ON TWO-ECHELON MULTI-SERVER QUEUE WITH BALKING AND LIMITED INTERMEDIATE BUFFER

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Abstract In this paper we study two echelon multi-server tandem queueing systems where customers arrive according to a poisson process with two different rates. The service rates at both echelons are independent of each other. The service times of customers is assumed to be completed in two stages. The service times at each stage are exponentially distributed. At the first stage, the customers may balk (i.e. reject to join the waiting line) when all servers are busy. The higher echelon has a limited buffer space. The steady state queue size distribution has been obtained for both stages. We investigate the properties of a Hessenberg matrix which is required for the complete specification of the generating function for ready use.

Key Words Tandem Queue, Balking, Multi-Server. Finite Buffer, Hessenberg Matrix, Queue Size Distribution

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INTRODUCTION

The problem of tandem queue with finite buffer has received the attention from the early stages of queueing theory. The earliest study made by Avi-Itzhak and Yadin [4] dealt with two single server stations without intermediate queue where customers followed the poisson distribution and the service times were arbitrarily distributed in both stations. This study was extended to a sequence of service stations with arbitrary input and regular service time by Avi-Itzhak [3] to derive waiting time distribution. A two stage single server network with finite intermediate buffer and blocking was attempted by Neuts [16]. He assumed that service times had a general distribution at the first stage and an exponential distribution at the second stage.

Konheim and Reiser [12] obtained an algorithmic solution to the state probabilities of the network having finite waiting room and blocking under Markovian assumptions. Gershwin [8] provided an efficient decomposition method for tandem queue with finite storage space and blocking by approximating the single K machine line by a set of K-1 two machine lines. Altiok [5] presented an approximate analysis of single server queues in series having finite waiting room and blocking under the assumption that service time follows phase-type distribution. Langaris [13] developed the finite set of balance equations for a two stage service system having single server at each stage, finite waiting space in both stages and blocking. Gun and Makowski [10] provided a matrix geometric solution for a two node tandem queueing system with phase type servers subject to blocking and failure.

Chao and Pinedo [7] obtained an expression for expected waiting time in the system for two single server stations in tandem without intermediate buffer where customers arrive in groups and arbitrary service is provided at both stations. Green [9] analysized multi-server system with two types of servers (primary and auxiliary) and two types of customers. The first type of customers are those who are satisfied with a service rendered by a primary server whereas second type customers request for combined service of primary and auxiliary together. Recently, Zuta and yechiali [18] studied a two echelon multi-server Markovian queueing system with a limited intermediate buffer. Abou-El-Ata and Showky [2] studied the single server for longer queues. Furthermore, Abou-El-Ata and Kotb [1] derived the solution of the state dependent queue M/M/1/N with general balk function, reflecting barrier, reneging and an additional server for longer queues. Recently, Mohanty et al. [15] analysed a multi-server queueing system with balking and reneging.

We consider a two stage multi-server queue in tandem with a limited intermediate buffer in which an arriving customer who finds all servers busy may balk or reject to join the queue. There are S_k servers at stage k (k = 1, 2). The service times of customers are exponentially distributed with mean rate m_k (k = 1, 2) at stage k. We further assume that there exists unlimited waiting capacity at the first stage but a finite buffer of size $M-S_2$ is provided at the second stage. At the first stage a customer is served separately at each queue and leaves the system with the probability of q while at the second stage a customer is forced to leave the system with the probability of 1 if the buffer is full.

The two dimensional continuous time Markov chain model has been developed. The co-ordinates of the system state (i, j) represent the number of customers waiting and being served at the corresponding stage respectively. The technique developed by Levy and Yechiali [14] and by Bocharov and Al Bores [6] has been employed to solve the problem.

We obtain expressions for the partial generating functions $\pi_j(z) = \sum_{i=0}^{\infty} P_{i,j} z^i$ to determine the steady state queue size distribution $P_{i,j}$ (i= 0, 1, 2, ...; j= 1, 2, ..., M). M being the maximum total number of customers at the second stage. To determine the partial generating function $\left\{\pi_j(z)\right\}_{j=0}^{m}$ we apply Hessenburg matrix approach which can be obtained from a set of linear equations [A(z) p(z) = b(z)] in the unknown generating functions. To investigate the properties of the Hessenburg matrix, we use M+ 1 roots of the determinant of A(z) lying in interval (0, 1) and the remaining M+1 in (1, È).

THE MODEL

Assume a two echelon multi-server queueing system, where customers arrive to the first lower echelon according to a poisson process with the arrival rate given by

$$\lambda_j = \begin{cases} \lambda & 0 \le j \le s_1 \\ \lambda\beta & s_1 + 1 \le j < \infty \end{cases} \qquad 0 < \beta < 1 \tag{1}$$

The arriving customers are served by S_1 identical servers at first echelon and the service times are exponentially distributed with parameter m_1. After being served first at lower echelon, a customer leaves the system with the probability q or moves on and requests for additional service at the second higher echelon with the probability of p = 1-q and are served by S_2 identical servers. The service times of customers at second echelon are exponentially distributed with parameter m_2.

The intermediate buffer between the two stages is limited to M-S_2. If a customer who requests for service at the higher echelon and finds that the buffer is full, leaves the system with the probability 1. We formulate the system as a two dimensional birth and death process. Let $P_{i,j}$ (i = 1, 2, ...; j = 1, 2, ..., M) be the probability that there are i(j) customers at the first (second) echelon.

THE BALANCE EQUATIONS GOVERNING THE MODEL

For j = 0 :

$$\lambda P_{0,0} = \mu_2 P_{0,1} + q\mu_1 P_{1,0} \quad (2a)$$

$$P_{i,0}(\lambda + i\,\mu_1) = \lambda P_{i-1,0} + \mu_2 P_{i,1} + q(i+1)\mu_1 P_{i+1,0}$$
$$1 \le i \le s_1 - 1 \quad (2b)$$

$$P_{i,0}(\lambda\beta + S_1\mu_1) = \lambda P_{i-1,0} + \mu_2 P_{i,1} + qS_1\mu_1 P_{i+1,0}$$
$$i = S_1 \quad (2c)$$

$$P_{i,0}(\lambda\beta + S_1\mu_1) = \lambda\beta P_{i-1,0} + \mu_2 P_{i,1} + qS_1\mu_1 P_{i+1,0}$$

$$S_1 + 1 \le i < \infty \quad (2d)$$

For $i \le j \le S_2 - 1$:

$$P_{0,j}(\lambda + j\mu_2) = (j+1)\mu_2 P_{0,j+1} + q\mu_1 P_{1,j} + P\mu_1 P_{1,j-1}$$
$$(2e)$$

$$P_{1,j}(\lambda + j\mu_2 + i\mu_i) = \lambda P_{i-1,j} + (j+1)\mu_2 P_{i,j+1}$$
$$+ q(i+1)\mu_1 P_{i+1,j} + p(i+1)\mu_1 P_{i+1,j-1}$$
$$1 \le i \le S_1 - 1 \quad (2f)$$

$$P_{i,j}(\lambda\beta + j\mu_2 + S_1\mu_1) = \lambda P_{i-1,j} + (j+1)\mu_2 P_{i,j+1}$$
$$+ qs_1\mu_1 P_{i+1,j} + pS_1\mu_1 P_{i+1,j-1}$$
$$i = S_1 \quad (2g)$$

$$P_{i,j}(\lambda\beta + j\mu_2 + S_1\mu_1) = \lambda\beta P_{i-1,j} + (j+1)\mu_2 P_{i,j+1}$$
$$+ qS_1\mu_1 p_{i+1,j} + pS_1\mu_1 P_{i+1,j-1}$$
$$S_1 + 1 \le i < \infty \quad (2h)$$

For $S_2 \le j < M - 1$:

$$P_{0,j}(\lambda + S_2\mu_2) = S_2\mu_2 P_{0,j+1} + q\mu_1 P_{i,j} + p\mu_1 P_{1,j-1}$$
$$(2i)$$

$$P_{i,j}(\lambda + S_2\mu_2 + i\mu_1) = \lambda P_{i-1,j} + S_2\mu_2 P_{i,j+1}$$

$$+ q(i+1)\mu_1 P_{i+1,j} + p(i+1)\mu_1 P_{i+1}$$

$$1 \le i \le S_1 - 1 \quad (2j)$$

$$P_{i,j}(\lambda\beta + S_2\mu_2 + S_1\mu_1) = \lambda P_{i-1,j} + S_2\mu_2 P_{i,j+1}$$

$$+ qS_1\mu_1 P_{i+1,j} + pS_1\mu_1 P_{i+1,j-1}$$

$$i = S_1 \quad (2k)$$

$$P_{i,j}(\lambda\beta + S_2\mu_2 + S_1\mu_1) = \lambda\beta P_{i-1,j} + S_2\mu_2 P_{i,j+1}$$

$$+ qS_1\mu_1 P_{i+1,j} + pS_1\mu_1 P_{i+1,j-1}$$

$$S_1 + 1 \le j \langle \infty \quad (2l)$$

For j = M :

$$P_{0,M}(\lambda + S_2\mu_2) = \mu_1 P_{1,M} + p\mu_1 q_{1,M-1} \quad (2m)$$

$$P_{i,M}(\lambda + S_2\mu_2 + i\mu_1) = \lambda P_{i-1,M} +$$

$$(i+1)\mu_1 P_{i+1,M} + p(i+1)\mu_1 P_{i+1,M-1}$$

$$1 \le i \le S_1 - 1 \qquad (2n)$$

$$P_{i,M}(\lambda\beta + S_2\mu_2 + S_1\mu_1) = \lambda P_{i-1,M} +$$

$$S_1\mu_1 P_{i+1,M} + p\, S_1\mu_1 P_{i+1,M-1}$$

$$i = S_1 \qquad (2o)$$

$$P_{i,M}(\lambda\beta + S_2\mu_2 + S_1\mu_1) = \lambda\beta P_{i-1,M} +$$

$$+ S_1\mu_1 P_{i+1,M} + pS_1\mu_1 P_{i+1,M-1}$$

$$S_1 + 1 \le i < \infty \qquad (2p)$$

The Generating Functions The solution of the set of Equations 2 depends on the knowledge of the values of the boundary probabilities $\{P_{i,j}\}$. To obtain the boundary probabilities, we shall use the following partial generating function:

$$\pi_j(z) = \sum_{i=0}^{\infty} P_{i,j} z^i, \qquad 0 \le j \le m$$

which is the marginal generating function of the number of customers at stage 1, when there are j customers at stage 2 for $0 \le j \le M$. Thus

$$\pi_j(1) = \sum_{i=0}^{\infty} P_{i,j} = P_j$$

gives the marginal probability of j customers at the second stage.

First we consider j= 0. On multiplying 2a-2d by z^i and summing for all i, we get

$$\pi_0(z) \quad z) + S_1\mu_1(1-q/z)\Big] - \mu_2\pi_1(z)$$

$$= b_0(z) \qquad (3a)$$

where

$$b_0(z) = (q/z-1)\mu_1 \sum_{i=1}^{S_1-1} iP_{i,0}z^i + S_1\mu_1(1-q/z)$$

$$\sum_{i=0}^{S_1-1} P_{i,0}z^i + \lambda(1-\beta)(z-1)\sum_{i=0}^{S_1-1} P_{i,0}z^i \qquad (3b)$$

For $1 \le j \le S_2 - 1$, using 2c-2h, in similar manner, we get

$$\frac{pS_1\mu_1}{z}\pi_{j-1} + \pi_j(z)\lambda\beta(1-z) + j\mu_2 + S_1\mu_1(1-q/z)$$

$$-(j+1)\mu_2\pi_{j+1}(z) = b_j(z) \qquad (3c)$$

where

$$b_j(z) = (q/z-1)\mu_1 = \sum_{i=0}^{S_1-1} iP_{i,j}z^i + (1-q/z)S_1\mu_1$$

$$\sum_{i=0}^{S_1-1} P_{i,j}z^i + \frac{p\mu_1}{z}\sum_{i=1}^{S_1-1} iP_{i,j-1}z^i - \frac{pS_1\mu_1}{z}\sum_{i=0}^{S_1-1} P_{i,j-1}z^i$$

$$+\lambda(\beta-1)\sum_{i=0}^{S_1-1} P_{i,j}z^i + z\lambda(1-\beta)\sum_{i=0}^{S_1} P_{i,j}z^i$$

$$(3d)$$

Similarly for $S_2 \le j \le \mu_1 - 1$, we obtain

$$-\frac{pS_1\mu_1}{z}\pi_{j-1}(z) + \pi_j(z)\left[\lambda\beta(1-z) + S_2\mu_2 + S_1\mu_1(1-q/z)\right]$$

$$-S_2\mu_2\pi_{j+1} = b_j(z) \qquad (3e)$$

where

$$b_j(z) = \mu_1(q/z-1)\sum_{i=1}^{S_1-1} iP_{i,j}z^i + S_1\mu_1(1-q/z)$$

$$\sum_{i=0}^{S_1-1} P_{i,j}z^i + \lambda(\beta-1)\sum_{i=0}^{S_1-1} P_{i,j}z^i + z\lambda(1-\beta)\sum_{i=0}^{S_1-1} P_{i,j}z^i$$

$$+\frac{p\mu_1}{z}\sum_{i=1}^{S_1-1} iP_{i,j-1}z^i - \frac{pS_1\mu_1}{z}\sum_{i=0}^{S_1-1} P_{i,j-1}z^i \qquad (3f)$$

Finally, for j= M,

$$-\frac{pS_1\mu_1}{z}\pi_{M-1}(z)+\pi_M(z)\left[\lambda\beta(1-z)+S_2\mu_2\right.$$

$$\left.+S_1\mu_1(1-1/z)\right]=b_M(z) \quad (3g)$$

where

$$b_M(z)=(\frac{1}{z}-1)\mu_1\sum_{i=1}^{S_1-1}iP_{i,M}z^i+(1-1/z)S_1\mu_1$$

$$\sum_{i=(}^{S_1-1} \quad \cdots \sum_{i=1}^{S_1-1}iP_{i,M-1}z^i-\frac{pS_1\mu_1}{z}$$

$$S_1-1 \quad)\sum_{i=0}^{S_1-1}P_{i,M}z^i$$

$$+\lambda z(1-\beta)\sum_{i=0}^{S_1-1}P_{i,M}z^i \quad (3h)$$

There are S_1(M+ 1) "boundary" probabilities $\{P_{i,j}\}$ (i = i, 2, ..., S_1-1; j = 0, 1, ..., M) which are to be determined in order to get the expressions for $b_j(z)$. Once these probabilities are obtained, we have (M+ 1) unknown generating functions $p_j(z)(0\le j\le M)$. The solution of these (M+1) unknown equations gives the result for $p_j(z)(0\le j\le M)$ which provides the entire probability distribution {Pi,j}($0\le i\le\infty; 0\le j \langle M$).

To construct S_1(M+ 1) equations with S_1(M+ 1) unknown, we use the set of balance equatins 2a, 2e, 2i and 2m where i= 0; $0\le j\le M$ and Equations 2, 2f, 2j and 2n, where $1\le i\le S_1-1$; $0\le j\le M$ which provide a set of (S_1-1)(M+ 1) linear equations with S_1(M+ 1) unknown. We also need additional (M+ 1) equations in the above unknown probabilities $\{P_{i,j}\}$.

Equations 3a, 3c, 3e and 3g can be written in matrix form as

$$A(z)\,\underline{\pi}(z)=\underline{b}(z) \quad (4)$$

where p(z) is a vector of the (M+1) generating function $p_j(z)$, $0\le j\le M$; A(z) is an (M+1) dimensional square matrix and b(z) is an (M+1) dimensional vector whose components are defined by Equations 3b, 3d, 3f and 3h.

The coefficients of the matrix A(z) in Equation 4 are represented in Figure 1, where $q(z)=\lambda\beta(1-z)+S_1\mu_1(1-q/z)$. The column or rows numbered from 0 to M represent the number of customers.

Applying Cramer's rule, for each value of z such that A(z) is non-singular, we obtain

$$\left|A(z)\right|\pi_j(z)=\left|A_j(z)\right|, 0\le j\le M \quad (5)$$

where the matrix $A_j(z)$ is derived from A(z) by replacing the i^{th} column of the matrix by b(z).

For every àzà À 1, the system A(z) p(z)= b(z) always yields a solution. It follows that whenever A(z) is singular, and therefore, $A_j(z)$ and Equation 5 hold good in this case as well for any root of A(z)= 0 and for each j. Hence we can write $\left|A_j(z_k)\right|$ = 0 which is a linear equation in the unknown probabilities $\{P_{i,j}\}$ (0<i<S_1-1; 0<j<M).

The Solution Now we solve unknown probabilities $\{P_{i,j}\}$ $\left[0\le i\le S_1-1; 0\le j\le M\right]$ as follows:

A set of (S_1-1)(M+1) independent equations in the above S_1(M+ 1) "boundary" probabilities are already derived. We will investigate additional (M+ 1) equations by considering the equation in àA(z)à = 0 and will show that the polynomial in z, àA(z)à = 0 has (M+1) distinct roots in (0,1). Let us denote these roots by z_k(M+1), k = 1, 2, ..., M+1 where z_{M+1}(M+1) = 1. Note that each root z_k = z_k(M+1) results Equation 5 in $\left|A_m(z_k)\right|=0, 1\le k\le M+1$, in the unknowns $\{P_{i,j}\}$ $(0\le i\le S_1-1; 0\le j\le M)$.

	0	1	2	---	S_2-1	S_2	S_2+1	---	M-2	M-1	M
0	q(z)	$-\mu_2$	0		0	0	0		0	0	0
1	$\frac{-pS_1\mu_1}{z}$	q(z) $+\mu_2$	$-2\mu_2$		0	0	0		0	0	0
2	0	$\frac{-pS_1\mu_1}{z}$	q(z) $+2\mu_2$		0	0	0		0	0	0
|											
S_2-1	0	0	0		q(z) $(S_2-1)\mu_2$	$-S_2\mu_2$	0		0	0	0
S_2	0	0	0		$\frac{-pS_1\mu_1}{z}$	q(z) $+S_2\mu_2$	$-S_2\mu_2$		0	0	0
S_2+1	0	0	0		0	$\frac{-pS_1\mu_1}{z}$	q(z) $+S_2\mu_2$		0	0	0
|											
M-2	0	0	0		0	0	0		q(z) $+S_2\mu_2$	$S_2\mu_2$	0
M-1	0	0	0		0	0	0		$\frac{-pS_1\mu_1}{z}$	q(z) $+S_2\mu_2$	$-S_2\mu_2$
M	0	0	0		0	0	0		0	$\frac{-pS_1\mu_1}{z}$	q(z) $+S_2\mu_2$ $\frac{-pS_1\mu_1}{z}$

Figure 1. The coefficient matrix A(z)

The equation obtained from the root $z_{M+1}(M+1) = 1$ is redundant. This can be verified by examining the matrix $A_M(z)$ obtained from A(z) by replacing its last column with the vector b(z). Indeed, for z = 1, the sum of all terms in each column of A(z) is zero. Also the sum of the elements of b(1) is zero. Therefore $\left|A_m(1)\right|$ is the zero polynomial, so that the equation $\left|A_M(z_{M+1})\right| = 0$ isredundant. Thus we need an additional equation in order to find solution for the required $\{P_{i,j}\}$.

The first echelon can consider classical queue where one-dimensional probabilities for i customers, are the marginal probabilities of our two dimensional system, that is $P_i = \sum_{j=0}^{M} P_{i,j}$, $0 \le i < \infty$. From Equations 2a to 2p, for $i \le 0$ ve $\lambda P_i = N_{i+1}\mu_1 P_{i+1}$ and $N_i = Min(i, S_1)$.

The well known result for classical M/M/S_1 queue with balking (Hiller and Liberman [11]) gives

$$P_0 = \left[\sum_{n=0}^{S_1-1}\left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \frac{1}{S_1!} + \left(\frac{\lambda\beta}{\mu_1}\right)^{S_1}\left[\frac{1}{1-\frac{\lambda\beta}{s_1\mu_1}}\right]\right]^{-1} \tag{4a}$$

since $P_0 = \sum_{j=0}^{M} P_{0,j}$, therefore Equation 4a with $\left\{P_{0,j}\right\}_{j=0}^{M}$ on the left hand side completes the set of $S_1(M+1)$ equations in the $S_1(M+1)$ boundary probabilities $\{P_{i,j}\}$.

After calculating "boundary" $\{P_{i,j}\}$ and expressions for $\left\{\pi_j(z)\right\}_{j=0}^{M}$ we can determine the mean total number of customers L_1 and L_2 at the two echelons as:

$$L_1 = \sum_{i=1}^{\infty} iP_i = P_0\left[\frac{(\rho_1)^{S_1}(\rho_2)}{S!\,(1-\rho_2)^2}\right] + (\rho_3)^n\frac{1}{n!}P_0 + \frac{1}{S!}(\rho_1)^{S_1}\frac{1}{(1-\rho_2)} \tag{4b}$$

where $\lambda\beta/\mu_1 = \rho_1, \lambda\beta/S_1\mu_1 = \rho_2$ and $\lambda/\mu_1 = \rho_3$ and P_0 is given by Equation 4a and

$$L_2 = \sum_{j=1}^{M} iP_j = \sum_{j=1}^{M} j\pi_j(1) \tag{4c}$$

Illustration: Set S_1 = 3, S_2 = 2, M = 2. In this case there are six "boundary" probabilities $\{P_{i,j}\}$ {i = 0,1,2; j = 0,1,2}. Also Equations 2a, 2b, 2m and 2n reduce to

$$\lambda P_{0,0} = \mu_2 P_{0,1} + q\mu_1 P_{1,0} \tag{4d}$$

$$P_{0,1}(\lambda+\mu_1) = \lambda P_{0,0} + \mu_2 P_{1,1} + 2q\mu_1 P_{2,0} \tag{4e}$$

$$P_{0,2}(\lambda+2\mu) = \mu_1 P_{1,2} + p\mu_1 P_{1,1} \tag{4f}$$

$$P_{1,2}(\lambda+2\mu_2+\mu_1) = \lambda P_{0,2} + 2\mu_1 P_{2,2} + 2p\mu_1 P_{2,1} \tag{4g}$$

Equation 4a get the form

$$P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1} =$$

$$\left[1 + \frac{\lambda}{\dots} \dots + \frac{1}{6}\left(\frac{\lambda\beta}{\mu}\right)^3\left(\frac{1}{1-\lambda\beta/3\mu_1}\right)\right]^{-1} \tag{4h}$$

Also

A(z) =

$$\begin{bmatrix} \lambda\beta(1-z) + 3\mu_1(1-q/z) & -\mu_2 & 0 \\ -\frac{3p\mu_1}{z} & \lambda\beta(1-z) + \mu_2 + 3\mu_1(1-q/z) & -2\mu_2 \\ 0 & \frac{3p\mu_1}{z} & \lambda\beta(1-z) + 2\mu_2 + 3\mu_1(1-q/z) \end{bmatrix}$$

Putting $z = z_0$ in b(z), we obtain

$$b_0(z_0) = (3D-C)P_{0,0} + z_0(2D-C)P_{1,0} + z_0^2(D-C)P_{2,0} \quad (4i)$$

$$b_1(z_0) = -3BP_{0,0} - z_0 2BP_{1,0} + z_0^2 BP_{2,0}$$

$$+ (3A-C)P_{0,1} + z_0(2A-C)P_{1,1} + z_0^2(A-C)P_{2,1} \quad (4j)$$

$$b_2(z_0) = -3BP_{0,1} - z_0 2BP_{1,1} + z_0^2 BP_{2,1}$$

$$+ (3A-C)P_{0,2} + z_0(2A-C)P_{1,2} + z_0^2(A-C)P_{2,2} \quad (4k)$$

where

$$A = \left[1 - \frac{1}{z_0}\right], \qquad B = \frac{\rho\mu_1}{z_0}$$

$$C = \lambda(1-\beta)(1-z_0), \qquad D = (1-q/z_0)\mu_1$$

$$q_1(z_0) = \lambda(1-z_0) + 3\mu_1(1-q/z_0)$$

Replacing the third column of $A(z_0)$ by $b(z_0)$ yields the matrix $A_1(z)$. The equation $|A_1(z_0)| = 0$ ...

$$-9B^2\left[q_1(z_0) + D(3D-C)\right] P_{0,0} + 3B\{q_1(z_0)$$

$$\left[(3A-C) - (q_1(z_0) + \mu_2\right] 3D\mu_2\}P_{0,1}$$

$$-Bz_0\left[q_1(z_0)3(2A-C) - 2(q_1(z_0) + \mu_2 + 6B\mu_2\right] P_{1,1}$$

$$-3B^2 z_0\left[2q_1(z_0) + 3(2D-C)\right] P_{1,0}$$

$$+(3A-C)\left[q_1(z_0)(q_1(z_0) + \mu_2) + 3B\mu_2\right] P_{0,2}$$

$$+z_0^2 3B^2\left[q_1(z_0) + B - 3(D-C)\right] P_{2,0}$$

$$+z_0(2A-C)\left[q_1(z_0)(q_1(z_0) + \mu_2) + 3B\mu_2\right] P_{1,2}$$

$$+z_0^2 B\left[q_1(z_0)(q_1(z_0) + \mu_2) + 3(A-C)\right] + 3B\mu_2]P_{2,1}$$

$$+z_0^2(A-C)\left[q_1(z_0)(q(z_0) + \mu_2) + 3B\mu_2\right] P_{2,2} \quad (4l)$$

Equations 4d, 4e, 4f, 4g, 4h and 4i give the solution of desired "boundary" probabilities for this case.

The Interlacing Theorem: We transform the basic A(z) into an equivalent matrix H(z) called Hessenberg form (Wilkinson [11]). To obtain Hessenberg matrix H(z), we perform elementary operations on the rows of the matrix A(z) as follows: We add the m^{th} row to the $(m-1)^{st}$ row, then we add the $m-1^{st}$ row to the $(m-2)^{nd}$ and so on up to the first row, thus we obtain its i^{th} row $0 \le i \le M$ as the sum of the rows from i to M in the original matrix A(z). The resulting matrix is characterized by the fact that the elements below the secondary diagonal are all zero. This type of matrix is called an upper Hessenberg matrix see Figure 2. All the elements of the 0^{th} row of the Hessenberg matrix obtained from A(z) are the same, and equal to

$$P(z) \equiv \lambda\beta(1-z) + S_1\mu_1\left[1 - \frac{1}{z}\right] = q(z) - S_1\mu_1\frac{p}{z}$$

Let us denote the determinant of the square sub-matrix of H(z) to comprise its first n rows and n columns (i = 0, 1, ..., n-1 and j = 0, 1, ..., n-1) by B(z). Let $G_n(z)$ denote the determinant of the above n-dimensional sub-matrix with the exception that the lower right element is P(z) instead of $P(z) + k_n\mu_2$ where $k_n = \min(n, S_2)$. Now we have

$$B_n(z) = (P(z) + k_{n-1}\mu_2)B_{n-1}(z) + \frac{pS_1\mu_1}{z} C_{n-1}(z) \quad (5a)$$

$$C_n(z) = P(z)\, B_{n-1}(z) + \frac{pS_1\mu_1}{z}\; C_{n-1}(z) \qquad (5b)$$

Using 5a and 5b, we get

$$C_n(z) = B_n(z) - k_{n-1}\mu_2 B_{n-1}(z) \qquad (5c)$$

and

$$C_{n-1}(z) = B_{n-1}(z) - k_{n-2}\;\; \mu_2 B_{n-2}(z) \qquad (5d)$$

Putting the value of $C_{n-1}(z)$ in Equation 5a, we have

$$B_n(z) = \left[(P(z) + k_{n-1}\mu_2 + \frac{pS_1\mu_2}{z}) B_{n-1}(z) - \frac{pS_1\mu_1}{z} \right] K_{n-2}\;\; \mu_2 B_{n-2}(z) \qquad (5e)$$

Rewriting Equation 5e as

$$B_n(z) = f_{n-1}(z) B_{n-1}(z) - g_{n-2} B_{n-2}(z) \qquad (5f)$$

where

$$f_n(z) = P(z) + k_n\mu_2 + \frac{pS_1\mu_1}{z} \qquad (5g)$$

$$g_n(z) = \frac{pS_1\mu_1}{z} k_n\mu_2 \qquad (5h)$$

By factorizing P(z), we write

$$B_n(z) = P(z) D_n(z), \qquad 1 \le n \le M+1$$

where $D_n(z)$ is the determinant of the matrix determining $B_n(z)$ with all elements of the first row equal to 1. Thus dividing Equation 5f by $P_n(z)$, we can obtain a recursion formula for $D_n(z)$ given by

$$D_n(z) = f_{n-1}(z) D_{n-1}(z) - g_{n-2}(z) D_{n-2}(z) \qquad (5i)$$

Our aim is to prove that

$$\left| A(z) \right| = \left| H(z) \right| = B_{M+1}(z) = P(z) D_{M+1}(z)$$

has M+ 1 real roots in the interval (0,1). It indicates that P(z) has a single root in (0,1). It is noted that P(z) has two roots z_1= 1 and $z_2 = \frac{S_1\mu_1}{\lambda\beta}$ since z_1>1; therefore P(z) has a single root in (0, 1], so that D_{M+1} (z) has M real roots in (0, 1]. This shows that B_{M+1}(z) has (M+ 1) real roots in that interval.

Therefore, we will prove that for every $1 \le n \le M+1$, D_n(z) provides n-1 real roots in (0, 1] and will indicate that all n roots are different.

D_n(z) is a rational function with z^{n-1} as its denominator and a polynomial $\overline{D}_n(z)$ of degree 2n-2 as its numerator, such that

$$D_n(z) = \frac{\overline{D}_n(z)}{z^{n-1}}$$

it will be proved that D_n(z) has its (n-1) roots in the open interval (0, 1) while the remaining (n-1) roots are in (1-È).

Theorem: $\overline{D}_n(z) = \sum_{i=0}^{2n-2} (-1)^i d_i(n) z^i$, where all d_i(z) are of the same sign and nonzero. Furthermore, for any two successive polynomials of $\overline{D}_{n-1}(z)$ and $\overline{D}_n(z)$, the coefficients d_o(n-1) and d_o(n) will have opposite signs.

Proof: Since B_1(z) = P(z), we have $D_1(z) \equiv 1$. Also B_2(z) = P(z) f_1(z), so that we have

$$D_2(z) = f_1(z) = P(z) + \mu_2 + \frac{pS_1\mu_1}{z}$$

$$= \lambda\beta(1-z) + S_1\mu_1(1 - \frac{1}{z}) + \mu_2 + \frac{pS_1\mu_1}{z}$$

$$= \frac{\lambda\beta z^2 + z(\lambda\beta + S_1\mu_1 + \mu_2) - S_1\mu_1 + pS_1\mu_1}{z}$$

$$= \frac{\left\{ -\lambda\beta z^2 + z(\lambda\beta + S_1\mu_1 + \mu_2) - qS_1\mu_1 \right\}}{z} = \frac{\overline{D}_2(z)}{z}$$

It is clear that $\frac{\overline{D}_2(z)}{z}$ has alternating signs, no

	0	1	2	---	S_2-1	S_2	S_2+1	---	M-2	M-1	M
0	p(z)	p(z)	p(z)		p(z)	p(z)	p(z)		p(z)	p(z)	p(z)
1	$\frac{-pS_1\mu_1}{z}$	p(z) $+\mu_2$	p(z)		p(z)	p(z)	p(z)		p(z)	p(z)	p(z)
2	0	$\frac{-pS_1\mu_1}{z}$	p(z) $+2\mu_2$		p(z)	p(z)	p(z)		p(z)	p(z)	p(z)
|											
S_2-1	0	0	0		p(z) (S_2-1)	p(z)	p(z)		p(z)	p(z)	p(z)
S_2	0	0	0		$\frac{-pS_1\mu_1}{z}$	p(z) $S_2\mu_2$	p(z)		p(z)	p(z)	p(z)
S_2+1	0	0	0		0	$\frac{-pS_1\mu_1}{z}$	p(z) $+S_2\mu_2$		p(z)	p(z)	p(z)
|											
M-2	0	0	0		0	0	0		p(z) $+S_2\mu_2$	p(z)	p(z)
M-1	0	0	0		0	0	0		$\frac{-pS_1\mu_1}{z}$	p(z) $+S_2\mu_2$	p(z)
M	0	0	0		0	0	0		0	$\frac{-pS_1\mu_1}{z}$	p(z) $+S_2\mu_2$

Figure 2. The Hessenberg matrix H(z)

zero coefficients, and $d_0(z) = -gS_1\mu_1$ has an opposite sign to that of $d_o(1) = 1$.

Now Equation 5g can be rewritten for $f_1(z)$ as

$$f_{n-1}(z) = -\frac{\lambda\beta z^2 + z(\lambda\beta + S_1\mu_1 + K_{n-1}\mu_2) - qS_1\mu_1}{z} \quad (5j)$$

The validity of the theorem for n > 2 can be proved by induction principle on the same line as done by Zuta and Yachiali [18].

CONCLUSION

In this study, two echelon multi-server tandom queuing system where customers arrive according to a poisson fashion with two different rates, is developed. The analysis is done by exploiting the properties of Hessenberg matrix whose determinants yield polynomials with interesting interlacing properties. The service of customers are assumed to be completed in two stages. At the first stage the customers may balk when all servers are busy while the second stage has a finite buffer. Such type of situation may occur in communication and computer systems wherein jobs need service of two kinds by primary and secondary auxiliary servers with a provision of intermediate limited buffer. The incorporation of balking at first stage of service make our model more feasible in realistic situations.

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