MAXIMUM LOAD AND MINIMUM VOLUME STRUCTURAL OPTIMIZATION

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Abstract A bi-criteria optimization is considered whose objectives are the maximization of the load sustained by a structure and the minimization of the structure's volume. As the objectives are conflicting, the solution to the problem is of the Pareto type. The problem is elaborated for a thin-walled column of cruciform cross-section, prone to flexural and torsional buckling. A numerical example is also presented.

Key Words Bi-criteria, Optimization, Buckling, Column, Pareto Solution, Structural, Optimization

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INTRODUCTION

The majority of the structural optimization problems presented in the literature [2,5,12] are single-criterion ones with the objectives of either the maximization of the structure's carrying capacity or the minimization of its cost or volume. When searching for the strongest structure (Problem 1), its cost or material volume is predetermined and kept constant. When, on the other hand, the minimization of the structure's cost or volume is carried out (Problem 2), then the bearing capacity of the structure or its external load is predetermined and kept constant.

The aim of this paper is to consider a bi-criteria optimization problem that combines the two above mentioned problems. More specifically the combined problem is to find the structure that, within the frames of various side and behavioral constraints, would be the strongest and would have the least possible volume.

Clearly, the demands of the maximum strength and the minimum material volume are conflicting and, therefore, the optimum solution to the problem, in the ordinary sense of the word, does not exist. Instead, the problem can be approached by applying the Pareto optimality [4,13].

Pareto solutions usually constitute a large set of designs whose usefulness can be ranked according to one or more additional criteria. Such a ranking is carried out in this paper, and the solution to the problem, as understood in this paper, is the Pareto optimum whose rank, according to an adopted criterion, is the highest.

Although, as stated earlier, the majority of structural optimization problems are formulated as single-criterion ones, variety of

multiple-criteria problems have also been stated, solved and published in the literature [1, 3,4,6,7,13]. In most of these papers, the objective functions adopted by the authors were the structure's volume and displacements of the selected points, or the allowable load and displacements.

Little work is reported concerning multi-criterion optimization with the objective functions reoriented towards the maximization of the allowable load applied to the structure and the minimization of its volume [10,11]. Such a type of the reoriented bi-criterion problem is presented in this paper.

The procedure followed is as follows. First the bi-criterion problem is stated more formally and the adopted criterion for ranking Pareto solution is described. Then, the same problem is specified for a thin-walled column of cruciform cross-section. And finally, a numerical example is solved.

GENERAL FORMULATION OF THE PROBLEM

Consider a structure whose material, topology and loading pattern (but not the value of the load factor) are known and whose overall geometry, as well as the cross-sectional dimensions are specified by a set of preassigned parameters and by a set x of design variables x_1, x_2, ..., x_n. Let the feasible domain D of the design variables be delineated by m side and behavioral inequality constraints $g_i(x) \geq 0$. The problem is to find such feasible design variables that describe the structure capable of sustaining possible the largest load, using possibly the least amount of material. Since the objectives are conflicting, there exist only Pareto optimum solutions x^* to the problem.

The solution $x^* \in D$ is a Pareto solution to the problem considered if and only if there is no such $x \in D$ which gives simultaneously greater or equal ($\geq$) allowable external load and less or equal ($\leq$) volume than x^* does, and additionally makes at least one of the two relations: $\geq$ and $\leq$, a sharp inequality.

Most common is the existence of many Pareto solutions for any bi-criterion problem, from which, usually, one or more so called preferred solutions $`x^*$ could be selected. The preferred solution is the one which satisfies an adopted additional criterion.

Our aim is just to determine the preferred Pareto optimum solution(s) $`x^*$, which can be stated more formally as follows: find $`x^* \in D$ that maximizes load factor $f_1(x)$, minimizes the structure's volume $f_2(x)$, and satisfies the adopted criterion for being classified as the preferred solution.

Before stating the criterion of preference that will be used in this paper, let us introduce two performance functions $f'_1(x)$ and $f'_2(x)$, defined as follows:

$$f'_1(x)=100[f_1(x)-f_{1max}]/(f_{1min}-f_{1max})$$

$$f'_2(x)=100[f_2(x)-f_{2min}]/(f_{2max}-f_{2min})$$

where f_{1max} and f_{1min} are the maximum and minimum values, respectively, of the load factor function $f_1(x)$ for $x \in D$, f_{2max} and f_{2min} are the maximum and minimum values, respectively, of the structure's volume function $f_2(x)$ for $x \in D$.

Each of the performance functions map the design variables space (DVS) 0, x_1, ... , x_n into the performance function space (PFS) $0'f'_1f'_2$. Thus, each point $x \in$ DVS has its image $x' \in$ PFS, each Pareto solution $x^* \in$ DVS is mapped into its image $x^{*'}$ in PFS, and the whole feasible domain D is mapped into its image set D'.

Consider a specific point $x' \in$ PFS, denoted further by x'_I, whose coordinates f'_1 and f'_2 (the values of the performance functions) are both

equal to zero. Zero coordinates indicate that the objective functions $f_1(x)$ and $f_2(x)$ are simultaneously best fulfilled by a certain hypothetical point $x_I \hat{u} DVS$.

However, such a point x_I, representing an ideal solution, does not exist because the objectives $f_1(x)$ and $f_2(x)$ are conflicting. Nevertheless, despite non-existence of the point x_I, it is convenient to consider its counterpart $x'_I \hat{u} PFS$ and treat it as a reference point in ranking the Pareto solutions. The ranking can be performed in accordance with the following criterion.

The Pareto solution $x^*_I \hat{u} DVS$ is deemed to be better than Pareto solution x^*_2 if the distance, measured in PFS, between the images $x^*_1{}'$, and x'_I is smaller than the distance between $x^*_2{}'$ and x'_I.

In the case of the bi-criteria problem considered in the paper, the distance between $x^{*'}$ and x'_I is given by

$$r = \sqrt{f_1(x^*)^2 + f_2(x^*)^2}$$

Pareto solution x^* is treated as the preferred one (denoted further by $`x^*$) if the distance between it image $`x^{*'}$ and x'_I is the least of the distances between the images $x^{*'}$ of other Pareto solutions and point x'_I.

The problem outlined above will now be specified for the optimization of a thin-walled column of cruciform cross-section.

COLUMN OPTIMIZATION

Consider a pinned-pinned column of length L, of the thin-walled cruciform cross-section shown in

Figure 1, subjected to a central load P. The column, due to its cross-sectional shape and dimensions ($t \grave{A} L$), is prone to flexural and torsional buckling [14].

The flexural buckling load, P_{FB}, expressed in terms of dimensionless quantities $x = t/b$ and $h = b/L$ is equal to $P_{FB} = p^2 E L^2 h^4 x(1 + x^2 - x^3)/12$ while the torsional buckling load, P_{TB}, is given by $P_{TB} = 2EL^2h^2x^3(2-x)/[(1+n)(1+x^2-x^3)]$, where E and n denote the modules of elasticity and Poisson's ratio, respectively.

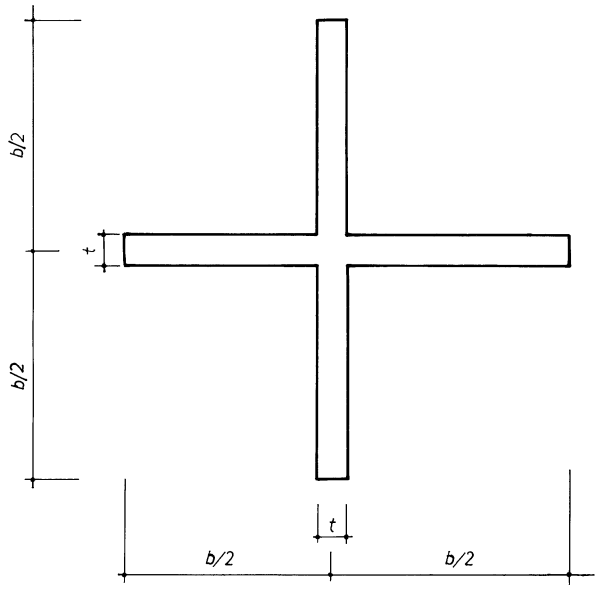

Figure 1. The cross-section of a column.

In some circumstances, yield of the column's material may occur before or simultaneously with buckling. The yield force, P_Y, is given by $P_Y = s_Y A = s_Y L^2 h^2 x (2-x)$, where s_Y is the yield stress and A is the cross-sectional area of the column.

The typical single-criterion optimization problems that can be formulated with reference to the above column are as follows.

Problem 1. For a given material volume V_1 and prescribed column's length L, find x and h that in the feasible domain D maximize the load P applied to the column.

Problem 2. For a given load P and column's length L, find x and h that minimize the column's volume V in the feasible domain D.

Formally, the two problems can be stated in

the following manner:

Problem 1. Find $\mathbf{x} = \{x_1\ x_2\} = \{x\ h\}$ that maximizes $f_1(x)=P=\min(P_{FB},P_{TB},P_Y)$ subject to $g_1(\mathbf{x}) = V_1 - L^3h^2x(2-x) = 0$ (volume is kept constant)

$$\left.\begin{array}{ll} g_2(\mathbf{x}) = x_{min}-x \le 0 \\ g_3(\mathbf{x}) = x-x_{max} \le 0 \\ g_4(\mathbf{x}) = h_{min}-h \le 0 \\ g_5(\mathbf{x}) = h-h_{max} \le 0 \end{array}\right\} \text{(side constraints)}$$

Problem 2. Find $\mathbf{x} = \{x_1\ x_2\} = \{x\ h\}$ that minimizes $f_2(x) = V = L^3xh^2(2-x)$ subject to $g'_1(\mathbf{x}) = P-\min(P_{FB}, P_{TB}, P_Y) = 0$ (the applied load is kept constant)

$g'_2(\mathbf{x}) = x_{min}-x \le 0$

$g'_3(\mathbf{x}) = x-x_{max} \le 0$ (side constraints)

$g'_4(\mathbf{x}) = h_{min}-h \le 0$ $\quad g'_5(\mathbf{x}) = h-h_{max} \le 0$

The bi-criterion problem, suggested in this paper, specified for the column considered, is to find x and h that make the column possibly the strongest, using possibly the least amount of material.

Stating it formally, the problem is to determine $\hat{x}^* = \{\hat{x}^*\ \hat{h}^*\}$ that constitutes the preferred solution within the Pareto solutions to the bi-criterion problem:

find $\mathbf{x} = \{x_1\ x_2\} = \{x\ h\}$ that

maximizes $f_1(\mathbf{x})=\min(P_{FB}, P_{TB}, P_Y)$,

minimizes $f_2(\mathbf{x}) = L^3 h^2x(2-x)$, satisfying the constraints $g_2(\mathbf{x})$, $g_3(\mathbf{x})$, $g_4(\mathbf{x})$ and $g_5(\mathbf{x})$.

The problem can be solved in a variety of ways. One option would be the use of the weighting method [4].

The weighting method transforms the vector optimization problem into the scalar problem whose solution coincides with the Pareto solution to the original problem. The scalar problem consists in finding the minimum point $\mathbf{x}^*$ of a single-valued objective function $z(\mathbf{x})$, formed as the sum of products $w_if_i(\mathbf{x})$, $i= 1,2,\ldots,n$, of non-negative weighting coefficients w_i's and the objective functions $f_i(\mathbf{x})$'s. The weighting coefficients are, usually, normalized so that $\sum w_i=1$. Consequently, in our bi-criterion case, $z(x)$ would be $w_1f_1(x) + w_2f_2(x) = w_1f_1(x) + (1-w_1)f_2(x)$.

The scalar minimization of $z(x)$, carried out for various values of weighting coefficients w_i, leads to creating a set of Pareto solutions $\mathbf{x}^*$. The solutions obtained are, subsequently, subjected to a ranking procedure, performed according to an adopted preference criterion. The criterion may be, for example, the above mentioned criterion of the minimum distance r of the image $\mathbf{x}^*$, of the Pareto solution $\mathbf{x}^*$ from the ideal solution's image x_I', measured in the objective functions space Of_1f_2.

Another option, well suited for solving the problem posed in this paper, is to determine and plot the contour lines of functions $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$. Their inspection, accompanied by the inspection of the directions of gradient vectors ∇f_1 and ∇f_2 permits, as will be shown in the numerical example, to recognize easily the set of Pareto solutions $\mathbf{x}^*$ in D. Next, the recognized Pareto solutions $\mathbf{x}^*$ are subjected to ranking, carried out in accordance with the criterion of the minimum distance r of the image $\mathbf{x}^*$' from $\mathbf{x}_I'$ in the objective function space $Of_1'f_2'$ or performance function space $Of_1'f_2'$.

The problem can also be solved by minimization, in the design variable space, of a scalar function $r(\mathbf{x}) = \sqrt{f_1'(x)^2 + f_2'(x)^2}$, $\mathbf{x} \in D$, that describes the distance of any feasible design's image (Pareto solutions' $\mathbf{x}^*$, included) from $\mathbf{x}_I'$. The numerical problem that follows is solved by using the last two methods.

Numerical example. The problem is solved for the following data: $E = 205$ GPa, $\nu = 0.3$, $L = 3.0$ m, $\sigma_Y = 350$ MPa, $x_{min} = 0.015$, $x_{max} = 0.055$, $h_{min} = 0.025$, $h_{max} = 0.075$.

The objective functions $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ for the above data are of the form

$f_1(\mathbf{x}) = \min\{1.5174^{*}10^6 h^4 x(1+ x^2-x^3),\ 2.8385^{*}10^6 h^2 x^3(2-x)/(1+ x^2-x^3);\ 3150h^2x(2-x)\}$

$f_2(\mathbf{x}) = 27.0h^2x(2-x).$

The feasible domain and contour lines of the functions $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are shown in Figure 2.

Examination of Figure 2 indicates that within rectangle CDEF the contour lines of $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are parallel which means that the gradient vectors ∇f_1 and ∇f_2 have the same directions. For any point $\mathbf{x} \in$ CDEF the increase of the value of $f_1(\mathbf{x})$ is possible in any direction which makes an angle $a \in (-p/2, p/2)$ with vector ∇f_1, while the decrease of $f_1(\mathbf{x})$ is possible only in the direction forming with vector $-\nabla f_2$ an angle $b \in (-p/2, p/2)$. Since $a \cap b$ is an empty set, there exists no such a direction from any point $\mathbf{x} \in$ CDEF which would allow for simultaneous increase of $f_1(\mathbf{x})$ and decrease of $f_2(\mathbf{x})$. Consequently, any point $x \in$ CDEF is the Pareto solution.

Let us now examine ridge BC of function $f_1(\mathbf{x})$. From any point $\mathbf{x}$ of the ridge, the increase of $f_1(\mathbf{x})$ can be obtained if a step is made in the direction contained in the cone of angle g, ($g < p$), formed by the tangents to the contour lines of $f_1(\mathbf{x})$ at $\mathbf{x}$ belonging to the ridge. At the same time, the decrease of $f_2(\mathbf{x})$ can be obtained if a step is made from x in the direction that forms an angle $b \in (-p/2, p/2)$ with the negative gradient of $f_2(\mathbf{x})$. Since $g \cap b = \phi$, there is no such a direction along which a step made would simultaneously increase $f_1(\mathbf{x})$ and decrease $f_2(\mathbf{x})$. Thus, any $\mathbf{x} \in$ BC is the Pareto solution.

By similar reasoning, supplemented by the effect of the constraint $g_2(\mathbf{x})=0$, one can also find that all of the points $\mathbf{x}$ belonging to segment AB are the Pareto solutions to the problem.

Among the determined Pareto solutions, there are two which, at the same time, constitute solutions to the single-criterion optimization problems. These solutions are represented in Figure.2 by points E(x= 0.055, h= 0.075) and A(x= 0.015, h= 0.025), respectively.

From the design variable space, let us now move to the performance function space Of^1f^2.

The image of the feasible domain D (area AHEG) in the performance function space $O'f^1f^1$ is presented in Figure. 3. The collection of points along the curve A'B'C'F'D'E' represents the image of the Pareto solution set. The point closest to the origin O' of the reference frame, denoted in Figure 3 by letter Z', is an image of the preferred solutions. In the DVS, the preferred solutions are represented by the points belonging to line PR, whose extremities P and R have the following coordinates (0.03333, 0.06792) and (0.03780, 0.06386), respectively.

For all the preferred solutions $\mathbf{x}^*$, the values of the objective functions are the same: $f_1(\mathbf{x}^*)=0.953$ MN and $f_2(\mathbf{x}^*)=0.00816 m^3$.

Comparing the maximum load sustained by the preferred design $\mathbf{x}^*$ and its volume with the P and V pertaining to other characteristics of the feasible designs, it can be found that $P(\mathbf{x}^*) = 0.953$ MN$= 0.503\ f_{1max} = 0.503(x_E) = 107.207^{*}f_{1min} = 107.207P(x_A)$, $V(\mathbf{x}^*) = 0.00816 m^3 = 1.991\ f_{2min} = 1.991V(\mathbf{x}_A) = 0.502\ f_{2max} = V(\mathbf{x}_E)$.

The present numerical example has also been solved by computer minimization of a scalar function $r(\mathbf{x}) = r(x, h) = \sqrt{f_1(\mathbf{x})^2 + f_2(\mathbf{x})^2}$, carried out in the design variable space. The results obtained are visualized in Figure 4, using the MATLAB 3-D graphics [8]. The Figure 4,

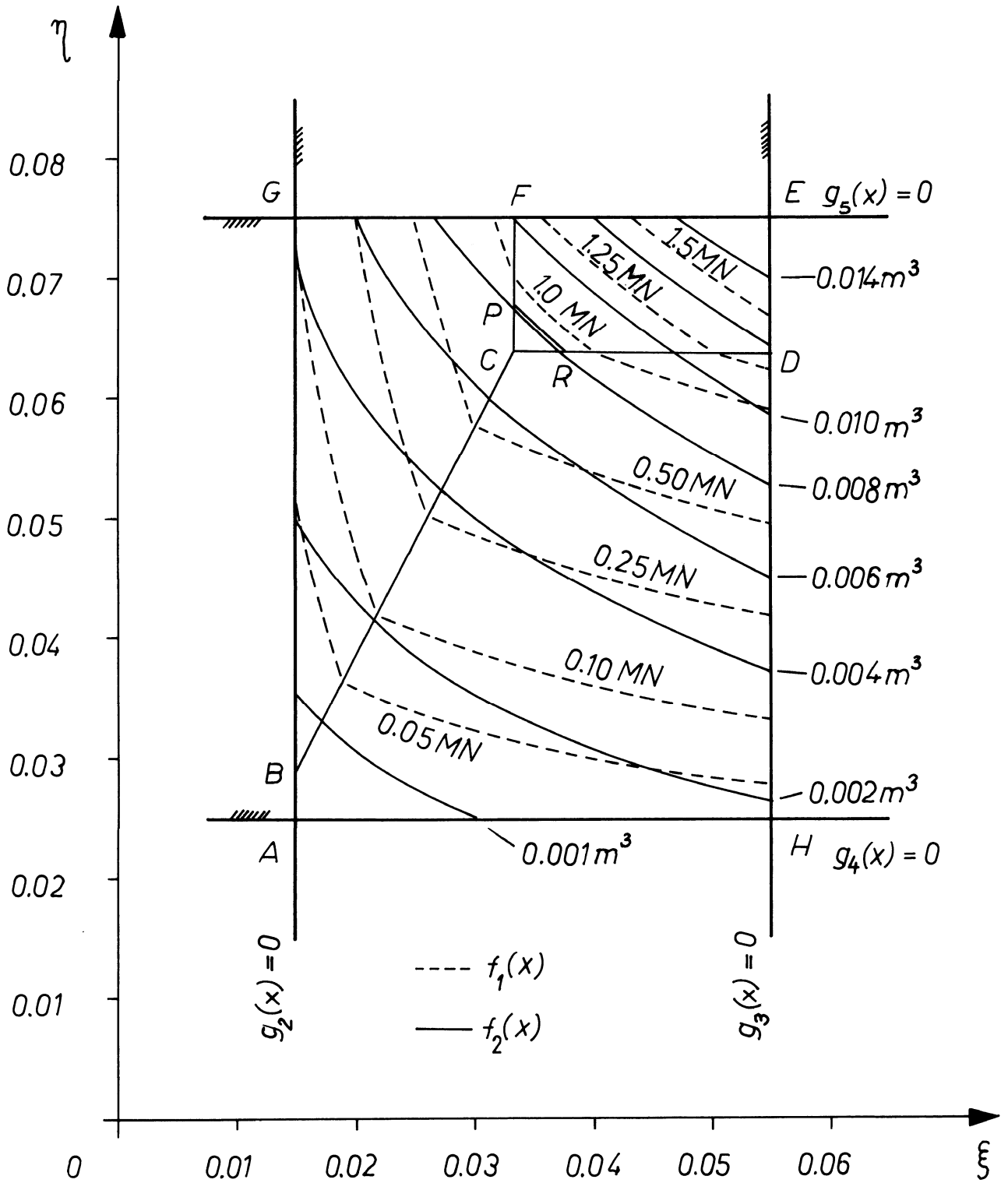

Figure 2. The feasible domain and contour lines of the objective functions $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$.

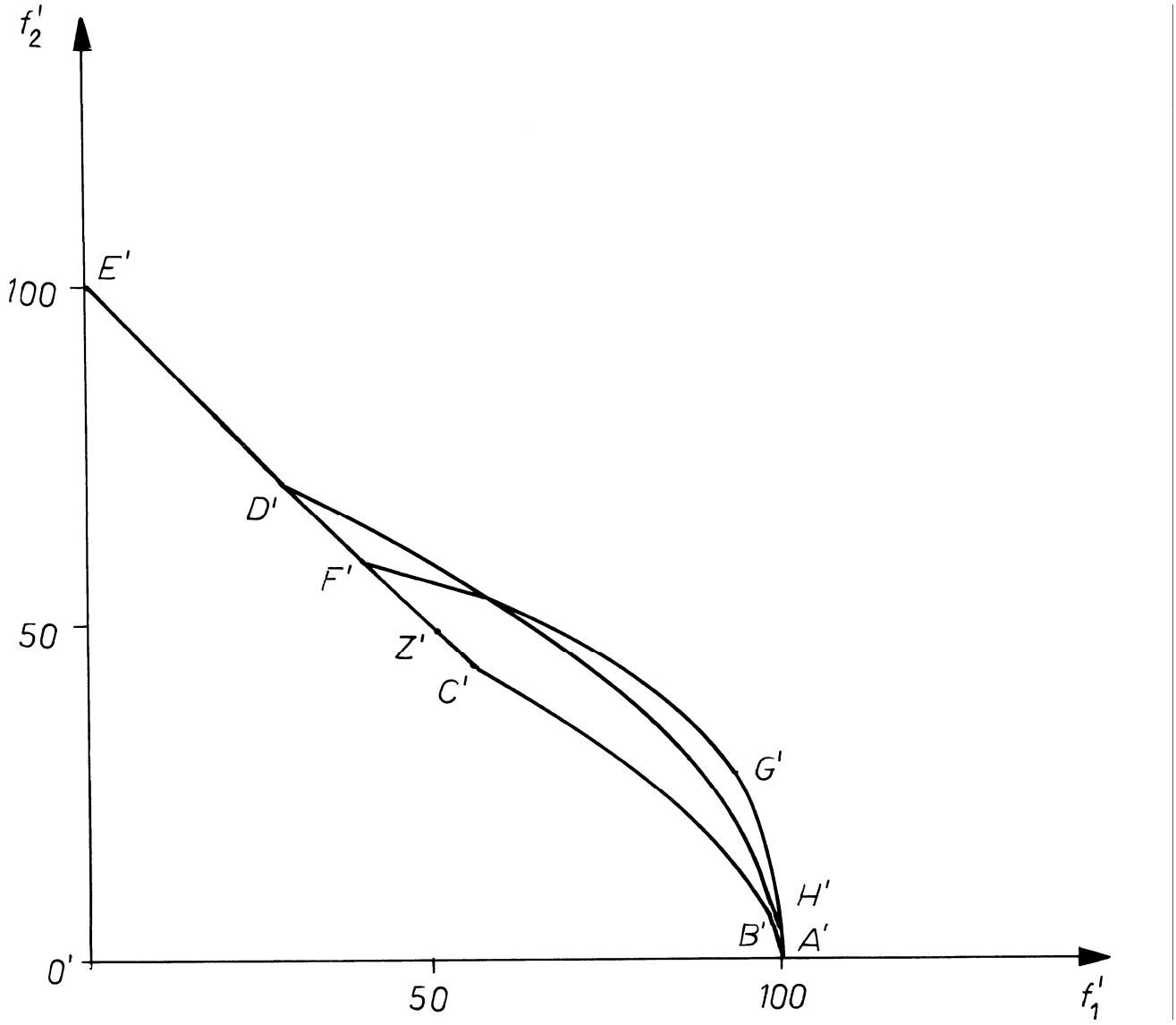

Figure 3. The image of the feasible domain in the performance function space.

shows the surface representing function r(x, h) and the shade map of r(x, h). The darkest region of the map refers to the set of the preferred solutions $\mathbf{x}^*$.

CONCLUDING REMARKS

Although structural optimization problems are by nature multiple-criteria, they are usually not treated as such. Designers frequently reduce them to single-criterion ones, choosing most often as the objectives either the maximization of the structure's strength or the minimization of the structure's cost. The main reason is the relative complexity of the solution procedures for multi-criteria problems.

Avoidance of formulating the optimization problems as multi-criteria ones in many cases is not justified. Firstly, multi-criteria formulations better reflect practical design reality than single-criterion problems do and; secondly, the computations involved in solving the former problems might not necessarily be complex. This

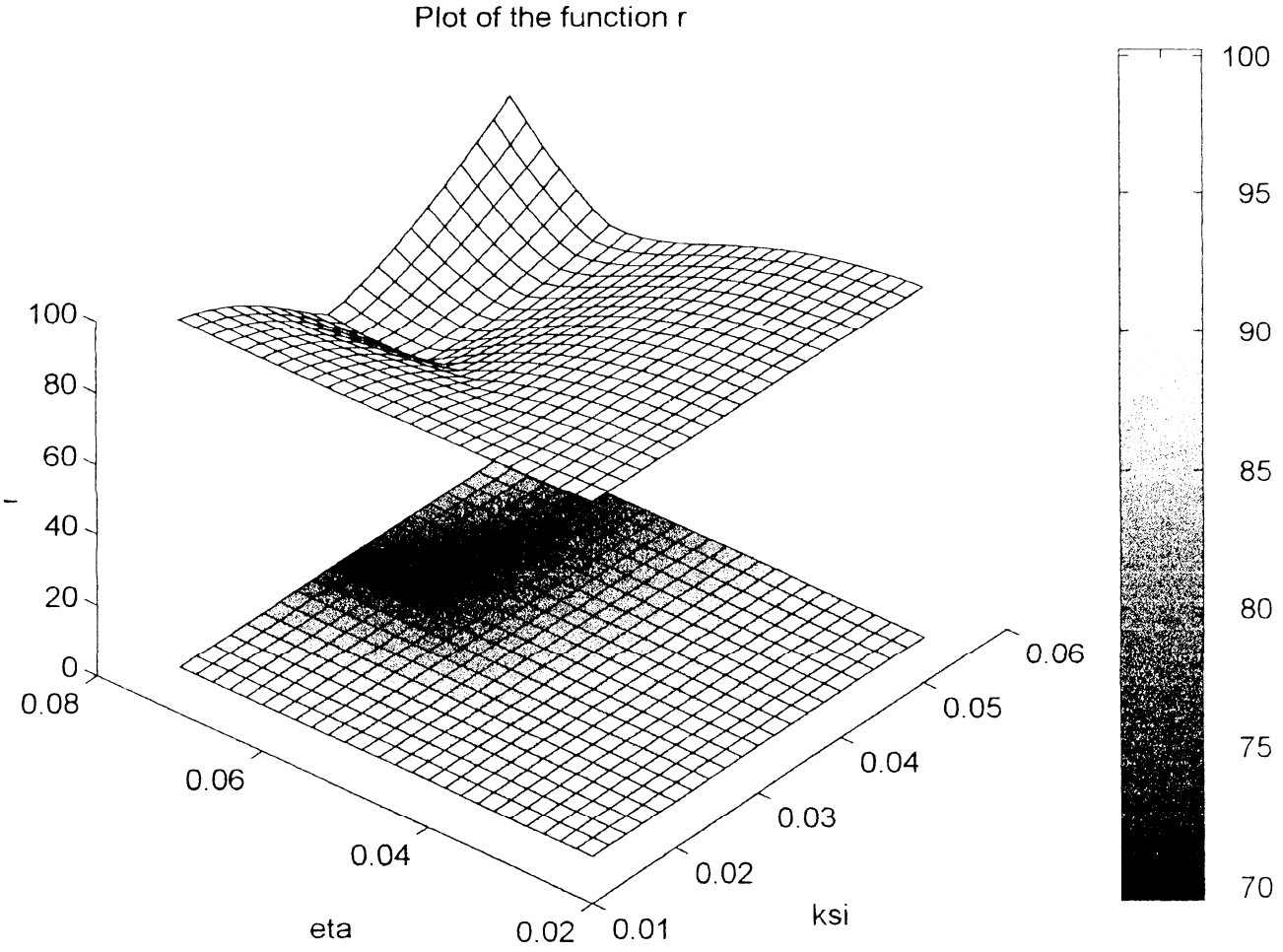

Figure 4. The surface representing function r(x,h) and shade map of r(x,h).

is especially true when both the number of the objective functions and the number of the design variables are kept low. Though the problem presented here is only bi-criteria, it takes into account quite essential objectives such as the maximization of the load factor for the structure and the minimization of the structure's volume. By formulating the problem in terms of two design variables only, as it was done in the numerical example, the problem was made amenable to solving by inspection of the contour lines of the objective functions.

An interesting feature of the bi-criteria problem considered in the paper is a certain degree of fuzziness with which the objectives of optimality are stated. Namely, unlike in the akin single-criterion problems, neither the value of the load to be sustained by the structure, nor the material volume need to be prescribed. Instead, we may simply say that the structure should be possibly the strongest and be made of the least amount of material. Clearly, however, the values of the load sustained and the structure's volume are not quite arbitrary; they are implicitly bracketed by the constraints imposed on the design variables.

LIST OF SYMBOLS

The following symbols are used in the paper

A= Cross-sectional area of a column
b= Width of the cruciform cross-section of a column
E= Modulus of elasticity
$f_1(\mathbf{x})$= load factor function
$f_2(\mathbf{x})$= structure' volume function
$f'_1(\mathbf{x})$= performance function
$f'_2(\mathbf{x})$= performance function
$g_i(\mathbf{x})$= i-th constraint function of Problem 1 and Problem 3
$g'_i(\mathbf{x})$= i-th constraint function of Problem 2
L= Length of a column
P= Central load applied to a column
P_{BP}= torsional buckling load
P_{TB}= flexural buckling load
P_Y= yield force
t= Thickness of the wall of a column
V= Column's volume
x_i= i-th design variable
D= Feasible domain
2D'= Image of feasible domain
$\mathbf{x}$= Vector of design variables
$\mathbf{x}'$= Image of x in the performance function space
$\mathbf{x}^*$= Pareto optimum solution
$\mathbf{x}^{*'}$= image of x^* in the performance function space
$\grave{\mathbf{x}}^*$= preferred Pareto optimum solution
$\grave{\mathbf{x}}^{*'}$= image of the preferred Pareto optimum solution in the performance function space
$\mathbf{x}_I$= ideal solution
$\mathbf{x}_I'$= image of x_I in the performance function space
χ= t/b=dimensionless quantity
η= b/l=dimensionless quantity
σ_Y= yield stress.

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