EFFICIENT OPTIMUM DESIGN OF STRUCTURES WITH FREQUENCY RESPONSE CONSTRAINTS USING HIGH QUALITY APPROXIMATION

E. Salajegheh

Department of Civil Engineering
University of Kerman
Kerman, Iran

Abstract An efficient technique is presented for optimum design of structures with both natural frequency and complex frequency response constraints. The main idea is to reduce the number of dynamic analysis by introducing high quality approximation. Eigenvalues are approximated using the Rayleigh quotient. Eigenvectors are also approximated for the evaluation of eigenvalues and frequency responses. A two point approximation is used to approximate the functions under consideration. After the substitution of the approximate functions into the orginal optimization problem, the dynamic analysis of the structure is not necessary in the specified move limits. The sensitivities of the eigenvalues and eigenvectors are calculated analytically. In the problems under consideration repeated eigenvalues occur. Thus the sensitivities are evaluated for repeated eigenvalues.

Key Words Frequency Response Constraints, Optimum Design, Two Point Approximation

چکیده روشی کارا جهت طرح بهینه سازه ها با محدودیت های فرکانس طبیعی ر فرکانس مختلط پاسخ ارائه شده است. هدف اصلی کاهش تعداد تحلیل دینامیکی سازه با معرفی توابع تقریبی با کیفیت بالا می باشد. مقادیر ویژه با روش رایلی تقریبی می شوند. بردارهای ویژه نیز جهت محاسبه مقادیر ویژه و توابع پاسخ سازه تقریبی می شوند. روش دو نقطه ائی جهت تقریبی نمودن توابع استفاده شده است. پس از جایگزینی روابط تقریبی در مسئله اصلی بهینه سازی، نیازی به تحلیل دینامیکی سازه در محدوده مشخص شده نمی باشد. مشتق مقادیر و بردارهای ویژه محاسبه شده اند. در مسائل مورد بحث مقادیر ویژه تکراری وجود دارد که مشتق این نوع توابع نیز محاسبه شده است.

INTRODUCTION

Numerical optimization methods are usually iterative and a great number of function evaluations are required to reach the optimum solution. Modern optimization techniques employ some sort of approximation concepts. The functions that are computationally expensive to evaluate, are approximated in each design iteration. The method of approximation is vital as the number of iterations in the optimization process is based on the quality of approximation. A survey of approximation techniques was presented in [1]. In particular, optimum design of structures with approximate frequency constraints requires accurate estimation of the frequencies. This is due to the fact that the eigenvalues and eigenvectors are highly nonlinear functions in terms of the design variables. A review of the methods with frequency constraints was presented in [2].

A hybrid frequency constraint approximation was presented in [3] using the Taylor serious expansion. Reference [4] employed the approximation of frequencies with respect to some intermediate design variables. Because of the inherent nonlinear characteristics of natural frequency constraints, reference [5] employed a second order Taylor series approximation of the eigenvalues in each design cycle. However, the computational procedures were similar to those of the first order approaches. To enhance the quality of approximation, the Rayleigh quotient approximation was developed in [6] by constructing first order approximations of the modal strain and kinetic energies, independently. This approach was extended in [7] to consider the effects of changes in the eigenvectors during each cycle. A linear approximation was used for both natural frequency and dynamic responses [8,9]. Then a linear programming method was employed to solve each of the optimzation problems under consideration. This method requires many cycles to converge and thus the total number of dynamic analysis will increase.

In the present study, the Rayleigh quotient approximation is used to approximate the eigenvalues using a two point function approximation. To predict the values of eigenvalues, the eigenvector, stiffness and mass matrices are approximated independently. In addition, by employing the approximate relations of the eigenvalues and eigenvectors, the approximate functions are established for the frequency responses. The gradients of the functions including repeated eigenvalues are evaluated analytically and implemented in the computer program. Examples of two and three dimensional structures with static and dynamic constraints are solved and the results are compared with those of published work.

APPROXIMATE PROBLEM PRESENTATION

The optimization problem with frequency and frequency response constraints is mathematically formulated as: Find the vector of design variables $X=[X_1,X_2,\ldots, X_{nd}]$, that will

minimize $W(X)$ (weight) (1 a)

subject to:

$\lambda_i^L \leq \lambda_i \leq \lambda_i^U$ (frequency constraints) (1b)

$A_{ij}^L \leq A_{ij} \leq A_{ij}^U$ (frequency response constraints) (1c)

$X_k^L \leq X_k \leq X_k^U$ (Side constraints) (1 d)

The subscripts i, j and k represent the response degree of freedom, force input degree of freedom and design variable. respectively. The superscripts U and L indicate the upper and lower limits, respectively.

For a general structure, the eigenvalues and associated eigenvectors are obtained by the following equation:

$$K\phi_i = \lambda_i M\phi_i \tag{2}$$

where K and M are the stiffness and mass matrices of the structure, respectively. λ_i is the ith eigenvalue obtained from this equation through eigenvalue analysis and ϕ_i is the associated eigenvector of the ith eigenvalue.

Premultiplying Eqation 2 by ϕ^T_i and evaluating λ_i, we have

$$\lambda_i = \frac{\phi_i^T K \phi_i}{\phi_i^T M \phi_i} \tag{3}$$

Equation 3 is known as Rayleigh quotient.

In the present work, the vector ϕ and matrices K and M are approximated in terms of some intermediate variables Z. The intermediate variables are chosen as the vector of cross sectional properties (area and moment of inertia) or their reciprocals. These approximate relation are expressed in the following form:

$$\phi (Z) \approx \phi(Z_0) + \nabla\phi(Z_0).\ \delta Z \tag{4a}$$

$$K (Z) \approx K (Z_0) + \nabla K (Z_0).\delta Z \tag{4b}$$

$$M (Z) \approx M (Z_0) + \nabla M (Z_0).\ \delta Z \tag{4c}$$

in which Z_0 is the point about which the Taylor series expansion is created and $\delta Z = Z-Z_0$.

To enhance the quality of approximation, the following intervening variables are used [10].

$y_i = z^r_i$, i = 1, nd (5)

where r represents the nonlinearity index, which is different at each iteration, but is the same for all variables. The nonlinearity index is determined by matching the function value of the current design point with the previous design point. On the other hand, r is numerically calculated such that the difference of the exact and approximate functions at the previous point Z_o becomes zero. For example, for the function $\phi(Z)$, we have

$$\phi(Z_0) - \{\phi(Z_1) + \frac{1}{r}\sum_{i=1}^{nd} z_{i,1}^{1-r}\frac{\partial\phi(Z_1)}{\partial Z_i}(z^r_{i,0} - z^r_{i,1})\} = 0 \quad (6)$$

r can be any positive or negative real number (not equal to zero). Then the two point approation for ϕ (Z) can be expressed as

$$\phi(Z) \approx \phi(Z_1) + \frac{1}{r}\sum_{i=1}^{nd} z_{i,1}^{1-r}\frac{\partial\phi(Z_1)}{\partial z_i}(z^r_i - z^r_{i,1}) \quad (7)$$

Although, the creation of the approximation function is based on a linear approximation, it has the property of the higher terms, which increases the quality of approximation. The same approximate relations are established for k and M matrices.

The other constraint used in this study is the frequency response of the structure. The complex frequency response function is expressed as

$$H_{ij}(\Omega) = R_{ij}(\Omega) + iI_{ij}(\Omega) \quad (8)$$

where $H_{ij}(\Omega)$ is the complex frequency response function, $R_{ij}(\Omega)$ is the real part and $I_{ij}(\Omega)$ is the imaginary part of the frequency response function. The magnitude of this frequency response function used in Eqation 1c is

$$A_{ij} = \sqrt{R_{ij}^2 + I_{ij}^2} \quad (9)$$

The expressions for $R_{ij}(\Omega)$ and $I_{ij}(\Omega)$ are as follows:

$$R_{ij}(\Omega) = \sum_{r=1}^{N}\phi_{ir}\phi_{jr}\left(\frac{\lambda_r - \Omega^2}{(\lambda_r - \Omega^2)^2 + 4\xi_r^2\lambda_r\Omega^2}\right) \quad (10)$$

and

$$R_{ij}(\Omega) = \sum_{r=1}^{N}\phi_{ir}\phi_{jr}\left(\frac{-2\xi_r\Omega\lambda_r^{0.5}}{(\lambda_r - \Omega^2)^2 + 4\xi_r^2\lambda_r\Omega^2}\right) \quad (11)$$

where ϕ_{ir} is the ith element of the rth eigenvector, ϕ_{ji} is the jth element of the rth eigenvector, λ_r is the rth eigenvalue, Ω is the excitation frequency in radians per second and ζ_r is the rth modal damping ratio.

The quantities λ and ϕ in Eqations 10 and 11 should be obtained from the dynamic analysis. Again the approximations of these relations are employed, thus a nonlinear explicit relation is obtained for A_{ij}.

With these approximation, an approximate nonlinear explicit optimization problem is stablished which can be solved without carrying out the dynamic analysis of the structure. Since the constraints are approximate, a move limit should be imposed to control the quality of approximations. The solution of this approximate problem is one design cycle and the result is a starting point for the next iteration. The process is repeated until the design problem converges.

SENSITIVITY ANALYSIS

The sensitivities of λ, ϕ, K and M are obtained analytically. The structures under consideration produce repeated eigenvalues, the sensitivities of which are different from those of nonrepeated eigenvalues. The details of evaluating all these sensitivities can be found in the literature and the method used in the present work to evaluate the eigenvalues and eigenvectors is briefly explained.

Non Repeated Eigenvalue

By differentiating Equation 2 with respect to Z_k, we have [11]

$$\frac{\partial K}{\partial z_k}\phi_i + K\frac{\partial \phi_i}{\partial z_k} = \frac{\partial \lambda_i}{\partial z_k} M\phi_i + \lambda_i \frac{\partial M}{\partial z_k}\phi_i + \lambda_i M \frac{\partial \phi_i}{\partial z_k} \quad (12)$$

Premultiplying Equation 12 by ϕ^T_i and using Equation 2 and the normalization property that

$$\phi^T_i M\phi_i = 1, \quad (13)$$

the following expression is obtained for the eigenvalue sensitivity

$$\frac{\partial \lambda_i}{\partial Z_k} = \phi^T_i \left(\frac{\partial K}{\partial Z_k} - \lambda_i \frac{\partial M}{\partial Z_k}\right)\phi_i \quad (14)$$

The eigenvector sensitivity is assumed to have the form [11]

$$\frac{\partial \phi_i}{\partial Z_k} = P_i + c_i \phi_i \quad (15)$$

where P_i and c_i are two unknowns to be determined. Substituting Equation 15 into Equation 12 yields

$$F_i P_i = -\frac{\partial F_i}{\partial Z_k}\phi_i \quad (16)$$

where $F_i = K - \lambda_i M$

Since the matrix F_i is singular, P_i can not be found by solving the linear Eqations 16. However, if the row and the column associated with the largest component of the eigenvector ϕ_i, are deleted from the matrix F_i and the same row deleted from the vector of right hand side of Equation 16, then the system of Equations 16 can be solved for P_i.

Taking the derivative of the orthogonality condition 13 with respect to z_i results in the value of c_i

$$c_i = -\phi^T_i MP_i - \frac{1}{2}\phi^T_i \frac{\partial M}{\partial Z_k}\phi_i \quad (17)$$

The Repeated Eigenvalue

The eigenvalue sensitivity for the repeated eigenvalue λ_i with multiplicity n is found from the following eigenvalue equation [12]

$$\left[\bar{\phi}_i^T \left(\frac{\partial K}{\partial z_K} - \lambda_i \frac{\partial M}{\partial z_K}\right)\bar{\phi}_i - \lambda_i' I\right] a_i = 0 \quad (18)$$

where $\bar{\phi}_i$ is composed of the n eigenvectors associated with the repeated eigenvalues λ_i through λ_{i+n-1}; λ_i' is the eigenvalue of Equation18 and is also proved to be the eigenvalue sensitivity for λ_i; the matrix I represents the unit matrix with dimensions of n×n; and a_i is the eigenvector associated with the eigenvalue λ_i'.
If the eigenvalue obtained from Equation 18 is not a repeated eigenvalue, then the unique eigenvector ϕ_i associated with λ_i is determined from

$$\phi_i = \bar{\phi}_i a_i \quad (19)$$

The eigenvector sensitivity for the repeated eigenvalue is found from Equation 15 if ϕ_i is replaced with $\bar{\phi}_i$ and c_i is considered as a vector. P_i can be evaluated from Equation 16 provided n appropriate rows and columns are omitted from both sides of Equation 16. The elements of the vector c_i can be found by taking the derivative of Equation 13. The ith element of c_i is

$$c_{ii} = -\phi^T_i\left(\frac{1}{2}\frac{\partial M}{\partial z_k}\phi_i + MP_i\right) \quad (20)$$

The other elements in vector c_i are evaluated as [12]

$$c_{li} = \frac{\phi_l^T\left[\frac{\partial^2 K}{\partial z_k^2} - 2\frac{\partial \lambda_i}{\partial z_k}\frac{\partial M}{\partial z_k} - \lambda_i \frac{\partial^2 M}{\partial z_k^2}\right]\phi_i + 2\phi_l^T \frac{\partial F_i}{\partial z_k} P_i}{2\left(\frac{\partial \lambda_i}{\partial z_k} - \frac{\partial \lambda_l}{\partial z_k}\right)} \quad (21)$$

Evaluation of ∇ K and ∇ M

The only unknown quantities in these relations are the derivatives of K and M matrices. These derivatives are normally calculated at the element level, i.e.,

$$\frac{\partial K}{\partial Z_k} = \sum_{ne} \frac{\partial k}{\partial Z_k} \tag{22}$$

$$\frac{\partial M}{\partial Z_k} = \sum_{ne} \frac{\partial m}{\partial Z_k} \tag{23}$$

where k and m are element matrices and ne is the number of finite elements. These element derivatives can be either calculated analytically or by using first order finite difference approximations.

NUMERICAL EXAMPLES

The method was implemented in a computer program and the results of three examples are presented here. The DOT program [13] was used to solve each of the approximate nonlinear design problems.

Ten - Bar Truss

The ten-bar truss shown in Figure 1 is chosen from [8]. The areas of the members are taken as design variables with initial values of 6.452 cm². The material properties are considered as Young's modulus E = 68.95 Gpa; Possions's ratio ν=0.3 and the mass density ρ = 0.0277 kg/cm³. The modal damping ratio is assumed to be zero. The structure is assumed to take two static loadings of 444822 Newtons each at nodes 2 and 4 in the negative y direction. For dynamic loading, the structure is subjected to a harmonic force with magnitude of 4.45 Newtons and an excitation frequency of 20 Hz at node 2 in the y direction. The constraints include both static and dynamic displacements. The static displacement constraints, considered as the deflections of nodes 2 and 4 in the negative y direction, subjected to the previously defined static loadings, must be less than 94.0 cm and 43.18 cm, respectively. The frequency response constraints require that the amplitudes of the responses at nodes 2 and 4 in the y direction be less than 2.28×10^{-5}cm, respectively. The lower and upper limits on the design variables are considered as 0.645 cm² and 12.90cm², respectively.

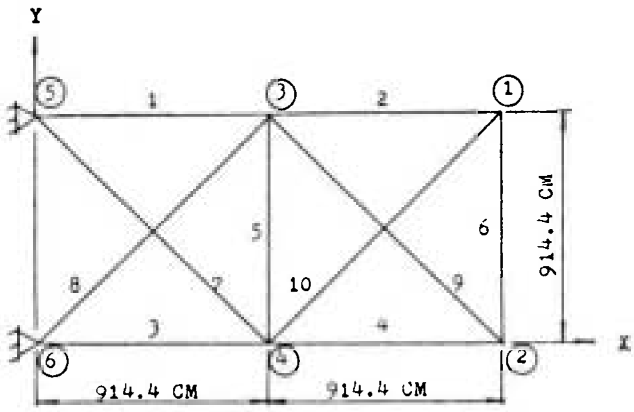

Figure 1. Ten-Bar Truss.

The static displacements are also approximated in each design cycle with reciprocal design variables, using a similar relation as in Equation 7.

Total number of design iterations for this problem was 6 and for each design iteration only one static and free vibration analysis was carried out. However, this problem was solved in [8] and 19 iterations completed the optimization process. Although, in the present approach, a nonlinear problem is solved in each design cycle, the computational cost of optimization is negligible as compared to the cost of analysis. The results are presented in Table 1 together with those of [8].

Five-Storey Plane Frame

This frame is shown in Figure 2 and is chosen from [8]. The design variables considered are the cross sectional areas and bending moments of inertia of the elements. The variables are linked as indicated in Table 2. The initial areas and moments of inertia are

TABLE 1: Results for Ten-Bar Truss.

Element no	Ref. 8 (cm^2)	Present method (cm^2)
1	10.425	10.501
2	0.669	0.645
3	9.371	9.289
4	4.848	4.850
5	0.645	0.645
6	0.645	0.645
7	5.375	5.290
8	7.048	6.459
9	6.859	7.012
10	0.707	0.645
No. of cycles	19	6

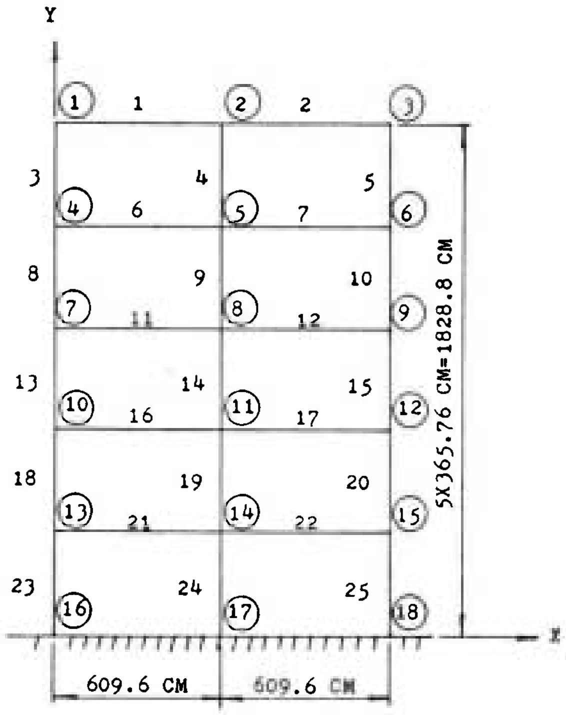

Figure 2. Five- Storey Plane Frame.

taken as 6.452 cm^2 and 41623 cm^4, respectively for all the horizontal members. For vertical members, the initial areas and moments of inertia are 12.90 cm^2 and 83246 cm^4, respectively. The material proerties are Young's modulus E = 68.95Gpa; Possions's ratio ν=0.3 and the mass density ρ = 0.0107 kg/cm^3. The rotational mass moment of inertia at each free node is 0.565 N-cm-s^2. The modal damping ratio is assumed to be zero.

TABLE 2. Design Variable Linking of Frame.

Design variable no	Variable	Element no
1	A	1.2
2	I_z	6,7
3	A	11,12
4	I_z	16,17
5	A	21,22
6	I_z	1,2
7	A	6,7
8	I_z	11,12
9	A	16,17
10	I_z	21,22
11	A	3,4,5,8,9,10,13,14,15,18 19,20, 23,24,25
12	I_z	3,4,5,8,9,10,13,14,15,18, 19,20,23,24,25

The structure is subjected to two static loaings of 22241Newtons each at nodes 1 and 4 in the x direction. The dynamic loading is a harmonic input force with a magnitude of 4.45 Newtons and an excitation frequency of 20 Hz acting at node 1 in the x direction. the static displacement constraints, specified to be the static deflections at nodes 1 and 4 in the x direction, are less than 3.175 cm and 2.54 cm, respectively. The frequency response constraint needs the response at node 1 in the x direction to be less than 1.27×10^{-5} cm. The side constraints for design variables 1, 3, 5, 7 and 9 range from 0.645 cm^2 to 12.904 cm^2. For design variables 2, 4, 6, 8 and 10, the lower and upper limits are 4162 cm^4 and 83246 cm^4 respectively. The bounds for design variable 11 are 1.29 cm^2 and 25.81 cm^2. The allowable change for design variable 12 is between 8325 cm^4 and 166492 cm^4.

The static displacement are also approximated in this problem. A total of 8 design cycles is needed to obtain the optimum solution. However, the optimum solution of this problem was obtained in [8] with 22 iterations. The results are presented in Table 3.

72-Bar Space Truss

Figure 3 shows the configuration of this structure

TABLE 3: Results for Five-story Frame.

Design variable no. (cm^2)	Ref. 8 (cm^2)	Present method (cm^2)
1	1.424	1.391
2	83246	83246
3	0.645	0.645
4	83246	83246
5	0.645	0.645
6	83246	83246
7	0.812	0.802
8	83246	83246
9	0.645	0.645
10	83246	83246
11	11.917	12.040
12	166492	166492
No. of cycles	22	8

which is chosen from [9]. Two cases are considered by choosing two different dimensions in the y direction. The length of 120 in. along the y direction forms a geometrically symmetric structure which produces repeated eigenvalues; the other dimension of 200 in. constructs an unsymmetric structure.

Case 1- Non Repeated Eigenvalue

This case is the unsymmetric 72-bar space truss with 48 translational degrees of freedom. No repeated eigenvalues are found in the structure. The cross sectional areas of the members are the design variables with the initial value of 1.0 in^2 for all members. The design variables are linked to have 16 variables as indicated in Table 4. The lower and upper limits on the design variables are 0.1 and 2.0 in^2. respectively. The mass density of the material is assumed to be 0.0002588 $lb\text{-}s^2/in^4$ and the mass of 0.001 $lb\text{-}s^2/in$ is lumped at each free node. (The U.S. units are used in this example to compare the results).

Both natural frequency and frequency response constraints are considered. Two natural frequency constraints are specified. It is assumed that the 3rd natural frequency would be less than 27 Hz and the 4th natural frequency greater than 50 Hz. A unit harmonic force with excitation frequency of 30 Hz is

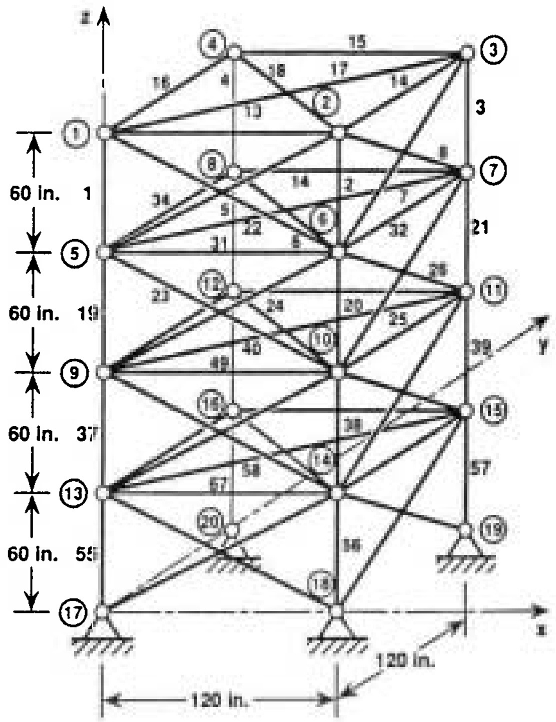

Figure 3. 72- Bar Space Truss

TABLE 4. Design Variable Linking of 72-Bar Truss.

Design variable no	Element no
1	1-4
2	13-16
3	5-12
4	17,18
5	19-22
6	31-34
7	23-30
8	35,36
9	37-40
10	49-52
11	41-48
12	53,54
13	55-58
14	67-70
15	59-66
16	71,72

applied at node 1 in the y direction for the forced response input. Four frequency response constraints are specified. Only magnitudes are considred. The frequency response amplitude at nodes 1, 2, 3 and 4 in the y direction must be less than 0.9×10^{-4},

TABLE 5. Results for 72-Bar Truss.

D.V. no.	Case - 1 (in^2)		Case - 2 (in^2)	
	Ref. 9	Present	Ref. 9	Present
1	0. 422	0. 390	0. 229	0. 312
2	0. 122	0. 100	0. 902	1. 012
3	0. 100	0. 100	0. 615	0. 589
4	0. 100	0. 100	1. 212	1. 215
5	0. 186	2. 020	0. 165	0. 166
6	1. 689	1. 710	0. 158	0. 160
7	0. 100	0. 100	0. 853	0. 841
8	0. 663	0. 719	0. 175	0. 180
9	0. 323	0. 311	0. 141	0. 138
10	0. 100	0. 100	0. 136	0. 140
11	0. 100	0. 100	0. 683	0. 711
12	0. 100	0. 100	0. 149	0. 138
13	0. 212	0. 221	0. 138	0. 137
14	0. 100	0. 100	0. 136	0. 137
15	0. 100	0. 100	0. 593	0. 602
16	0. 100	0. 100	0. 148	0. 144
No. of cycle	18	7	13	6

0.8×10^{-4}, 0.8×10^{-4} and 0.9×10^{-4} in., respectively.

The number of required dynamic analysis was 7 while 18 iterations was required by the method presented in [9]. The final results are presented in Table 5. The optimum weight was found to be 255.5 lb.

Repeated Eigenvalue

This case has a configuration similar to the first case except that the dimension in the y direction is 120 in. Since the structure is completely symmetric in the x and y directions, repeated eigenvalues occur. The loading is the same as that of case 1. The natural frequency constraints require that the 2nd natural frequency which is repeated be less than 27 Hz and the 3rd natural frequency be greater than 32 Hz. Four frequency response constraints are chosen at nodes 1, 2, 3 and 4 with the degree of freedom in the y direction. The amplitude limits are set as 0.75×10^{-4}, 0.9×10^{-4}, 0.9×10^{-4} and 0.75×10^{-4} in., respectively. A total of 6 iterations needed to complete the optimization of this case as compared to the 13 iterations required by Reference 9. The results are given in Table 5. The optimum weight is 437 lb.

CONCLUSIONS

The main aim of the present approach was to reduce the number of required dynamic analyses in the optimization process. The quantities that were obtained through the analysis of the structure, were approximated in such a way that the approximate functions would be close to the actual functions. The results clearly showed that the number of dynamic analysis required to complete the optimization was reduced substantially. The reduction in the number of cycles was basically due to the creation of the high quality approximations for the dynamic responses. Although the approximate problems were nonlinear as compared to the linear problems created by previous researches the overall cost of optimization was greatly reduced. For practical design problems, when the cost of analysis is a great portion of the overall cost, the computational work will be reduced.

REFERENCES

1. J. F. M. Barthelemy and R. T. Haftka, "Approximation Concepts for Structural Design -A Review" *Structural Optimization*, (1993), 129-144.
2. R. V. Grandhi, "Structural Optimization with Frequency Constraints -A Review ", *AIAA J.*, 31, (1993) 2296-2303.
3. T. H. Woo, "Space Frame Optimization Subject to Frequncy Constraints", *AIAAJ.*, 25 (1987) 1396-1404.
4.. G. N. Vanderplaats, and E. Salajegheh, "An Efficient Approximation Technique for Frequency Constrints in Frame Optimization ", *Int. J. Num. Meth. Eng.*, 26, (1988) 1057-1069.
5. H. Miura, and L. A. Schmit, "Second Order

Approximation of Natural Frequency Constraints in Structural Synthesis", *Int. J. Num. Meth. Eng.*, 13, (1978) 337-351.

6. R. A. Canfield, "High Quality Approximation of Eigenvalues in Strudtural Optimization", *AIAA J.*, 28, (1990) 1116-1122.
7. E. Salajegheh, "Optimum Design of Structures with Frequency Constraints" *Asian J. Struct. Eng.* 1, (1995), 243-254.
8. T. Y. Chen, "Design Optimization with Static and Dynamic Displacement Constraints", *Structural Optimization*, 4 (1992) 179-185.
9. T. Y. Chen, "Optimum Design of Structures with both Natural Frequency and Frequency Response Constraints" *Int. J. Num. Meth. Eng.*, 33 (1992) 1927-1940.
10. L. Wang, R.V. Grandhi, "Efficient Safety Index Calculations for Structural Relibility Analysis", *Comp. & Struct.* 52, (1994) 103-111.
11. R. B. Nelson, "Simplified Calculation of Eigenvector Derivatives ", *AIAA J.*, 14, (1976) 1201-1205.
12. W. C. Mills-Curran, "Calculation of Eigenvector Derivatives for Structures with Repeated Eigenvalues", *AIAAJ.*, 26 (1988) 867-971.
13. G. N. Vandrplaats, "DOT Users Manual" VMA Engineering. CA, USA, (1991).