphi u}] = \int_v [B_\varphi]^T [e][B_u] dv \qquad (23d)$$

$$[K_{\varphi\varphi}] = \int_v [B_\varphi]^T [K][B_\varphi] dv \qquad (23e)$$

$$\{F_m\} = \int_{s_p} [N_\varphi]^T \{q\} ds \qquad (23f)$$

$$\{F_m\} = \int_v [N]^T [H]^T \{f_b\} dv +$$
$$\int_{S_f} [N]^T [H]^T \{f_s\} ds + [N]^T [H]^T \{f_c\} \qquad (23g)$$

Where f_c represents the concentrated force and all the other terms are expressed previously.

Another form of equation of motion can be expressed by eliminating the vector $\{\varphi\}$ between Equations 22a and 22b as

$$[M_{uu}]\{\ddot{u}\} + [K_{eq}]\{u\} = \{F_m\} + [K_{u\varphi}][K_{\varphi\varphi}]^{-1} \{f_q\} \qquad (24)$$

Where

$$[K_{eq}] = [K_{uu}] + [K_{u\varphi}][K_{\varphi\varphi}]^{-1}[K_{\varphi u}] \qquad (25)$$

5. THE EIGENVECTOR AND EIGENVALUE DERIVATIVES

The standard eigenvalue problem for an n-degree of freedom un-damped system may be formulated as

$$([K] - \zeta[M])\{y\} = 0 \qquad (26)$$

Where $[K]$, $[M]$, ζ and $\{y\}$ are the stiffness matrix, mass matrix, the eigenvalue and the eigenvector, respectively. Differentiating Equation 26 with respect to any arbitrary parameter and pre-multiplication by the term $\{y\}^T$ gives

$$\{y\}^T [K]\{y\} + \{y\}^T [K]\{y\}' =$$
$$\zeta'\{y\}^T [M]\{y\} + \zeta\{y\}^T [M]'\{y\} + \zeta\{y\}^T [M]\{y\}' \qquad (27)$$

Where, the symbol (') represents derivative with respect to the arbitrary parameter. Transposing both sides of Equation 26 and post-multiplying both sides by the term by $\{y\}'$ gives

$$\{y\}^T [K]\{y\}' = \zeta\{y\}^T [M]\{y\}' \qquad (28)$$

Using the orthogonality condition

$$\{y\}^T [M]\{y\} = 1 \qquad (29)$$

and substituting Equations 29 and 28 into Equation 27 gives

$$\zeta' = \{y\}^T [K]'\{y\} - \zeta\{y\}^T [M]'\{y\} \qquad (30)$$

Equation 30 can be rewritten as

$$\frac{\partial \zeta^i}{\partial b_j} =$$
$$\frac{\partial K_{mn}}{\partial b_j} y_m^i y_n^i - \zeta^i \frac{\partial M_{mn}}{\partial b_j} y_m^i y_n^i \quad m,n = 1,2,...,N \qquad (31)$$

Where N is the number of DOFs, b_j can be any of physical or geometrical properties of the structure and the superscript i represents the i_{th} mode of vibration. Note that, no summation is implemented on the upper indices.

Equation 31 can be used for performing the initial sensitivity analysis for finding the regions within the structure that changing the parameter b_j has the most influence on the i_{th} natural frequency.

The derivative of Equation 31 with respect to the arbitrary parameter b_k can be expressed as

$$\frac{\partial^2 \zeta^i}{\partial b_j \partial b_k} = 2 \frac{\partial y_r^i}{\partial b_k} (\frac{\partial K_{rs}}{\partial b_j} - \zeta^i \frac{\partial M_{rs}}{\partial b_j}) y_s^i +$$
$$y_r^i (\frac{\partial^2 K_{rs}}{\partial b_j \partial b_k} - \zeta^i \frac{\partial^2 M_{rs}}{\partial b_j \partial b_k} - \frac{\partial \zeta^i}{\partial b_k} \frac{\partial M_{rs}}{\partial b_j}) y_s^i \qquad (32)$$

It is obvious that the second derivative of each eigenvalue depends on the first derivative of the corresponding eigenvector. For calculating the eigenvector derivative, the modal superposition method is used. Rewriting Equation 26 gives

$$(K_{mn} - \zeta M_{mn}) \overset{i}{y}_n = 0 \qquad m,n = 1,2,...,N \qquad (33)$$

The derivative of Equation 33 with respect to b_k is

$$(K_{mn} - \zeta M_{mn}) \frac{\partial \overset{i}{y}}{\partial b_k} = -\frac{\partial (K_{mn} - \zeta M_{mn})}{\partial b_k} \overset{i}{y} \qquad (34)$$

Approximating the eigenvector derivative by a truncated series of the low-frequency mode shapes gives

$$\frac{\partial \overset{i}{y}_n}{\partial b_k} = a_{ikl} \overset{l}{y}_n \qquad (35)$$

Where summation is implemented on the upper index l. Substituting Equation 34 in 35 and Multiplying both sides by the term $\overset{p}{y}_m$, in the case of $p \neq i$ gives

$$a_{ikl} (\overset{p}{y}_m K_{mn} \overset{l}{y}_n - \zeta \overset{p}{y}_m M_{mn} \overset{l}{y}_n) = \\ -\overset{p}{y}_m \frac{\partial (K_{mn} - \zeta M_{mn})}{\partial b_k} \overset{i}{y}_n \qquad (36)$$

From the orthogonality condition, if l = p then the left side of Equation 36 is zero, and in the case of $l \neq p$ Equation 36 is simplified to

$$a_{ikp} = \frac{\overset{p}{y}_m \left(\frac{\partial K_{mn}}{\partial b_k} - \overset{i}{\zeta} \frac{\partial M_{mn}}{\partial b_k} \right) \overset{i}{y}_n}{(\overset{i}{\zeta} - \overset{p}{\zeta})} \qquad (37)$$

Equation 37 gives all of the coefficients a_{ikl} in equation 35 except the terms a_{iki}. For calculating these terms the orthogonality of the eigenvector

with respect to the mass matrix is used as

$$\overset{i}{y}_m M_{mn} \overset{i}{y}_n = 1 \qquad (38)$$

The derivative of Equation 38 with respect to b_k using the truncated series of Equation 35 can be written as

$$2 \overset{i}{y}_m M_{mn} a_{ikl} \overset{l}{y}_n = -\overset{i}{y}_m \frac{\partial M_{mn}}{\partial b_k} \overset{i}{y}_n \qquad (39)$$

The left side of Equation 39 is nonzero only for i = l, in this case

$$a_{iki} = \frac{1}{2} \overset{i}{y}_m \frac{\partial M_{mn}}{\partial b_k} \overset{i}{y}_n \qquad (40)$$

6. THE DIRECT AND INVERSE APPROXIMATED EIGENVALUE PROBLEMS

The first and second derivatives of eigenvalue were calculated in the previous section. Using these derivatives, Taylor expansion may be used for approximating the change of natural frequencies due to an arbitrary change in physical or geometrical properties of structure. In addition to the direct problem, the inverse eigenvalue problem can be formulated based on the Taylor expansion by solving a set of linear or quadratic equations.

Defining the matrix S as

$$S_{ij} = \frac{\partial \overset{i}{\zeta}}{\partial b_j} \qquad (41)$$

Then, the Taylor expansion by using only the first term can be expressed as

$$\Delta \overset{i}{\zeta} = \frac{\partial \overset{i}{\zeta}}{\partial b_j} \Delta b_j \qquad (42)$$

or

$$\{\Delta\zeta\} = [S]\{\Delta b\} \tag{43}$$

Equation 42 represents the direct approximated eigenvalue problem. The inverse eigenvalue problem using the first approximation in Taylor expansion may be formulated as

$$\{\Delta b\} = [S]^{-1}\{\Delta\zeta\} \tag{44}$$

The Taylor expansion using the first two terms is expressed as

$$\Delta\overset{i}{\zeta} = \frac{\partial\overset{i}{\zeta}}{\partial b_j}\Delta b_j +$$

$$\frac{\partial\overset{i}{\zeta}}{\partial b_k}\Delta b_k + \frac{1}{2}[\frac{\partial^2\overset{i}{\zeta}}{\partial b_j^2}(\Delta\Delta_j)^2 + \frac{\partial^2\overset{i}{\zeta}}{\partial b_k^2}(\Delta\Delta_k)^2 + \tag{45}$$

$$\frac{\partial^2\overset{i}{\zeta}}{\partial b_j\partial b_k}(\Delta\Delta_j\Delta b_k) + \frac{\partial^2\overset{i}{\zeta}}{\partial b_k\partial b_j}(\Delta\Delta_j\Delta b_k)]$$

Where b_j and b_k can be any of the physical or geometrical parameters. In a particular case that the change of eigenvalue by changing only one design parameter is studied, Equation 45 can be simplified to

$$\Delta\overset{i}{\zeta} = \frac{\partial\overset{i}{\zeta}}{\partial b_j}\Delta b_j + \frac{1}{2}\frac{\partial^2\overset{i}{\zeta}}{\partial b_j^2}(\Delta\Delta_j)^2 \tag{46}$$

Equation 46 can be rewritten in the matrix form as

$$\{\Delta\zeta\} = [S1]\{\Delta b\} + \frac{1}{2}[S2]\{\Delta b^2\} \tag{47}$$

Where

$$S1_{ij} = \frac{\partial\overset{i}{\zeta}}{\partial b_j} \qquad S2_{ij} = \frac{\partial^2\overset{i}{\zeta}}{\partial b_j^2}$$

The direct approximated eigenvalue problem is formulated by setting the parameters Δb_j to known values in Equation 46 and the inverse problem may be formulated by solving a set of quadratic algebraic equations when the parameters $\Delta\overset{i}{\zeta}$ are set to known values and the parameters Δb_j are unknown.

7. NUMERICAL RESULTS AND DISCUSSION

7.1. Verification of the Finite Element Model In order to verify the accuracy of the proposed finite element model and self developed computer programs, three case studies are taken into consideration: (a) an 8-layer graphite/epoxy cross-ply laminate [90/0/90/0/0/90/0/90]; (b) a PZT-5A piezoelectric layer at the top surface of an 8-layer graphite/epoxy cross-ply laminate [90/0/90/0/0/90/0/90/p]; (c) an 8-layer graphite/epoxy laminate covered with two PZT-5A piezoelectric layers at the top and bottom surfaces [p/90/0/90/0/0/90/0/90/p]. The plates are square of side length L = 1 and the side-to-thickness ratio L/h = 100. All the layers (the piezoelectric and composite material layers) have the same thickness, the material properties are:

$E_1 = 181$Gpa, $E_2 = 10.3$ GPa, $G_{12} = 7.17$ GPa, $G_{23} = 2.87$ GPa $\nu_{12} = 0.28$, $\nu_{23} = 0.33$, $\rho = 1580$Kg/m^3, $E_3 = E_2$, $G_{13} = G_{12}$.

The piezoelectric material properties are shown in Table 1.

The results for the fundamental frequency of plate with simply supported boundary conditions on all sides obtained by the presented FE model are compared with the results of two different analytical methods in the literature: a 3D state space uncoupled solution proposed by Xu, et al [12] and a 2D analytical model based on sandwich formulation using layer-wise first order shear deformation theory proposed by Benjeddou, et al [13]. It can be seen that there is a good agreement between the presented results and the previously reported results in the literature.

7.2. The Direct Approximated Eigenvalue Problem A four-layered cantilever plate with cross-ply stacking sequence [0/90/90/0] is

	E (N/m^2)	ν	ρ(kg/m^3)	e_{31}(c/m^2)	e_{32}(c/m^2)	K_{33}(F/m)
G-1195N	63e9	0.3	7600	22.86	22.86	1.5e-8
PZT-5A	61e9	0.35	7750	7.209	7.209	1.5e-8

considered. The plate is square with length set as 0.4_m and the thickness of each layer is set to 1_{mm}. the material properties are:

$E_1 = 132.5$ GPa, $E_2 = 10.8$ GPa, $G_{12} = 5.7$ GPa, $G_{23} = 3.4$ GPa $\nu_{12} = 0.24$, $\nu_{23} = 0.49$, $\rho = 1540$Kg/m^3, $E_3 = E_2$, $G_{13} = G_{12}$.

G-1195N piezoelectric patches are used for bonding the top and bottom surfaces of the laminated plate in a configuration as depicted in Figure 1. The material properties of the piezoelectric layer are shown in Table 2. The thickness of each piezoelectric layer is set to be 0.1_{mm}.

The first ten natural frequencies of the laminated plate before and after adding the piezoelectric patches are shown in Table 3. It can be seen that adding the piezoelectric patches changes the value of natural frequencies.

In the first step, the thickness of each piezoelectric patch is considered as the design
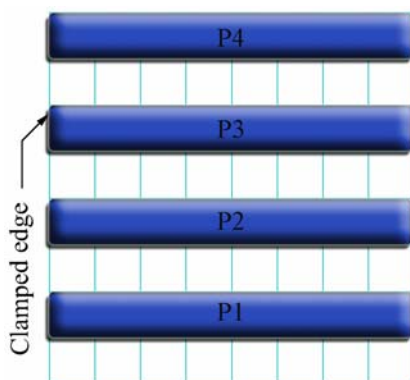
Figure 1. Patch configuration.

variable and the direct eigenvalue problem is solved for investigating the effect of these parameters on each natural frequency. As an example, the change of the first natural frequency due to changing the piezoelectric layer thickness from 0.1_{mm} to 1_{mm} in each patch using the first order approximation is shown in Figure 2.

Figure 3 shows the same results obtained by the second order approximation.

In order to study the accuracy of the proposed approximated method, the change of the first natural frequency due to changing the thickness of the piezoelectric patch number 1 from 0.1mm to 1mm is calculated by FE (as exact solution). Figure 4 compares the results obtained by the first and second order approximations with the exact solution. Figure 5 shows the error of the first and second order approximations for the results depicted in Figure 4. It is obvious that the error of the second order approximation for changing the natural frequency up to 2 % is almost negligible.

In the next step, the fiber angle in each layer is taken into consideration as the design parameter. A four-layered cantilever plate with angle-ply stacking sequence [15/45/75/105] and geometry and material properties like the previous problem is considered. The piezoelectric patch configuration is shown in Figure 1and each piezoelectric layer thickness is set to 0.1mm in all patches. The first and second derivatives of the first four eigenvalues are calculated and the direct approximated eigenvalue problem is solved based on the first and second order approximations in Taylor expansion for finding the change of frequencies due to an arbitrary change in fiber angles. As an example the change of the first natural frequency due to change of fiber angles up to 50° based on the first order approximation is depicted in Figure 6. Figure 7 represents the same

TABLE 2. Comparison of the Results for the Fundamental Frequency (Hz) Obtained by the Proposed FE Model and the Results in the Literature.

Configuration	3D* [12]	2D** [13]	Present Study
(a)	333.02	340.86	337.45
(b)	290.38	300.64	298.80
(c)	268.86	283.93	270.28

* Uncoupled
** Exact sandwich formulation

TABLE 3. The First Ten Natural Frequencies of the Laminated Plate.

| Mode No.	Without Piezoelectric Patches (Hz)	With Piezoelectric Patches (Hz)
1	35.312	34.36
2	47.399	48.52
3	127.03	133.47
4	221.32	215.47
5	235.78	233.24
6	295.87	303.86
7	306.11	322.42
8	437.88	460.44
9	583.02	601.2
10	618.91	613.33

results obtained by the second order approximation.

In order to study the accuracy of the proposed approximated method, Figure 8 compares the results obtained by the first and second order approximations for calculating the change of the first natural frequency due to change of fiber angles in the second layer with the exact results obtained by FE simulation. Figure 9 shows the error of approximated solution for first and second order methods. It is obvious that the error of second order approximation for frequency changes of up to 4 % is negligible.

As another example, consider an arbitrary modification in the stacking sequence from [15/45/75/105] to [25/90/80/105] for the plate with geometry and material properties like the previous example. The change of the first three natural frequencies due to this modification obtained by the first and second order approximations are compared with the results obtained by the exact solution in Table 4. It can be seen that a good agreement is obtained by the proposed approximated methods.

7.3. The Inverse Approximated Eigenvalue Problem As explained previously, the great advantage of the proposed approximated method is the ability of formulating the inverse eigenvalue problem. That means finding the necessary changes in geometrical or physical properties of the structure for achieving the desired changes in natural frequencies. In this section the inverse eigenvalue problem based on the first and second order approximations is investigated. The design
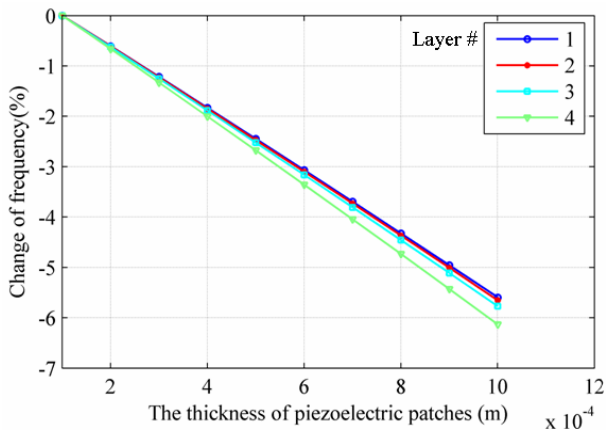
Figure 2. Change of the first natural frequency due to changing the piezoelectric patch thickness using the first order approximation.
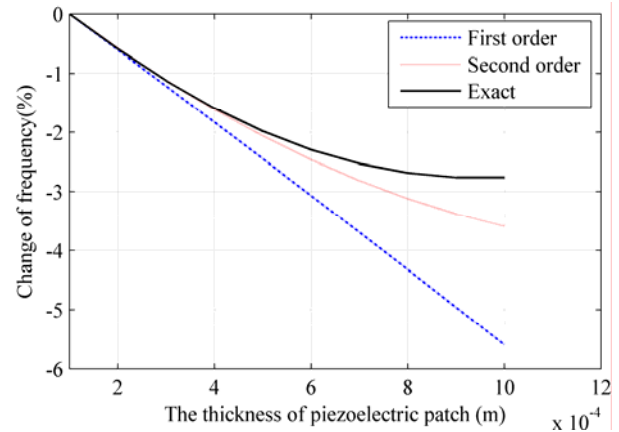
Figure 4. Comparison of the approximated and exact results for the first natural frequency due to changing the thickness of the piezoelectric patch number 1.
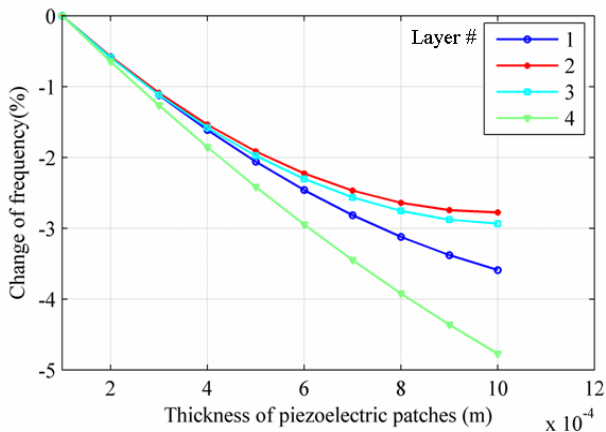
Figure 3. Change of the first natural frequency due to changing the piezoelectric patch thickness using the second order approximation.
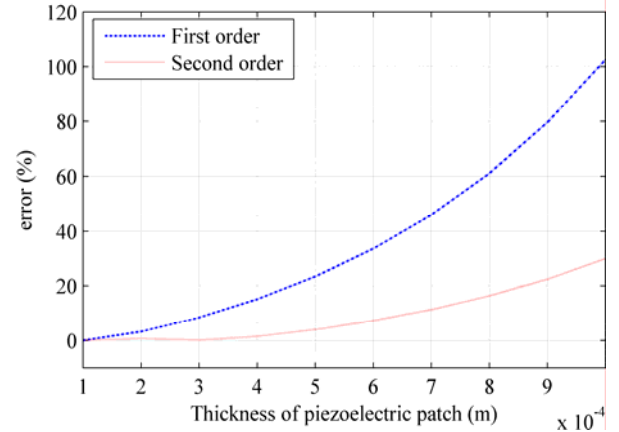

Figure 5. Error in results using the first and second order approximations.

parameter is selected to be the piezoelectric layer thickness in each patch. A four-layered cantilever plate with stacking sequence [0/90/90/0] is considered, the first two natural frequencies of this plate are shown in Table 5. The problem is formulated as finding the necessary change in the thickness of piezoelectric patches P2 and P4 (Figure 1) which causes an arbitrary decrease in the first natural frequency with the amount of 3 % and an increase in the second frequency by 5 %.

Using the first order approximation yields to a linear set of two equations with the parameters

of Δt_2 and Δt_4 as the unknowns which results in changes of $\Delta t_2 = 0.4\text{mm}$ and $\Delta t_4 = 0.1\text{mm}$. Using the second order approximation yields to a set of two quadratic equations which results in $\Delta t_2 = 0.33\text{mm}$ and $\Delta t_4 = 0.2\text{mm}$. By implementing these changes on the initial model the change of the first two natural frequencies are calculated and compared with the desired values in Table 5.

It is obvious that the approximated method can perform the desired modification with a good accuracy.
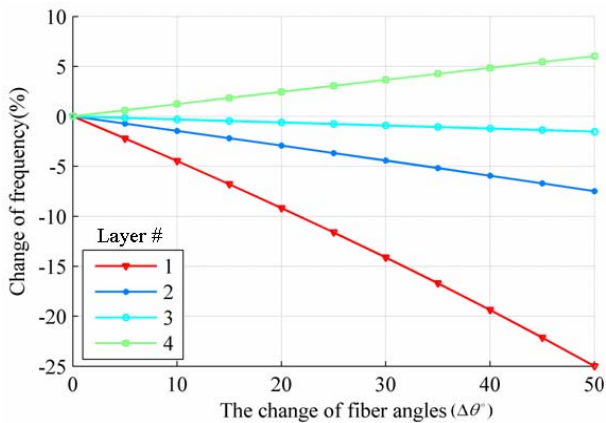
Figure 6. Change of the first natural frequency due to change of fiber angles in different layers based on the first order approximation.
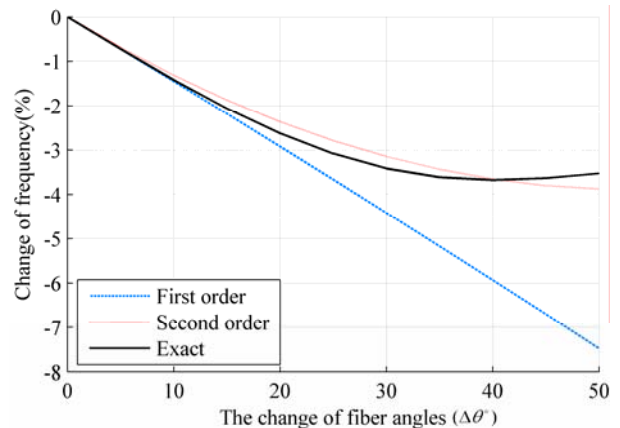
Figure 8. Comparison of the approximated and exact results for the first natural frequency due to change of fiber angles in the second layer.
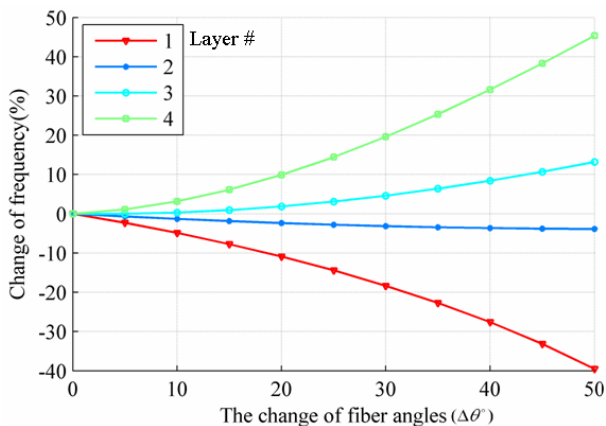
Figure 7. Change of the first natural frequency due to change of fiber angles in different layers based on the second order approximation.
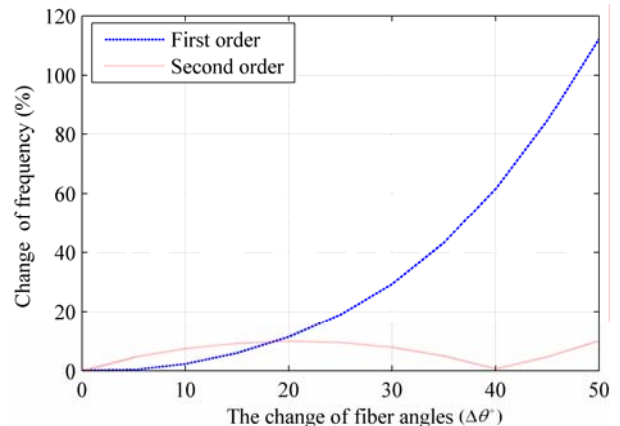
Figure 9. Error in results using the first and second order approximations.

8. CONCLUSION

This paper deals with the modification of free vibration behavior of a laminated composite plate with piezoelectric patches. Based on the classical laminated plate theory, the finite element formulation for composite plate with collocated piezoelectric patches are derived and by using this formulation, the first and second order approximations of Taylor expansion are resulted for solving the direct and the inverse eigenvalue problems. Several numerical examples are carried out to show the application and the accuracy of the present method and an applied design problem of dynamic behavior modification is solved to show the efficiency of the presented method on solving the inverse eigenvalue problem. Studying the results in Table 4, Figure 4 and Figure 8 (which compares the results of the first and second order approximations with the finite element solution) shows the accuracy of the presented algorithm. It is obvious from Table 5 that the presented method is so efficient in dynamic behavior modification, that its being formulated as an inverse eigenvalue problem. The maximum error for the second order approximation in the inverse method is less than 1 %.

TABLE 4. Change of the First three Natural Frequencies (Hz) Due to an Arbitrary Modification in Stacking Sequence.

Mode Num.	Initial*	Modified** (Exact)	Modified (F.O.)	Modified (S.O.)	Error % (F.O.)	Error % (S.O.)
1	18.54	16.71	16.37	16.89	-2.03	1.01
2	43.91	44.02	44.77	44.05	1.7	0.07
3	117.07	106.08	104.9	107.01	-1.11	0.09

F. O: First Order, S. O: Second Order
* [15/45/75/105]
** [25/90/80/105]

TABLE 5. The Natural Frequencies Obtained by the Inverse Eigenvalue Problem Based on the First and Second Order Approximations.

	$\Delta\omega$	Initial (Hz)	Modified (Exact)	Modified (F.O.)	Error % (F.O.)	Modified (S.O.)	Error % (S.O.)
ω_1	-3 %	34.36	33.33	33.54	0.63	33.41	0.24
ω_2	5 %	48.52	50.95	51.35	0.78	50.47	-0.94

F. O: First order, S. O: Second order

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