A FINITE CAPACITY PRIORITY QUEUE WITH DISCOURAGEMENT

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Abstract In this paper we report on a study of a two level preemptive priority queue with balking and reneging for lower priority level. The inter-arrival and the service times for both levels follow exponential distribution. We use a finite difference equation approach for solving the balance equations of the governing queueing model whose states are described by functions of one independent variable. Hence the balance equations may be viewed as a set of simultaneous difference equations and can be solved by using appropriate techniques.

Key Words Queue, Balance Equation, Balking, Reneging, Priority Queue, Markovian Queue

چکیده در این مقاله مطالعه بر روی یک صف با دو سطح تقدّم با طفره رفتن وردّ حق تقدّم سطح پایین تر را گزارش می کنیم. زمانهای خدمت و interarrival برای هر دو سطح از توریع نمایی پیروی می کنند. از یک شیوه معادلهٔ تفاضل محدود برای حل معادلات تراز حاکم بر قالب صف که حالات آن با توابع یک متغیره بیان می شود استفاده می کنیم. لذا معادلات تراز را می توان به صورت یک دستگاه معادلات تفاضلی همزمان انگاشت که آن را می توان با استفاده از روشهای مناسب حل کرد.

INTRODUCTION

In queueing systems, the general behavior with respect to the number of customers at service facilities may be mathematically described by a set of finite difference equations. The scope of the technique is not limited to exponential distribution, but applicable to general distributions, if such distributions are approximated by a finite number of exponential "stages" [1]. For such type of difference equations (finite difference equations) with more than one independent variables, no general solution is known [2]. Of course this is valid only when the coefficients in these equations are constant, i.e. instantaneous service and arrival rates are fixed and independent of system states. For such queueing systems, the solution must be sought on an individual basis using generating functions method [2]. This method is possibly applicable to the solution of finite difference equations. It is often difficult to apply this approach to finite capacity systems.

In practical queueing situations, the arriving customers may be discouraged due to long queue. Such queueing models involve the concept of balking and reneging and have been studied by several researchers [3-6]. The main purpose of this paper is to demonstrate the use of an approach suggested by Brandwajn [7] for a two level system with finite capacity involving balking and reneging.

This queueing system has two level priority queue where level 1 customers have preemptive priority over level 2. The customers of level 2 are also dependent on the balking and reneging parameters. The arrival and service rates for level 2 customers depend upon the levels queue length. The service discipline for each level is FCFS. If the service and arrival rates are constant, the system described reduced to in classic form [8,9].

THE MATHEMATICAL MODEL

We assume that n_1 and n_2 are the respective current numbers of customers at level 1 and level 2 in the system. The level 1 and 2 customers are drawn from infinite and finite (say N_2) populations respectively.

We denote λ_1 and μ_1 as constant arrival and service rates for level 1 customers. The queue dependent arrival and service rates for customers of level 2 are given by $\lambda_2 e_{n_2}$ and $\mu_2+(n_2-1)\alpha$, resepctively, where e_{n_2} $(0\le e_{n_2}\le 1)$ is the balking probability and α is the parameter corresponding to reneging. It is assumed that type 2 customers arrive with constant rate until the queue size of type 2 customers reaches to **k**. The customers of type 2 balk with probability of e_{n_2} when there are $n_2>$ **k** customers of level 2 present in the system.

The system balance equations are as follows:

$$\mu_1 p(n_1+1, n_2)-[\lambda_1+\lambda_2+\mu_1]\, p(n_1, n_2)+\lambda_1 p(n_1-1, n_2)+\lambda_2 p(n_1, n_2-1)=0; \quad 1\le n_2\le k,\ n_1=1,2,\ldots \qquad (1.1)$$

$$\mu_1 p(n_1+1, n_2)-[\lambda_1+\lambda_2 e_{n_2}+\mu_1]\, p(n_1, n_2)+\lambda_1 p(n_1-1, n_2)+\lambda_2 e_{n_2-1}\, p(n_1, n_2-1)=0; \quad k\le n_2\le N_2-1,\ n_1=1,2,\ldots \qquad (1.2)$$

$$\mu_1 p(n_1+1, N_2)-(\lambda_1+\mu_1)\, p(n_1, N_2)+\lambda_1 p(n_1-1, N_2)+\lambda_2 e_{N_2-1}\, p(n_1, N_2-1)=0; \quad n_1=1,2,\ldots \qquad (1.3)$$

$$\mu_1 p(n_1+1, 0)-(\lambda_1+\lambda_2+\mu_1)\, p(n_1, 0)+\lambda_1 p(n_1-1, 0)=0; \quad n_1=1,2,\ldots \qquad (1.4)$$

$$\mu_1 p(1, 0)-(\lambda_1+\lambda_2+\mu_1)\, p(0, 0)+\mu_2 p(0, 1)=0; \qquad (1.5)$$

$$\mu_1 p(1, n_2)-[\lambda_1+\lambda_2+\mu_2+(n_2-1)\alpha]\, p(0, n_2)+\lambda_2 p(0, n_2-1)+(\mu_2+n_2\alpha)\, p(0, n_2+1)=0 \quad 1\le n_2\le k \qquad (1.6)$$

$$\mu_1 p(1, n_2)-[\lambda_1+\lambda_2 e_{n_2}+\mu_1+(n_2-1)\alpha]\, p(0, n_2)+\lambda_2 e_{n_2-1}\, p(0, n_2-1)+(\mu_2+n_2\alpha)\, p(0, n_2+1)=0 \quad k\le n_2\le N_2-1,\ n_1=0 \qquad (1.7)$$

$$\mu_1 p(1, N_2)-[\lambda_1+\mu_2(N_2-1)\alpha]\, p(0, N_2)+\lambda_2 e_{N_2-1}\, p(0, N_2-1)=0 \quad n_1=0,\ n_2=N_2 \qquad (1.8)$$

These equations must be complemented by the normalizing condition

$$\sum_{n_1, n_2} p(n_1, n_2)=1 \qquad (2)$$

Now we present a dircet solution method and outline its application to the set of difference Equations 1.1 to 1.8.

THE SOLUTION METHOD

The unknown probability distribution $p(n_1,n_2)$ is a function of two independent variables n_1 and n_2. The basic idea of the direct solution method presented here is to consider such a function of two variables as a set of functions of one variable. We consider $\mathbf{n_1}$ as this variable and $\mathbf{n_2}$ as an index to identify the function. thus $p(n_1, n_2)$ can be rewritten as

$$p_{n_2}(n_1) = p(n_1, n_2);\ n_1 = 0, 1, 2, \ldots;\ n_2 = 0, 1, 2, \ldots, N_2 \tag{3}$$

With the help of (3), we may consider the difference equations for $p(n_1,n_2)$ as a set of simultaneous difference equations that can often be solved by simply eliminating all the functions except one and solving the resulting difference equations for that function.

We note that Equation 1.4 involves, without any elimination, only one function $p_0(n_1)$ which may easily be solved, since with respect to our independent variable n_1, its coefficients are constant. Then $p_{n_2}(n_1)$ for $n_2 = 1,2,\ldots$ may be computed using (1.1) - (1.3) since it involves only the functions $p_{n_2}(n_1)$ and $p_{n_2-1}(n_1)$ which are already known.

We consider shift operator E and difference operator Δ s.t. $Ef(x) = f(x+1)$ and $\Delta f(x) = f(x+1) - f(x)$. If $\Delta f(x) = \phi(x)$ then $f(x) = \Delta^{-1}\phi(x)$. By using these notations, (1.1) - (1.4) can be rewritten as

$$\psi_{n_2} E p_{n_2}(n_1) = -\lambda_2 e_{n_2-1} E p_{n_2-1}(n_1);\ n_1 = 0, 1, 2, \ldots;\ n_2 = 0, 1, \ldots, N_2 \tag{4}$$

where $\psi_{n_2}(E) = \mu_1 E^2 - (\lambda_1 + \lambda_2 e_{n_2} + \mu_1)E + \lambda_1$

and $e_{n_2} = \begin{cases} 1 \text{ for } n_2 = 0, 1, 2, \ldots, k \\ 0 \text{ for } n_2 = N_2 \end{cases}$

For $n_2 = 0$ $\qquad \psi_0 E p_0(n_1) = 0;\ n_1 = 0, 1, 2, \ldots \qquad (5)$

The general solution of a homogeneous difference equation with constant coefficient

$$\psi_{n_2}(E)\, p_{n_2}(n_1) = 0 \tag{6}$$

may be expressed in terms of the roots of the characteristic equation $\psi_{n_2}(r) = 0$. The roots of this quadratic equation are

$$r(n_2) = \begin{cases} \dfrac{(\lambda_1 + \lambda_2 + \mu_1) \pm \sqrt{(\lambda_1 + \lambda_2 + \mu_1)^2 - 4\lambda_1\mu_1}}{2\mu_1} \\ n_1 = 0, 1, 2, \ldots;\ 1 \le n_2 \le k \\[2ex] \dfrac{(\lambda_1 + \lambda_2 e_{n_2} + \mu_1) \pm \sqrt{(\lambda_1 + \lambda_2 e_{n_2} + \mu_1)^2 - 4\lambda_1\mu_1}}{2\mu_1} \\ n_1 = 0, 1, 2, \ldots;\ k \le n_2 \le N_2 - 1 \\[2ex] \dfrac{(\lambda_1 + \mu_1) \pm \sqrt{(\lambda_1 + \mu_1)^2 - 4\lambda_1\mu_1}}{2\mu_1} \\ n_1 = 0, 1, 2, \ldots;\ n_2 = N_2 \end{cases} \tag{7}$$

The general solution of Equation 6 is of the form $\sum_{i=1}^{2} d_i r_i^{n_1}$ where d_i is an arbitrary constant. Since $p_0(n_1)$ must tend to zero as $n_1 \to \infty$, we reject the root $r_1(0) > 1$, i.e., we set d_1 to zero when remaining root $r_2(0) < 1$ i.e.

$$\frac{(\lambda_1 + \lambda_2 e_{n_2} + \mu_1) - \sqrt{(\lambda_1 + \lambda_2 e_{n_2} + \mu_1)^2 - 4\lambda_1\mu_1}}{2\mu_1} < 1,\ 1 \le n_2 \le N_2 \tag{8}$$

The solution of Equation 5 may be expressed as

$$p_0(n_1) = C_0\,[r_2(0)]^{n_1} \tag{9}$$

where C_0, is an arbitrary constant to be determined from Equations 1.5 - 1.8 and normalizing condition (2). The solution of Equation 4 is calculated as a sum of the general solution of a homogeneous Equation 6 and a particular solutions of Equation 4. Such a particular solution may be obtained as a result of the oepration

$$-\lambda_2 e_{n_2-1} E/\psi_{n_2}(E) p_{n_2-1}(n_1);\ 1 \le n_2 \le N_2 \tag{10}$$

When the function $p_{n_{2}-1}(n_1)$ is of the form $\sum_j g_j a_j^{n_1}$ where g_j and a_j are constants, (10) becomes

$$\sum_j g_j[(-\lambda_2 e_{n_2-1})a_j] / \psi_{n_2}(a_j)\, a_j^{n_1} \; ; \; 1 \le n_2 \le N_2 \qquad (11)$$

Using Equation 9 and the general solution of Equation 6, it is clear that $p_{n_2-1}(n_1)$ is of the form considered.

Using the fact that $\psi_k(r_2(k)) = 0$ and hence

$$\psi_{n_2}(r_2(k)) = r_2 k\, [\lambda_2 e_k - \lambda_2 e_{n_2}]; \; 1 \le n_2 \le N_2 \qquad (12)$$

In general, we have

$$p_{n_2}(n_1) = \sum_{i=0}^{N_2} C_i\, [r_2(i)]^{n_1} [\prod_{j=k}^{n_2-1} \frac{\lambda_2 e_j}{\lambda_2 e_{j+1}} - \lambda_2 e_i]$$

$$n_1 = 0, 1, 2, \ldots; \; N_2 = 0, 1, \ldots, k \qquad (13)$$

The solution exist only if

$$r_2(n_2) < 1 \forall\, n_2 \qquad (14)$$

We can easily show that (14) is equivalent to simple condition $\lambda_1 < \mu_2$.

The constant C_{n_2}; $n_2 = 0, 1, \ldots, N_2$ have to be determined using Equations 1.5-1.8 and the normalization condition 2 which may impose yet other conditions for the existence of the steady state probabilities. For each value of n_2 of conditional probability $p(n_1/n_2)$ must sum to unity.

Define a set of new constants G_{n_2} with $G_{n_2} = p(n_2)\, G_{n_2}$. From Equation 9 we obtain $G_0 = 1 - r_2(0)$. In general, using Equation 13, we get $G_{n_2} = f_{n_2}(1 - r_2(n_2))$ where f_{n_2} is determined from following recurrence relation

$$f_l = \{1 + g_l\, (1 - \gamma_2(l)/h_l\}^{-1} \qquad (15)$$

with

$$g_l = \sum_{i=0}^{l-1} t_{l,i} \text{ and } h_l = 1 - \sum_{i=0}^{l-1} (1 - r_2(i)) t_{l,i} \qquad (16)$$

where

$$t_{l,i} = \prod_{j=i}^{l-1} [\{\mu_2 + j\alpha\}/\lambda_2 (e_{j+1} - e_j)] \prod_{j=1}^{k-2} p(n_1 = 0/j+1) \qquad (17)$$

We have

$$p(n_1 = 0/k) = G_k/h_k \text{ and } f_0 = 1 \qquad (18)$$

The remaining constant H may be given as

$$H = \{1 + \sum_{n_2=1}^{N_2} \prod_{l=1}^{N_2} \frac{\lambda_2 e_{l-1}}{u(l)}\}^{-1} \qquad (19)$$

Now we consider the case when there is a finite queueing room N_1 positions for customers of level 1. In this particular case Equations 4-6 are valid only upto $n_1 = N_1 - 1$. For $n_1 = N_1$ a boundary equation must be added.

$$-[\lambda_2 e_{n_2} + \mu_1] p_{n_2}(N_1) + \lambda_1 p_{n_2}(N_1 - 1) = -\lambda_2 e_{n_2-1}\, p_{n_2-1}(N_1), \; 1 \le n_2 \le N_2 \qquad (20)$$

Since the maximum value of n_1 is now finite, both roots of the characteristic equation $r_1(n_2)$ and $r_2(n_2)$ have to be present in the general solution of homogeneous Equation 6.

This gives for $n_2 = 0$

$$p_0(n_1) = B_0\, [r_1(0)]^{n_1} + C_0\, [r_2(0)]^{n_1}; \; n_1 = 1, 2, \ldots, N_1 \qquad (21)$$

In general

$$p_0(n_1) = \sum_{i=0}^{n_2} \{B_i\, [r_1(i)]^{n_1} + C_i\, [r_2(i)]^{n_1}\} \prod_{j=1}^{N_2-1} \frac{\lambda_2 e_j}{\lambda_2 e_{j+1} - \lambda_2 e_i} \qquad (22)$$

The new arbitrary constant B_i are disposed of, so as to satisfy the boundary equations and remaining C_i's are determined as mentioned before, from Equations 1.5 to 1.8 and normalization condition.

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