RESEARCH NOTE

OPTIMAL CONTROL FOR DESCRIPTOR SYSTEMS: TRACKING PROBLEM

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Abstract Singular systems have been studied extensively during the last two decades due to their many practical applications. Such systems possess numerous properties not shared by the well-known state variable systems. This paper considers the linear tracking problem for the continuous-time singular systems. The Hamilton-Jacobi theory is used in order to compute the optimal control and associated trajectory. Two methods are presented for solving these trajectories. The first method uses the concept of the Drazin inverse, and the second involves the derivation and solution of a Riccati equation. Similar to the linear regulator problem, necessary and sufficient conditions for existence and uniqueness of a solution are stated.

Key Words Optimal Control, Riccati Equation, Singular Systems

| A±i /k¯A³Tµo£ nÞo¹ | Bi ³]±Ü jn±« o¼iA ·µj °j nj Ë¼ñÜ ºB´ªTv¼w ,¬A°Aoµ ºBµjoIInB½¥¼§j ³II **²k¼ña**
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INTRODUCTION

Consider the system of the form:

$$E X^3(t) = A x(t) + B u(t) \qquad x(t_Ü) = x_Ü \qquad (1)$$

where E, A and B are constant matrices. If ½E¼ = 0 then the system described by Equation 1 is called a singular system [1], degenerate system [2], generalized state space system [3], descriptor system [4], or semi-state system [5]. Systems which satisfy these properties can consist of both static and dynamic equations and need not be causal. These types of systems appear in many practical applications such as robotics, optimal control, electrical networks, economics, large scale interconnected systems, biology, power systems, neural delay systems, and aircraft dynamics with algebraic constraints.

It has taken considerable effort to extend the theories available for state variables to singular systems. The problem of deriving a continuous-time singular system from some initial state to a desired final state has been discussed in [1,2,7,8]. In [9,10,24] the pole placement of the singular systems has been investigated. In the area of optimal control, the linear singular regulator problem has been considered in [11,12] for the discrete-time case and in [2,6,13-19] for the continuous-time case. By using state feedback, the generalized Riccati equation for both time-invariant and time-varying cases has been obtained [20,21]. It was shown that for the case ½R¼ Ú0, the Riccati equation is symmetric (where R is the conventional quadratic weighting matrix), but where ½R¼=0, the Riccati matrix P will not be symmetric. Necessary and sufficient conditions for the existence and uniqueness of a solution have also been derived.

Our goal in this paper is to take some steps towards generalization of linear regulator problem, namely linear tracking problem for singular systems. In this case, we wish to find the control that causes the output to track a desired output state, $\Gamma(t)$. For this, we wish to find the control which minimizes a performance index, J. In section II we describe the formulation of the linear tracking problem for the singular systems. In addition, the Hamilton-Jacobi theory is applied in order to compute the optimal control and associated trajectory. Section III is devoted to the solution of these trajectories by two methods (namely direct and Riccati approaches). The condition for existence and uniqueness of a solution is also given.

FORMULATION OF THE PROBLEM

In this section, we extend the results obtained for the linear regulator problem [6] to the tracking problem; that is, the desired value of the state vector is not the origin. Given that:

$$E\,x^{0}(t) = A\,x(t) + B\,u(t) + w(t)$$
$$z(t) = C\,x(t) \qquad (2a)$$

where $w(t)$ is a deterministic input or plant noise, the matrix E is singular, with $x \in R^{n}$, $u \in R^{m}$, $w \in R^{n}$, $z \in R^{n}$. Additionally, we assume that the matrix $(sE-A)$ has a non-zero determinant for some value of s. This conditions, ensures that for appropriate initial conditions, Equation 2a possesses a unique solution [1] (namely it is tractable).

We wish to find the control which minimises the performance index,

$$J = \frac{1}{2}[\Gamma(t_f) - z(t_f)]^{T} E^{T} SE[\Gamma(t_f) - z(t_f)] + \frac{1}{2}\int_{t_0}^{tf} [\| \Gamma(t) - z(t)\|_Q^2 + \| u(t)\|_R^2]\,dt \qquad (2b)$$

where $\Gamma(t)$ is the desired output state vector. S, Q, R are symmetric matrices and at least positive semi-definite (p.s.d.). The final time t_f is fixed, $x(t_f)$ is free and the states and controls are not bounded. The Hamilton is as follows:

$$H[x(t), u(t), \lambda(t), t] = \frac{1}{2}\| \Gamma(t) - Z(t)\|_Q^2 + \frac{1}{2}\| u(t)\|_R^2 + \lambda^{T}(t)\,[Ax(t) + Bu(t) + w(t)] \qquad (3)$$

By using the calculus of variation and following the same approach as we did for the regulator problem [6], we obtain:

$$J = \frac{1}{2}[\Gamma(t_f) - z(t_f)]^{T} E^{T} SE[\Gamma(t_f) - z(t_f)] + \frac{1}{2}\int_{t_0}^{tf} \| \Gamma(t) - z(t)\|_Q^2 + \| u(t)\|_R^2 + \lambda^{T}(t)\,[Ax(t) + Bu(t) + w(t) - E\dot{x}(t)]\,dt \qquad (4)$$

With some manipulation we have the following:

$$\frac{\partial H}{\partial u} = 0 \qquad B^{T}\lambda(t) + Ru(t) = 0 \qquad (5a)$$

$$-\frac{\partial H}{\partial x} = E^{T}\dot{\lambda}(t) \qquad E^{T}\dot{\lambda}(t) = -C^{T}QCx(t) - A^{T}\lambda(t) + C^{T}Q\,\Gamma(t) \qquad (5b)$$

$$\frac{\partial H}{\partial \lambda} = E\dot{x}(t) \qquad E\dot{x}(t) = Ax(t) + Bu(t) + w(t) \qquad (5c)$$

or

$$\begin{bmatrix} E & 0 & 0 \\ 0 & E^{T} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -C^{T}QC & -A^{T} & 0 \\ 0 & -B^{T} & R \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & C^{T}Q & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w(t) \\ \Gamma(t) \end{bmatrix} \qquad (5)$$

With the initial and terminal condition,

$$x(t_0) = x_0, \quad \lambda(t_f) = C^{T}E^{T}SE\,[Cx(t_f) - F(t_f)] \qquad (6)$$

SOLUTIONS

The above non-homogeneous equation can be solved by two methods.

Method 1 (direct approach) - Assuming $|R| \neq 0$, we find that $u(t) = -R^{-1}B^T\lambda(t)$. By substituting for $u(t)$ in (5), we obtain the following:

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^TQC & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & C^TQ \end{bmatrix}\begin{bmatrix} w(t) \\ G(t) \end{bmatrix} \quad (7a)$$

Comparing with the regulator problem, we also have a forcing part whereas we had a homogeneous equation in the previous case. The above equation can be written as follows:

$$\bar{E}\,\bar{X}^{0}(t) = \bar{A}\,\bar{X}(t) + \bar{B}\,f(t). \quad (7b)$$

Assuming the function $f(t)$ is k-times continuously differentiable around initial time, t_0 we can state the necessary and sufficient conditions for the existence and uniqueness of solution for tracking problem in terms of the solution of original singular system. Being k-times differentiable is due to the solution of the singular system which is:

$$x(t) = e^{\hat{E}^D\hat{A}(t-t_0)}\hat{E}^D\hat{E}A\,x(t-t_0) + \int_{t_0}^{t} e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\hat{B}\,f(s)\,ds - (I - \hat{E}\hat{E}^D)\sum_{i=0}^{k-1}(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{B}\,f^{i}(t) \quad (8)$$

where

$(\hat{\ }) = (\lambda\bar{E} - \bar{A})^{-1}(.)$ for some λ, k is the index of Matrix E, $(.)^D$ is the Drazin inverse and $f^i(t)$ refers to the ith derivative.

Theorem 1 - Assuming $f(t)$ is k-times continuously differentiable around t_0, with consistent initial condition, the non-homogeneous Equation 7b is tractable (regular) iff the original singular Equation 1 is tractable.

Proof - By some manipulation we can easily obtain:

$$\begin{vmatrix} sE-A & BR^{-1}B^T \\ C^TQC & sE^T+A^T \end{vmatrix} = |(sE+A)^T|$$

$$|sE-A|\,.\,|I - BR^{-1}B^T(sE+A)^{-T}C^TQC\,(sE-A)^{-1}| \quad (9)$$

For simplicity let us write the above equation as follows:

$\Delta = \Delta_1\Delta_2\Delta_3$

Only if Part - It is easily seen that $\Delta \neq 0$ (by [1] it means Equation 7b has a unique solution) implies $\Delta_1 \neq 0$, (that is Equation 1 has a unique solution).

If Part - $\Delta_1 \neq 0$, implies $\Delta_2 \neq 0$, and since $\Delta_3 \neq 0$, therefore $\Delta \neq 0$, which gives the result.

Remark 1 - For the case $|R| = 0$ we cannot find u directly from Equation 5a. Thus we should solve the non-homogeneous Equation 5 directly as a singular system (see [1]).

Method 2 (Riccati approach) -

Case 1 - $|R| \neq 0$: For deriving Riccati equation we assume

$$\lambda(t) = P(t)\,Ex(t) - L(t) \quad (10)$$

Then

$$\dot{\lambda}(t) = \dot{P}(t)\,Ex(t) + P(t)\,E\dot{x}(t) - \dot{L}(t) \quad (11)$$

and

$$E^T\dot{\lambda}(t) = E^T\dot{P}(t)\,Ex(t) + E^TP(t)\,E\dot{x}(t) - E^T\dot{L}(t) \quad (12)$$

By a procedure similar to that of the singular regulator problem, we obtain the following equation:

$[E^T\dot{P}(t)\,E + E^TP(t)\,A + C^TQC + A^TP(t)$

$E + E^T P(t)\ BR^{-1} B^T P(t)\ E]\ x(t) +$

$[E^T P(t)\ BR^{-1} B^T L(t) + E^T P(t)\ w(t) - E^T \dot{L}(t)$

$- A^T L(t) - C^T Q\ \Gamma(t)] = 0 \qquad (13)$

Since Equation 13 holds for all non-zero x(t), the term premultiplying x(t) and second term must be zero. Therefore, we have the following two sets of equations:

$$E^T \dot{P}(t)\ E + E^T P(t)\ A + A^T P(t)\ E - E^T P(t)\ BR^{-1} B^T P(t)\ E + C^T Q\ C = 0$$

$$P(t_f)\ E = C^T E^T SEC \qquad (14)$$

and

$$E^T \dot{L}(t) + A^T L(t) - E^T p(t)\ BR^{-1} B^T L(t) - E^T P(t)\ w(t) + C^T Q\ \Gamma(t) = 0$$

$$L(t_f) = C^T E^T SE\Gamma(t_f) \qquad (15)$$

Thus, we need to solve a generalized Riccati Equation 14 for p(t) and singular Equation 15 for L(t) in order to compute $\lambda(t)$. The optimum control law is obtained from Equations 5a and 10. That is, we have:

$$u(t) = -R^{-1} B^T \lambda(t) = -R^{-1} B^T [P(t)\ Ex(t) - L(t)].$$

In [6] different methods for solving similar Riccati equations have been shown and necessary and sufficient conditions for existence and uniqueness of a solution have been stated, so details of the derivation are omitted.

Theorem 2 - The Riccati Equation 14 is regular iff the system Equation 1 is regular.

Proof - The result is analogous to the derivation of the regulator case in [6].

Theorem 3 - The system Equation 2 has a unique solution if the generalized Riccati Equation 14 and the singular system Equation 15 are regular.

Proof - Having Riccati Equation 14 and Equation 15 regular we can calculate P(t) and L(t). Therefore:

$$u(t) = -R^{-1} B^T [P(t)\ Ex(t) - L(t)]$$

can be considered.

Example - Given the singular system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$z(t) = [1 \quad 0]\ x(t)$$

we would like to minimize the cost function

$$J = \frac{1}{2} \left[\Gamma(t_f) - z(t_f)\right]^T E^T SE \left[\Gamma(t_f) - z(t_f)\right]$$

$$+ \frac{1}{2} \int_0^{t_f} \left\{ \| \Gamma(t) - z(t) \|_q^2 + \| u(t) \|_r^2 \right\} dt$$

where q and r are scalars. For the case S=0, we will have the following.

We first use Equation 14 to obtain Riccati equation for the above system. So we have

$$\dot{p}_{11}(t) - 2p_{11}(t) - \frac{p_{12}^2(t)}{r} + q = 0 \qquad p_{11}(t_f) = 0$$

$$p_{11}(t) - 2p_{12}(t) = 0 \qquad p_{12}(t_f) = 0$$

If we allow t_f to become infinite ($t_f = \infty$), we obtain the following solutions

$$p_{11} = 4r \sqrt{1 + \frac{q}{4r} - 4r}$$

$$p_{12} = \frac{1}{2} p_{11}$$

where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$

Now, we use Equation 15 to obtain L(t). Thus

$$l_1(t) - l_1(t) - \frac{p_{12}}{r} l_2(t) + q\ \dot{\Gamma}(t) = 0$$

$$l_1(t) - 2l_2(t) = 0$$

For t_f being very large, we obtain

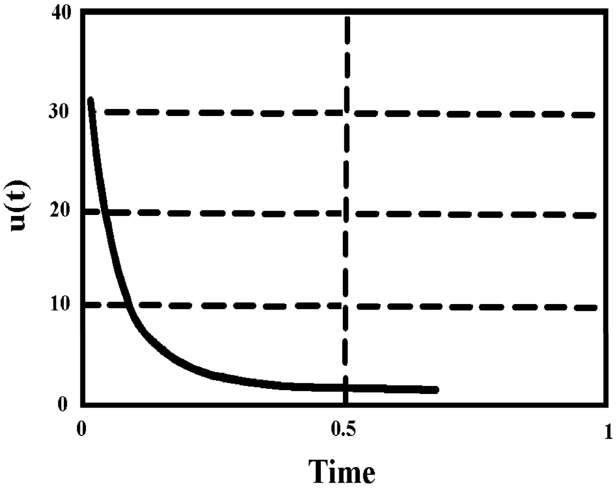

Figure 1. Optimal control.

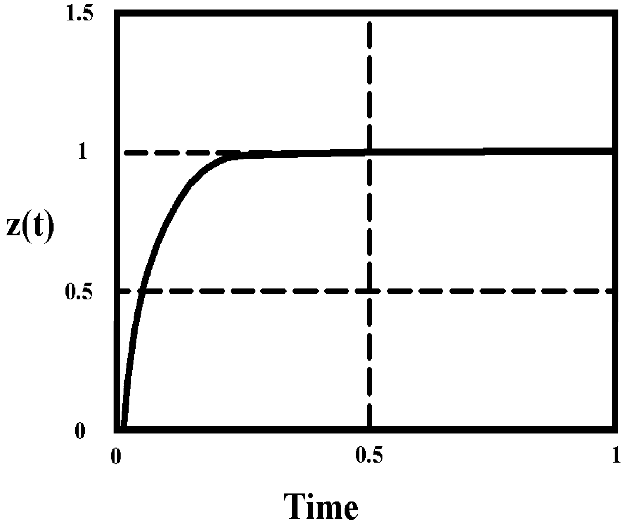

Figure 2. System output.

$$l_1(t) = \frac{1}{1 + \frac{p_{12}}{2\,r}}\; q\, G(t)$$

$$l_2(t) = \frac{1}{2}\, l_1(t)$$

where $L(t) = \begin{bmatrix} l_1(t) \\ l_2(t) \end{bmatrix}$

After determining P and L(t), the optimal control will be as following:

$$u(t) = -\frac{1}{r}\begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left(P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) - L(t) \right)$$

1 -We first consider G(t) to be a step function. Figures 1 and 2 show the optimal control and corresponding output where r= 1 and q= 1000 and x(0)=0.

2 -Now, consider G(t) be a sinusoidal function. Figures 3 and 4 show the optimal control and corresponding output for r=1 and q=10, which in this case, the output of system does not follow the desired trajectory. However by choosing r= 0.1 and q= 1000, the output follow the desired trajectory. Figures 5 and 6 show the optimal control and associated output for this choice.

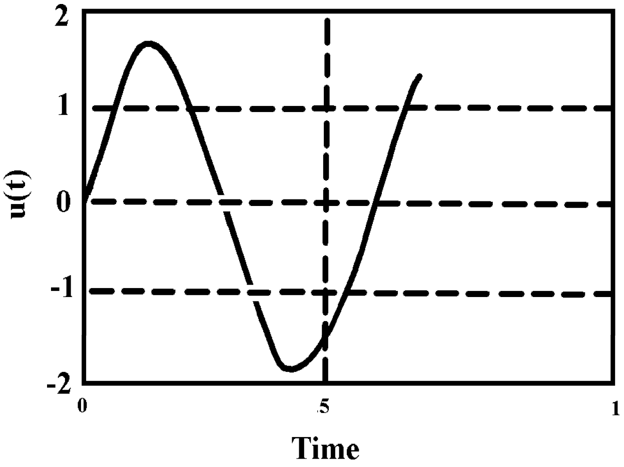

Figure 3. Optimal control.

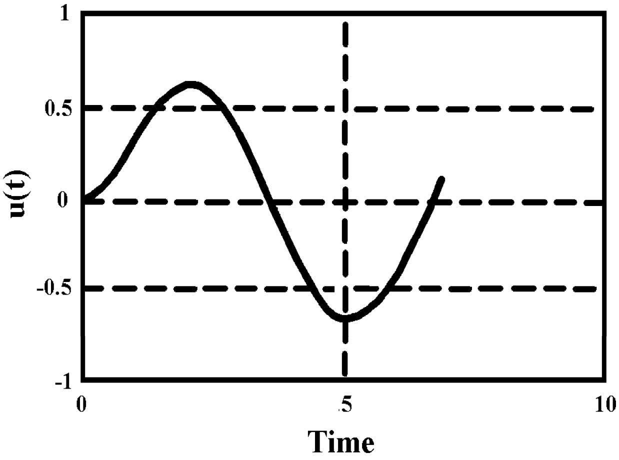

Figure 4. System output.

Lemma 1 - The system Equation 2 has a

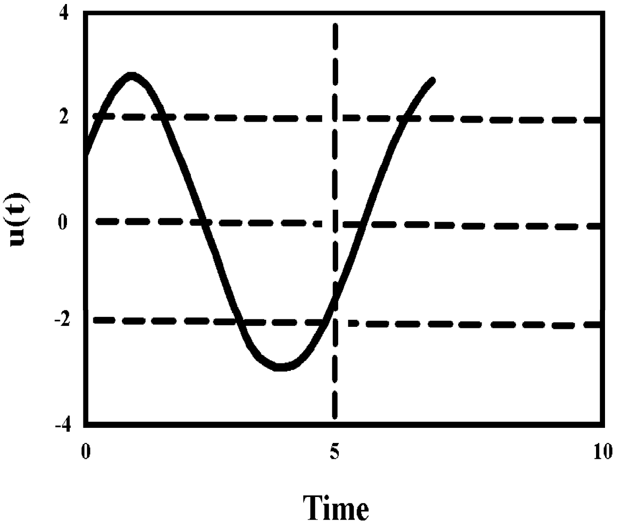

Figure 5. Optimal control.

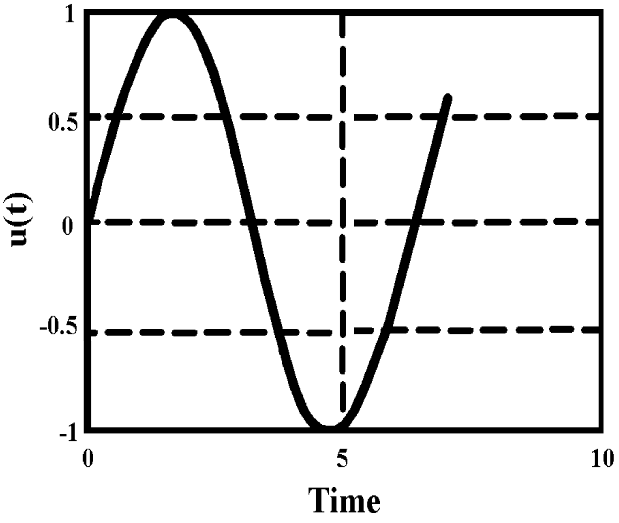

Figure 6. System output.

unique solution if singular systems 1, and 15 are regular.

Proof - Combining theorems 2 and 3 gives the result.

Assuming the matrix P is constant, the following theorem is immediate.

Theorem 4 - System Equation 15 has a unique solution iff there exists a scalar s such that the following matrix is invertible:

$$(sE^T + A^T - E^T PBR^{-1} B^T) \quad (16)$$

Proof - By [1], Equation 15 has a unique solution subject to an appropriate initial condition (regular) if $|sE^T + A^T - E^T PBR^{-1} B^T| \neq 0$.

Case 2 - $|R| = 0$

In this case we cannot find u(t) in terms of $\lambda(t)$. However, similar to the regulator case by choosing:

$$u(t) = P_1 \lambda(t) + P_2 x(t) \quad (17)$$

and substituting for u in processes of deriving Riccati equation, we obtain the following non-symmetric generalized Riccati and singular equations respectively.

$$E^T \dot{P}(t) E + E^T P(t) A + A^T p(t) E + E^T P(t) BP_1 P(t) E + E^T P(t) BP_2 + C^T Q C = 0 \quad (18)$$

and

$$E^T \dot{L}(t) + A^T L(t) + E^T P(t) BP_1 L(t) + C^T Q G(t) - E^T P(t) w(t) = 0 \quad (19)$$

The solution of the above equations can be found in [6] and [1] following some simplifications.

Remark 2 - Consider matrix E as constant. All derivations in this paper could hold for time varying cases. However, if matrix E is time varying, we have some modification as follows for the case $|R| \neq 0$.

$$\lambda(t) = P(t) E(t) x(t) - L(t)$$

$$\dot{\lambda}(t) = \dot{P}(t) E(t) x(t) + P(t) E(t) \dot{x}(t) \dot{L}(t) + P(t) \dot{E}(t) x(t)$$

and

$$E^T(t) \dot{\lambda}(t) = - C^T(t) Q(t) C(t) x(t) - A^T(t) \lambda(t) + C^T(t) Q(t) G(t) - \dot{E}^T(t) \lambda(t)$$

combining the above equations gives us the following equations:

$$E^T \dot{P}(t)\, E(t) + E^T(t)\, p(t)\, [A(t) + \dot{E}(t)] + [\dot{E}(t) + A(t)]^T\, p(t)\, E(t) - E^T(t)\, P(t)\, B(t)\, R^{-1}(t)\, B^T\, p(t)\, E(t) + C^T(t)\, Q(t)\, C(t) = 0$$

and

$$E^T(t)\, \dot{L}(t) + [A(t) + \dot{E}(t)]^T\, L(t) - E^T(t)\, P(t)\, B(t)\, R^{-1}(t)\, B^T(t)\, L(t) - E^T(t)\, P(t)\, w(t) + C^T(t)\, Q(t)\, G(t) = 0$$

It can easily be seen that by changing A(t) to $[A(t) + \dot{E}(t)]$ for the constant coefficient case the new equations for the time varying case have been obtained. This result can also be achieved for the case

¼R ¼= 0.

Remark 3 Although these results have been developed for continuous systems, they can easily be extended for the discrete cases as well.

CONCLUSION

The linear singular optimal tracking problem has been discussed and the Hamilton-Jacobi theory is used in order to compute the optimal control and associated trajectory. We have shown that the singular tracking problem is composed of two parts, a singular regulator part, and a prefilter to determine the optimal driving function from the desired value, G(t), of the system output. We also have obtained the generalized Riccati equation for both time invariant and time varying cases.

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