VIBRATION OF ROAD VEHICLES WITH NON-LINEAR SUSPENSIONS

E. Esmailzadeh and G. Nakhaie-Jazar

Department of Mechanical Engineering

B. Mehri

Department of Mathematical Sciences
Sharif University of Technology
Tehran, Iran

Abstract In order to investigate the effects of non-linear springs in vibrating behaviour of vehicles, the independent suspension of conventional vehicles could be modelled as a non-linear single degree of freedom system. The equation of motion for the system would be a non-linear third order ordinary differential equation, when considering the elasticity of ruber bushings in joints of shock absorber. It is desirable that the system be stable, i.e., periodic inputs periodic outputs. In order to obtain the conditions that guarantee the existence of periodic solutions, and therefore, stable responses, the Schauder's fixed point theorem has been implemented to a general third order equation. Thus, the adapted conditions could be used for a wide range of dynamical systems. For the numerical analysis, a rapid convergence method has been developed, and used to solve the model. The correctness of periodic conditions and the numerical algorithm have been demonstrated.

Key Words Non-linear Suspensions, Vehicle Dynamics, Periodic Conditions, Fixed Point Theorem, Green's Function

چکیده نشان داده ایم که چگونه می توان رفتار ارتعاشی یک خودرو با فنرهای غیر خطی را با بررسی یک مدل ریاضی و معادلات غیر خطی آن شناخت. با درنظر گرفتن ضریب سختی لاستیکهای دو انتهای کمک فنر، مدل ریاضی خودرو با یک معادله دیفرانسیل غیر خطی مرتبه سوم بیان می شود. از آنجا که درنظر است ارتعاشات خودرو پایدار باشد باید بکوشیم پارامترهای مدل را به گونه ای برگزینیم که واکنش آن به تحریک ورودی سینوسی، کراندار بماند و واگرا نشود. ازاین رو به یاری قضیه نقطه ثابت (Schauder)، شرطی را می جوییم که متناوب بودن (Periodicity) و پایداری مدل را ضمانت کند. برای این منظور، یک معادله فراگیر مرتبه سه را بررسی کرده ایم تا یافته هایمان برای دسته بزرگتری از دستگاههای مرتبه سه کارآمد باشد. افزون برآن، روشی عددی برای یافتن شرایط آغازین دستگاه که واکنش تناوبی پدید آورد، ساخته ایم که همگرایی آن بسیار مطلوب و سریع است.

INTRODUCTION

During the last few decades many scientific papers dealing with suspension analysis and design have been published, [1,2]. Attention has been especially paid to the theoretical study of the dynamic behaviour of active suspension systems, [3,4].

For an industrial application it seemed useful to study on passive suspension whose physical characteristics do not vary, but have non-linear character. Important features of the real world car suspension design problem are that only a fixed and limited suspension working space is available, and that such vehicles have to traverse road surfaces of widely differing roughnesses. These results have made it clear that the chief limitation of conventional fixed parameter passive suspension system arises from the need for compromise in the choice parameters between the demands of smoothness of the surfaces, vehicle attitude control with load changes and manoeuvring, and high speed handling quality, [5,6].

In order to facilitate such a compromise, the relationship between the extent of the stiffness and that of damping variations must be provided and the performance gains must be obtained, [7]. Using non-linear springs might be a way to overcome some of the limitations. With non-linear elements, the simplicity and stability of the system would be changed and therefore, its behaviour should be analysed.

The assumption of linear behaviour of mechanical elements, although simplifies the solution considerably, but is far too ideal for most real systems. Non-linear systems, being more of a realistic representation of the nature, could exhibit a somewhat complex behaviour. Their analysis requires a thorough investigation into the solution of the governing differential equations. These non-linear differential equations that, usually, do not provide any exact solutions and thus must be solved numerically, [8].

The most important step in studying non-linear dynamical systems is to obtain conditions which guarantee the existence of periodic solutions, and hence calculate these solusions by implementing suitable numerical techniques. The significance of periodic solutions, lies on the fact that these solutions represent the steady state response of the system. It is known that the most analysis of non-linear systems are on the following second order, whose periodic condition has earlier been discussed elsewhere, [8,9].

$$x'' + g_1(x)x' + g(x,x',t) = e(t) \qquad (1)$$

$$g(x, x', t) = g(x, x', t + \tau) \quad e(t) = e(t + \tau) \qquad (2)$$

We have modelled the front suspension of a conventional vehicle with a third order differential equation as a non-linear dynamic vibration system. Then, an attempt has been made to obtain the necessary and sufficient conditions for periodicity of the corresponding system response using Schauder's fixed point theorem. After obtaining the periodic conditions, the same differential equation has then been solved numerically. The obtained numerical solutions, not only demonstrate the response of the system, but also offer a means to check whether or not the proposed sufficient conditions are valid.

VEHICLE VIBRATION WITH NON-LINEAR SUSPENSION

Figure 1 represents the essential parts of the front suspension of a road vehicle showing the unsprung mass consisting of the tire, the wheel and the stub axle connected by a rubber bushing to a hydraulic shock absorber and the main spring. The other end of the shock absorber is connected by another rubber bushing to a sub-frame of the car body. A set of whishbone link arms at each end serve to stabilise the unit.

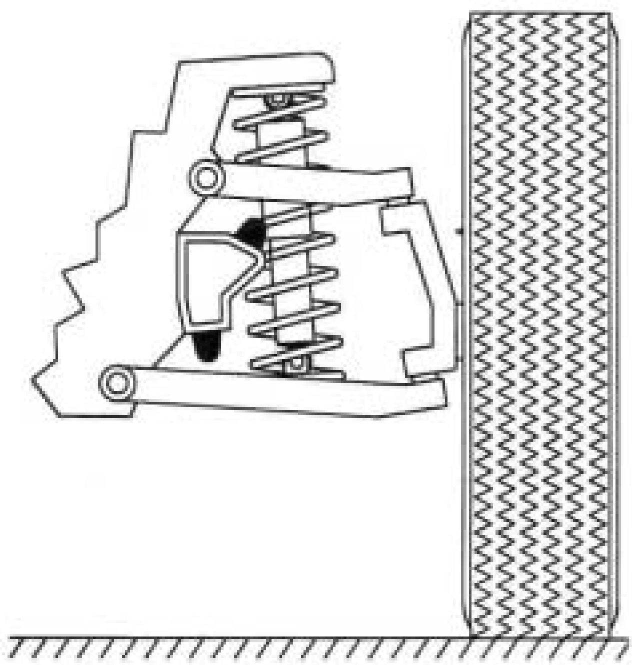

Figure 1. Essential parts of the front suspention of a conventional vehicle.

This mechanical model can be considered as a two degree-of-freedom dynamical system, [10]. The tire stiffness is assumed to be large enough and could simplify the system in the form of a single degree of freedom system.

The co-ordinates x, y and z represent the body motion, wheel excitation, and the displacement at the connection point of the rubber bushing and the hydraulic shock absorber respectively. The equations of motion may then be written as:

$$-k_1(x-y) - k_2(x-z)= M_p x'', \quad k_2(x-z)= c(z'-y') \qquad (3)$$

In order to obtain the relation between the input displacement y, and the output motion x, the variable z should be eliminated. Thus,

$$x''' + \frac{k_2}{c} x'' + \frac{k_1 + k_2}{M_p} x' + \frac{k_1 k_2}{M_p c} x = \frac{k_1 + k_2}{M_p} y' + \frac{k_1 k_2}{M_p c} y \qquad (4)$$

Assuming a periodic profile for the surface, the input displacement y could well be represented by a periodic function provided that the vehicle is travelling at a constant speed.

In case of linear springs and dampers, the solution of this third order differential equation could be easily derived. However, due to non-linear behaviour of real mechanical springs and shock absorbers, such an over-simplification is not always realistic. For the proposed model with non-linear elements, a third order non-linear differential equation is obtained.

Experiments show that with relatively large displacements, the spring rate may be expressed as: $a+b\delta^2$, where a is a positive constant for hard springs. On the other hand, for the case of a soft spring b ought to be negative. The factor δ represents the relative displacement of the two ends of the considered spring and therefore, for the main spring, $\delta = x-y$.

Considering both k_1 and k_2 to be non-linear functions of displacement x, while c being a function of speed x', then Equation 4 would become a simplified case of the following general equation,

$$x''' + g_1(x')x'' + g_2(x)x' + g(x, x',t)= e(t) \qquad (5)$$

where

$$g_1(x') = \frac{k_2}{c} \qquad g_2(x) = \frac{k_1 + k_2}{M_p} \qquad (6)$$

$$g(x, x',t) - e(t) = \frac{k_1 k_2}{M_p c} x - \frac{k_1 + k_2}{M_p} y' - \frac{k_1 k_2}{M_p c} y \qquad (7)$$

For a specific example a system with all elements being linear except the main spring k_1, may be considered. Hence, by assuming $k_1 = a+b\delta^2$ and $\delta = x-y$, Equation 4 may be rewritten as,

$$x''' + \frac{k_2}{c} x'' + \frac{a+k_2}{M_p} x' + \frac{b}{M_p c}(x-y)^2 x' \frac{a k_2}{M_p c} x + \frac{b k_2}{M_p c}(x-y)^2 x =$$

$$\frac{a+k_2}{M_p} y' + \frac{b}{M_p}(x-y)^2 y' + \frac{a k_2}{M_p c} y + \frac{b k_2}{M_p c}(x-y)^2 y \qquad (8)$$

MATHEMATICAL ANALYSIS

Consider the following class of non-linear differential equations

$$x'''+g_1(x')x'' + g_2(x)x' + g(x, x', t)= e(t) \qquad (9)$$

Equation 9 is a third order non-linear differential equation for which the exact solution in the general case, is not known. However, various numerical technqiues should be implemented in order to determine its approximate periodic solutions. The Schauder's fixed point theorem enables one to find the conditions for the existence of periodic solutions, whithout evaluating such answers.

Regarding Equation 9, the aim is to obtain conditions for periodicity of solution, with the same time period as that of the input excitation. The method presented here is based on the Schauder's fixed point theorem, [11].

Assuming g and e to be periodic functions of t, the necessary and sufficient conditions for Equation 9 to have a periodic solution x with the same time period τ and e are:

$$x^{(i)}(0)= x^{(i)}(\tau) \qquad i= 0, 1, 2 \tag{10}$$

Introducing the Green's function $G(t, s)$, the solution of Equation 9 can be expressed as:

$$x(t)= \int_0^{\tau} G(t,s) [g_1(x'(s))x''(s) + g_2(x(s))x'(s)+ g(x(s), x'(s),s) - e(s)]ds \tag{11}$$

If $$\int_0^{\tau} e(t)dt= 0 \tag{12}$$

Then the last condition in Equation 12 for the feasible $x(t)$, which satisfies the last condition in Equation 3, must be satisfied. Therefore,

$$\int_0^{\tau} g(x'_0(s), x'(s), s)ds= 0 \tag{13}$$

Equation 13 expresses the sufficient condition for periodicity of a solution of Equation 9. In order to find conditions that ensure the existence of $x_0(t)$ which satisfies Equations 11 and 13, the Schauder's fixed point theorem may be applied.

Let $C[0, \tau]$ be the space of all differentiable functions on $[0, \tau]$ equipped with the following norm:

$$x = Max\{x(t); \; t \in [0, \tau]\} \tag{14}$$

The complete normed linear space B could be defined in the following form:

$$B= C[0, \tau] \times C[0, \tau] \times C[0, \tau] \times R \tag{15}$$

The norm of B elements could be defined as:

$$(x, x', x'', h)= x + x' + x'' + h \tag{16}$$

On the space B, the operator U could be defined as following:

$$U(x, x', x'', h)= (x, x', x'', h) \tag{17}$$

where:

$$x^{(i)}(t)= h^{(i)}\int_0^{\tau} G^{(i)}(t,s) [g_1(x'(s))x''(s) + g_2(x(s))x'(s) + g(x(s), x'(s), s) - e(s)]ds \qquad i= 0, 1, 2 \tag{18}$$

$$h = h-\frac{1}{\tau}\int_0^{\tau} g(x(s), x'(s), s)ds \tag{19}$$

Hence, the operator U represents a continuous mapping from B into itself. A closed convex subset of B could be defined as:

$$S=\{(x,x',x'',h) \in B; x\leq K+x'\leq K+x''\leq K+h\leq(\nu+2m)\} \tag{20}$$

where,

$$x\geq n; \qquad \nu\geq 0; \qquad t \in [0,\tau] \tag{21}$$

$$m= Max\,\{MM_0\tau, MM_1\tau, MM_2\tau, F\} \tag{22}$$

$$F= Max\{g(x,x',t); t\in [0, \tau], x \leq K\} \tag{23}$$

$$M= Max\,\{g_1(x'(t))x''(t)+g_2(x(t))x'(t) + g(x(t), x'(t), t)-e(t); t \in [0, t], \; x^{(i)} \leq K\} \quad i= 0,1,2$$

$$M_i = Max\left\{\frac{\partial^i G(t,s)}{\partial t^i}; (t, s) \in [0, \tau] \times [0, \tau]\right\} i= 0,1,2 \tag{25}$$

If it is shown that the operator U has a fixed point in the set S, then there is a function x_0 for which:

$$U(x_0, x_0, \ddot{x}_0, h_0) = (x_0, x_0, \ddot{x}_0, h_0) \quad (26)$$

Considering Equations 18 and 19, then x_0 must satisfy both Equations 11 and 13. Consequently, x_0 would be the desired periodic solution of Equation 9.

According to Schauder's fixed point theorem, existence of a fixed point is proved if:

$$U(S) \subset S \quad (27)$$

It can be shown that if $(v+3m) \leq K$, for any

$(x, x', x'', h) \in S$

its corresponding transformation, i.e.,

(x, x', x'', h)

is also a member of S and the proof is completed.

Regarding the foregoing discussion, a theorem could be deduced:

Theorem: Considering Equation 9 together with periodic conditions given by (10), at least one solution with the same time period τ as functions g and e exists when:

$$(v+3m) \leq K; \qquad xg(x, x', t) > 0, \qquad t[0, \tau] \quad (28)$$

PERIODIC CONDITION FOR VIBRATION OF VEHICLE SUSPENSION

Suppose that the input excitation of the system with Equation 7 is represented by $y = y_0 Cos(2\pi t)$. Then the sufficient condition (28) for periodicity of the response of the system with $v = 0$ will be,

$$\frac{K_2}{c}K + \frac{a+k_2}{M_p}K + \frac{b}{M_p}K\,(K+y_0)^2 + \frac{a\,k_2}{M_pc}K + \frac{a+K_2}{M_p}(2\pi y_0) + \frac{b}{M_p}(K-y_0)^2(2\pi y_0) + \frac{a\,k_2}{M_pc}y_0 + \frac{bk_2}{M_pc}(K-y_0)^2 y_0 < \frac{K}{3}$$

(29)

Considering the following numerical values for the parameters involved

$K_2 = 100$ kgf cm; $c = 1000$ kgf.s cm; $a = 500$ kgf cm
$M_p = 5000$ kg; $b = 10$ kgf cm^3; $y_0 = 0.1$ cm (30)

it may be verified that the Inequality 29 is being satisfied for $K = 0.76$.

NUMERICAL PROCEDURE

By making certain that a periodic solution exists, the next step is to calculate this solution by the use of numerical methods. The differential equation is assumed that could be expressed as,

$$x''' + g(x, x', x'', t) = e(t) \quad (31)$$

in which all explicit functions of time are assumed to be periodic with the same period τ. The purpose of the present discussion is to calculate solutions of Equation 31 which are periodic with the same period τ. Hence, such a solution should satisfy the boundary Conditions 10.

Regarding the foregoing explanations, the problem reduces to finding proper values for

$$\alpha = x(0) \quad \beta = x'(0) \quad \gamma = x''(0) \quad (32)$$

such that the corresponding solution of Equation 33 would satisfy the following set of algebraic Equations:

$$\begin{aligned} \varphi(\alpha, \beta, \gamma) &= x(\alpha, \beta, \gamma, \tau) - \alpha = 0 \\ \theta(\alpha, \beta, \gamma) &= x'(\alpha, \beta, \gamma, \tau) - \beta = 0 \\ \psi(\alpha, \beta, \gamma) &= x''(\alpha, \beta, \gamma, \tau) - \gamma = 0 \end{aligned} \quad (33)$$

Now, one could guess the values of $(\alpha_0, \beta_0, \gamma_0)$ as the initial conditions. Then the Equation 33 could be solved numerically to evaluate $x(\tau), x'(\tau), x''(\tau)$. The validity of the initial guess could be checked with the following criterion:

$$\varphi(\alpha_0, \beta_0, \gamma_0)/\alpha_0 + \theta(\alpha_0, \beta_0, \gamma_0)/\beta_0 + \psi(\alpha_0, \beta_0, \gamma_0)/\gamma_0 < \varepsilon \qquad (34)$$

where ε is a convergence tolerance. If it is valid, then $(\alpha_0, \beta_0, \gamma_0)$ would be the proper set of initial conditions and the corresponding solution of Equation 31 would be τ-periodic. If it is not valid, the Newton-Raphson method could be applied to obtain a more feasible set of initial conditions $(\alpha_1; \beta_1, \gamma_1)$, such that,

$$\alpha_1 = \alpha_0 + \Delta\alpha_1, \quad \beta_1 = \beta_0 + \Delta\beta_1, \quad \gamma_1 = \gamma_0 + \Delta\gamma_1 \qquad (35)$$

Using Taylor series expansions and neglecting all second and higher order terms, one obtains,

$$\varphi(\alpha_1, \beta_1, \gamma_1) \approx \varphi(\alpha_0, \beta_0, \gamma_0) + \frac{\partial\varphi}{\partial\alpha}\Delta\alpha_1 + \frac{\partial\varphi}{\partial\beta}\Delta\beta_1 + \frac{\partial\varphi}{\partial\gamma}\Delta\gamma_1$$

$$\theta(\alpha_1, \beta_1, \gamma_1) \approx \theta(\alpha_0, \beta_0, \gamma_0) + \frac{\partial\theta}{\partial\alpha}\Delta\alpha_1 + \frac{\partial\theta}{\partial\beta}\Delta\beta_1 + \frac{\partial\theta}{\partial\gamma}\Delta\gamma_1$$

$$\psi(\alpha_1, \beta_1, \gamma_1) \approx \psi(\alpha_0, \beta_0, \gamma_0) + \frac{\partial\psi}{\partial\alpha}\Delta\alpha_1 + \frac{\partial\psi}{\partial\beta}\Delta\beta_1 + \frac{\partial\psi}{\partial\gamma}\Delta\gamma_1 \qquad (36)$$

where all derivatives are calculated at $(\alpha_0, \beta_0, \gamma_0)$. According to Equations 33, the proper set of increments $\Delta\alpha_1, \Delta\beta_1, \Delta\gamma_1$, could be calculated through the solution of Equations 36 with zero for the right hand sides. Finally, Equation 35 could be used to provide the improved initial conditions. This procedure could be repeated until the proper set of initial conditions satisfying Inequality 34 are determined.

For calculating the derivatives in Equation 36 effectively, one could find the following relations, with the use of Equations 33,

$$\frac{\partial\varphi}{\partial\alpha} = \frac{\partial x(\alpha, \beta, \gamma)}{\partial\alpha} - 1 \qquad \frac{\partial\varphi}{\partial\beta} = \frac{\partial x}{\partial\beta} \qquad \frac{\partial\varphi}{\partial\gamma} = \frac{\partial x}{\partial\gamma}$$

$$\frac{\partial\theta}{\partial\alpha} = \frac{\partial x'(\alpha, \beta, \gamma)}{\partial\alpha} \qquad \frac{\partial\theta}{\partial\beta} = \frac{\partial x'}{\partial\beta} - 1 \qquad \frac{\partial\theta}{\partial\gamma} = \frac{\partial x}{\partial\gamma}$$

$$\frac{\partial\psi}{\partial\alpha} = \frac{\partial x''(\alpha, \beta, \gamma)}{\partial\alpha} \qquad \frac{\partial\psi}{\partial\beta} = \frac{\partial x''}{\partial\beta} \qquad \frac{\partial\psi}{\partial\gamma} = \frac{\partial x''}{\partial\gamma} - 1 \qquad (37)$$

The partial derivatives of x, x' and x'' with respect to α, β or γ at some point are obtained by imposing perturbations on the corresponding initial conditions and then analysing the effects of such perturbations in $x(\tau)$, $x'(\tau)$ and $x''(\tau)$.

RESULTS

The periodicity condition (28) is illustrated in Figure 2. The appropriate domain of K could easily be found by assuming values for m and v.

With a computer program based on the foregoing numerical techniques, we have detected the feasible initial conditions for periodicity. The parameters of the system have been chosen to be identical to the numerical values given in (30). Then the proper initial conditions for periodic answer were computed as:

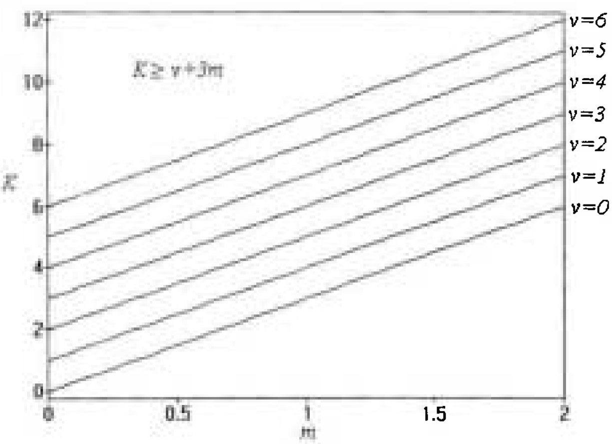

Figure 2. Illustration of the periodicity condition.

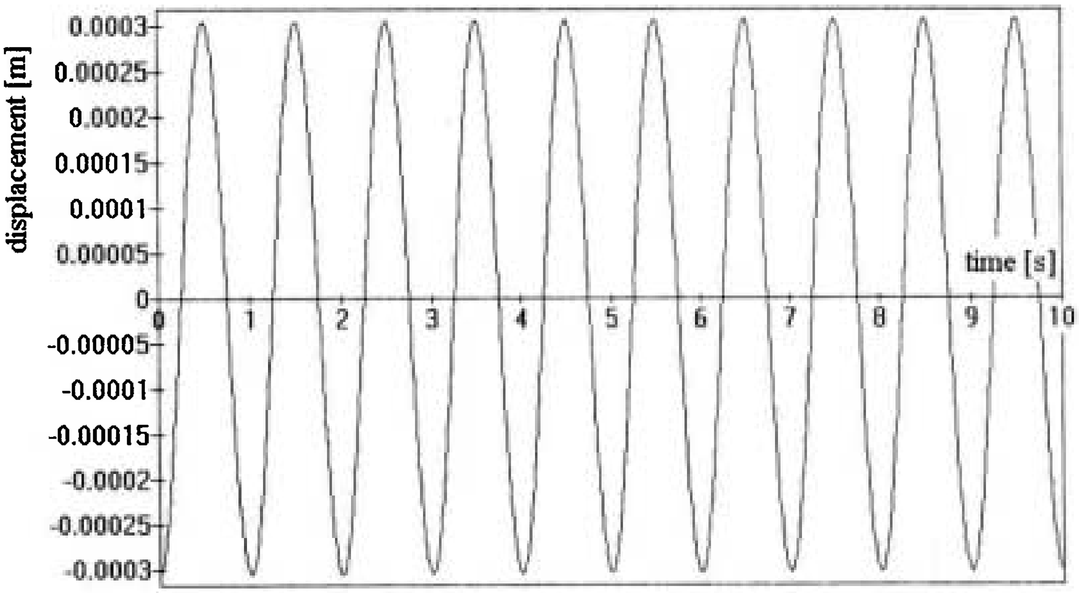

Figure 3. Time hisotry of the system for evaluated initial conditions.

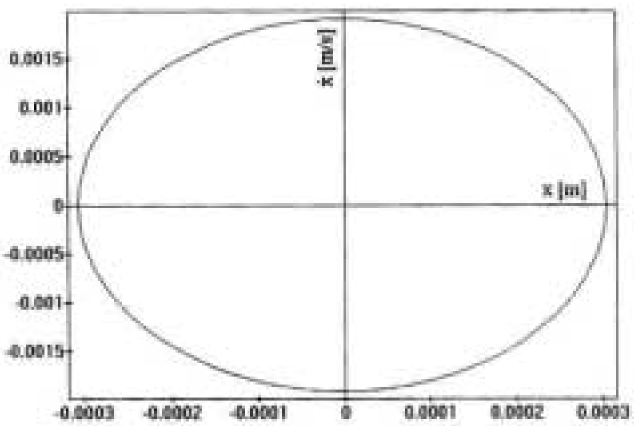

Figure 4. Phase plane trajectory for the periodic response of the system.

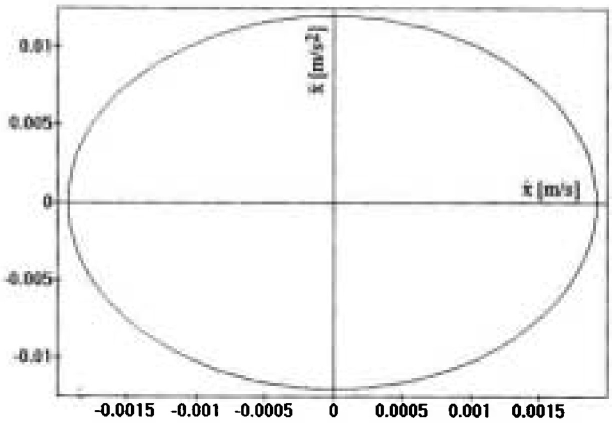

Figure 5. Plot of $\dot{x}$ - $\ddot{x}$ for the periodic response of the system.

$$x(0)= -3.05928\times10^{-4};\ x'(0)= 5.0073\times10^{-6}$$

$$x''(0)= 1.203694 \times 10^{-2} \tag{38}$$

for which:

$$x(\tau)= -3.05927\times10^{-4};\ x'(\tau)= 5.0091\times10^{-6}$$

$$x''(\tau)= 1.203694 \times 10^{-2} \tag{39}$$

These results compare very satisfactorily with the Conditions 10, and are in agreement with the previously computed value of *K*.

The time history of state variable have been evaluated and shown in Figure 3. It is demonstrated that the calculated initial conditions are proper. Figures 4,5 and 6 show the appropriate diagram in state planes respectively. It is interesting to note that the x-x'' diagram is in the form of a line segment passing through the origin, which is swept at each time period. Hence, the solution represents a harmonic response.

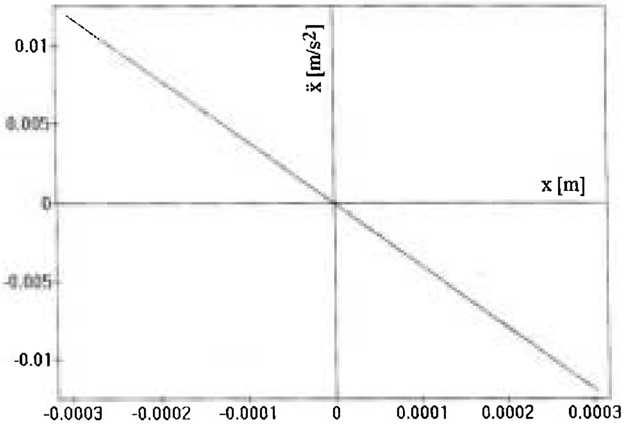

Figure 6. Plot of x - ẍ for the periodic response of the system.

CONCLUSIONS

A vehicle suspension could be modelled as a non-linear vibration system. Its governing differential equation could be explained by a non-linear third order differential equation. Existence of periodic and hence, stable responses which is not an obvious feature for the non-linear systems must be guaranteed. Thus the necessary and sufficient conditions for existence of periodic solutions for a general class of third order ordinary differential equations have been obtained. It has been shown that these conditions could be applicable for analysing the steady-state behaviour of the suspension system. Therefore, the suspension system could have a periodic response with a constant amplitude when it is subjected to the excitation of the road in the form of harmonic wave.

NOMENCLATURE

g_i, $i = 0,1,2$	functions appearing as the coefficients of differential equaitons
$e(t)$	forcing function
C	class of continuous and differentiable function
$x(t), x'(t), x''(t)$	state variables
$G(t,s)$	Green's function

S, B	Banach space
K	bounded domain of phase space
U	Operator
M	maximum of a function in a bounded region of phase space
t	Time
d, ∂	differential symbols
' (prime)	d/dt
k_i, $i = 1, 2$	stiffness coefficient of springs
c	damping coefficient of shock absorber
M_p	mass
τ	period
φ, θ, ψ	error functions
α, β, γ	initial conditions
Δ	increment in initial conditions

REFERENCES

1. E. Esmailzadeh, "Compact Self-Damped Pneumatic Isolators for Road Vehicles,", *ASME, Transactions, Journal of Mechanical Design*, vol. 102, No. 2, (1980), 270-278.
2. T. Ghalberg, "An Optimised Speed Controlled Suspension of a 2 d.o.f. Vehicle Travelling on a Randomly Profiled Road", *J. of Sound and Vibration*, (69) 6, (1979), 541-546.
3. E. Esmailzadeh, "Servo Valve-Controlled Pneumatic Suspensions", *Journal of Mechanical Engineering Science*, vol. 21, No. 1, (1979), 7-18.
4. G. Mastinu, "Passive Automobile Suspension Parameter Adaptation", *IMechE, C425/88*, (1988), 51-58.
5. E. Esmailzadeh, "Vibration Isolation System with Variable Natural Frequency", *International Journal of Mechanical Engineering Education*, vol. 6, No. 3, (1978), 125-129.
6. R. S. Sharp and S. A. Hassan, "An Evaluation of Passive Automotive Suspension Systems with Variable Stiffness and Damping Parameters", *Vehicle System Dynamics*, 15, (1986), 335-350.

7. H. B., Pacejka, "Non-Linearities in Road Vehicle Dynamics", *Vehicle System Dynamics*, 15, (1986), 237-254.
8. E. Esmailzadeh, et. al., "Periodic Behaviour of a Non-linear Dynamical System", *Nonlinear Dynamics Journal*, vol. 7, (1995), 335-344.
9. E. A., Jackson, Perspective of Non-linear Dynamics, Cambridge University Press, New York, (1991).
10. E. Esmailzadeh, "Design Synthesis of a Vehicle Suspension System Using Multi-Parameter Optimisation", *Vehicle System Dynamics*, 7, (1978), 83-96.
11. E. Esmailzadeh, et. al., "Periodic Solution of a Second Order, Autonomous, Non-linear System", *Nonlinear Dynamics Journal*, 10. (1996), 307-316.