VIBRATION ANALYSIS OF BEAMS TRAVERSED BY A MOVING MASS

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Abstract A detailed investigation into the analysis of beams with different boundary conditions, carrying either a moving mass or force is performed. Analytical and numerical techniques for determination of the dynamic behavior of beams due to a concentrated travelling force or mass are presented. The transformation of the familiar Euler-Bernoulli thin beam equation into a series of ordinary differential equations is demonstrated. These equations are solved numerically using fourth order Runge-Kutta and central difference expansion methods. It is observed that the results corresponding to either method of solution, with the assumption made (moving force or mass) are very close. Moreover, the moving force problem is solved using the finite element method. The inertial effect of the moving mass has been proven to be an important factor in the dynamic behavior of such structures. Finally, using the obtained dynamic deflection functions, values of maximum shear force and bending moment at each time step are calculated and variation of these parameters with time is demonstrated.

Key Words Moving Loads, Vehicles, Vibration of Bridges, Dynamics

چکیده در این مقاله تحلیل مشروحی در زمینه رفتار تیرها با شرایط مرزی گوناگون تحت اثر حرکت یک جرم و یا یک نیروی متمرکز ارائه گردیده و برای این منظور روشهایی تحلیلی و عددی مطرح شده است. به‌علاوه نشان داده شده که چگونه معادله اولر - برنولی برای یک تیر نازک در این حالت به یک سری معادلات دیفرانسیل معمولی تبدیل می گردد. در این مقاله معادلات مزبور توسط روشهای رونگ - کوتا و تفاضلات مرکزی حل شده اند. درنتیجه مشاهده شد که نتایج بدست‌آمده با استفاده از هریک از متدهای حل برای هرکدام از حالات جرم یا نیروی متحرک بسیار به یکدیگر نزدیک می‌باشد. ازطرف دیگر ملاحظه گردید که اینرسی جرم متحرک یک عامل بسیار مهم در رفتار دینامیکی تیر می باشد. در خاتمه با استفاده از توابع خیز دینامیکی بدست آمده مقادیر حداکثر نیروی برشی و لنگر خمشی در هرگام زمانی محاسبه‌شده و تغییرات این پارامترها برحسب زمان مورد بررسی قرار گرفته است.

INTRODUCTION

Engineers designing highway and railroad bridges and space installation facilities which are likely to be affected by sudden changes of mass, have renewed the investigation of the travelling mass problem. Bridges on which vehicles or trains travel and trolleys of overhead travelling cranes that move on their girders could be modelled as a moving mass on a simply supported beam.

Since the middle of the last century, when railroad bridge construction began, the problem of oscillation of bridges under travelling loads has interested engineers. Contributions to the solution of this problem were made by Sir George Stokes, Robert Willis [1] and many others. Stephen Timoshenko [2] considered the case of a pulsating load passing over a bridge. Sir Charles Inglis [3] in this systematic analysis of trains crossing a bridge, took into account many important factors such as the effects of moving loads, the influence of damping and the spring suspension of the locomotive.

For the case of a concentrated force moving with constant velocity along a beam, neglecting damping forces, Timoshenko [4] has determined a general solution to this problem and presented an expression for the critical velocity. In 1969, Stanisic and Hardin [5] tried to investigate the dynamic behavior of a simply supported beam carrying a moving mass. Although the findings were interesting, they were not easily applicable to different boundary conditions.

The analysis of dynamic behavior of bridges subjected to motion of a concentrated mass, has been performed by the present authors in [6] and [7] and a detailed discussion has been presented in [8].

In this paper, inertial effects of the moving mass are considered. It is assumed that structural damping and effects of rotary inertia and shear forces in the dynamic behavior of the beam are negligible. Furthermore, the beam is assumed to be homogeneous with constant sectional properties. The response of this resulting, to some extent, more realistically modeled system is determined by using both analytical and numerical techniques. Finally, a detailed comparison between the results for the case of moving mass with those corresponding to the travelling force is made. As an important design factor, variations of maximum bending moment and maximum shear force with time have been analyzed and corresponding graphs are plotted.

RESPONSE OF A UNIFORM BEAM TO A TRAVELLING LOAD

Consider the case of a concentrated load advancing along a beam with constant velocity. As can be expected, this load will produce larger deflections and stresses than the same load acting statically. The effect of such live loads on bridges and beams is of great practical importance. Figure 1 illustrates a vehicle of weight P, travelling on a simply supported beam at a constant velocity V.

It is assumed that the moving load exerts a constant vertical force on the beam and the inertial effect

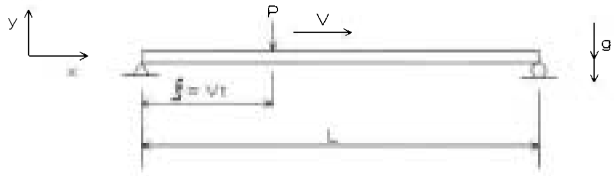

Figure 1. The model of the problem.

of the load is negligible. Moreover, any change in the gravitational potential energy is omitted. The beam is assumed to have zero velocity and deflection at time t=0, when the load is at the left-hand support. Therefore, it is clear that at any later time t, the distance of the load from the left support will be $\xi = Vt$.

In order to obtain the governing differential equation of motion for this system, the principle of virtual work may be applied,

$$\delta W_{Ii} + \delta W_{Ei} + \delta W_{pi} = 0 \qquad (1)$$

where,

δW_{Ii} = the virtual work associated with the beam distributed inertia

δW_{Ei} = the virtual work related to the elasticity forces,

δW_{Pi} = the virtual work associated with the vertical load.

These terms are considered to be the results of the virtual displacement, δy_i, for the ith natural mode of the beam vibration.

With a reasonable assumption, the transverse motion of the beam is:

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(t) \,.\, X_i(x) \qquad (2)$$

where ϕ_i 's are unknown functions of time, and X_i' s are deflection curves for the ith mode.

In the case of a simply supported beam, with mass per unit lenght m and the ith natural frequency, p_i, substituting Equation 2 in Equation 1 would lead to:

$$-m\frac{d^2\phi_i}{dt^2}\delta\phi_i - m p_i^2 \delta\phi_i - P\,[\sqrt{\frac{2}{L}}\,Sin\,(\frac{i\pi Vt}{L})]\,\delta\phi_i = 0$$

$$(i = 1\,,2\,,...) \qquad (3)$$

Thus, the governing differential equation for ϕ_i 's is:

$$\frac{d^2\phi_i}{dt^2}+p_i^2\,\phi_i=-\frac{P}{m}\sqrt{\frac{2}{L}}\,Sin(\frac{i\pi Vt}{L}) \quad (i=1,2,...) \qquad (4)$$

Using the Duhamel's integral and employing Equation 2, the solution is,

$$y(x,t)=\frac{-2PL^3}{m\pi^2}\sum_{i=1}^{\infty}\frac{Sin\,(i\pi x/L)}{i^2\,(i^2\pi^2a^2-V^2L^2)}\,Sin(i\pi Vt/L)$$
$$+\frac{2PL^4V}{m\pi^3a}\sum_{i=1}^{\infty}\frac{Sin\,(i\pi x/L)}{i^3\,(i^2\pi^2a^2-V^2L^2)}\,Sin(i^2\pi^2at/L^2) \qquad (5)$$

where: $a=\sqrt{EI/m}$; E = modulus of elasticity; I = moment of inertia of the beam cross section.

The first series in Equation 5 corresponds to the forced vibration of the beam, while, the second one pertains to its free vibrations. Since i^2 and i^3 appear in the denominators, it is expected that the terms which correspond to the higher modes have less contribution to the resulting deflection than those of lower modes.

BEAMS TRAVERSED BY MOVING MASSES

At this stage, inertial effects of the moving load are taken into consideration. Hence, it is expected that more realistic solutions for practical problems could be pursued such as cars travelling on a suspension bridge, as modelled in Figure 1. According to Euler-Bernoulli beam theory, the governing differential equation describing the lateral vibration of a beam, carrying the time varying force, $p(x, t)$, per unit lenght is, [9]:

$$EI\frac{\partial^4 y}{\partial x^4}+m\frac{\partial^2 y}{\partial t^2}=p\,(x,t) \qquad (6)$$

where,

$$P\,(x\,,t)=-M_p\left[\,g+\frac{\partial^2 y\,(\xi\,,t)}{\partial t^2}\right]\delta(x-\varepsilon) \qquad (7)$$

M_p = *mass of the moving load*

Assuming that the mass acceleration at any point is equal to the acceleration of the beam point coincident on the mass at the desired time, the total derivative term in Equation 7 may be substituted with the partial one. However, this assumption is valid only at relatively low speed of the mass motion. Adopting this assumption, Equation 7 would reduce to,

$$p\,(x,t)=-M_p[\,g+\frac{\partial^2 y\,(\xi\,,t)}{\partial t^2}]\,\delta(x-\xi) \qquad (8)$$

Similar to Equation 2, the loading function per unit length may be expressed as,

$$p\,(x\,,t)=\sum_{i=1}^{\infty}X_i(x)\,.\,\psi_i(t) \qquad (9)$$

For a simply supported beam, it may be shown that the normalized modal shape functions are,

$$X_i(x)\,]=\sqrt{\frac{2}{L}}Sin\left(\frac{i\pi x}{L}\right) \qquad (10)$$

Substituting Equation 2 in Equation 8 one obtains,

$$p\,(x\,,\,t)=-M_p\left[g+\frac{\partial^2}{\partial t^2}\left(\sum_{i=1}^{\infty}X_i(\xi)\,.\,\phi_i(t)\right)\right]\delta(x-\xi) \qquad (11)$$

Comparison of Equations 9 and 11 results in,

$$\sum_{i=1}^{\infty}X_i(x)\,\psi_i(t)=-M_p\left[g+\frac{\partial^2}{\partial t^2}\left(\sum_{i=1}^{\infty}X_i(\xi)\,\phi_i(t)\right)\right]\delta(x-\xi) \qquad (12)$$

In order to drop the Dirac delta function, one may multiply both sides of Equation 12 by $X_j\,(x)$ and integrate along the beam to yield,

$$-M_pg\int_0^L X_j(x)\,\delta(x-\xi)\,dx-M_p[\sum_{i=1}^{\infty}X_i(\xi)\,\ddot{\phi}_i(t)]\times$$
$$\int_0^L X_j(x)\,\delta(x-\xi)\,dx=\sum_{i=1}^{\infty}\psi_i(t)\int_0^L X_i(x)\,X_j(x)\,dx \qquad (13)$$

However, since the modal shape functions, $X_i\ (x)$ form a set of orthonormal functions on the beam length interval, the following holds true

$$\int_0^L X_i(x)\,X_j(x)\,dx = \delta_{ij} \tag{14}$$

Hence, Equation 13 reduces to,

$$-M_p g X_j(\xi) - M_p\left[\sum_{i=1}^{\infty} X_i(\xi)\,\ddot{\phi}_i(t)\right] X_j(\xi) = \psi_j(t) \tag{15}$$

Substituting Equation 15 into Equation 9 results in,

$$p\,(x\,,t) = -M_p\left[g + \sum_{j=1}^{\infty} X_j(\xi)\ddot{\phi}_j(t)\right] \times \sum_{i=1}^{\infty} X_i(\xi) X_i(x) \tag{16}$$

Finally, Equations 16 and 2 may be substituted into Equation 6 to yield,

$$EI\sum_{i=1}^{\infty} X_i^{iv}(x)\,\phi_i(t) + \rho A\sum_{i=1}^{\infty} X_i(x)\ddot{\phi}_i(t) =$$

$$-M_p\left[g + \sum_{j=1}^{\infty} X_j(\xi)\,\ddot{\phi}_j(t)\right] \times \sum_{i=1}^{\infty} X_i(\xi)\,X_i(x) \tag{17}$$

and with,

$$X_i^{iv}(x) - \beta_i^4 X_i(x) = 0 \tag{18}$$

Equation 17 takes on the form,

$$EI\sum_{i=1}^{\infty} \beta_i^4\, X_i(x)\,\phi_i(t) + \rho A\sum_{i=1}^{\infty} X_i(x)\ddot{\phi}_i(t) =$$

$$-M_p\left[g + \sum_{j=1}^{\infty} X_j(\xi)\,\ddot{\phi}_j(t)\right] \times \sum_{i=1}^{\infty} X_i(\xi)\,X_i(x) \tag{19}$$

Considering a finite and arbitrary number of mode shapes, N, (rather than an infinite number) the following equation should be satisfied for each mode.

$$EI\beta_i^4 X_i(x)\phi_i(t) + \rho A X_i(x)\ddot{\phi}_i(t) = -M_p\left[g + \sum_{j=1}^{\infty} X_j(\xi)\,\ddot{\phi}_j(t)\right] \times X_i(\xi)\,X_i(x) \tag{20}$$

or,

$$\rho A\ddot{\phi}_i(t) + M_p\left[\sum_{j=1}^{N} X_j(\xi)\ddot{\phi}_j(t)\right] \times X_i(\xi) + EI\beta_i^4\phi_i(t) = -M_p g X_i(\xi)$$

$$i = 1, 2, 3, \ldots, N \tag{21}$$

In order to solve the considered problem, Equation 21 should be analyzed. By solving this set of coupled linear differential equations, ϕ_i (t)'s could be evaluated. When substituting these functions in Equation 2, the desired solution for vibration of the beam, under different boundary conditions and with any number of modal shapes can be determined.

MOVING MASS ON A SIMPLY SUPPORTED BEAM

In order to compare the results obtained for vibration of the beam carrying a moving mass, with those obtained for the case of travelling force a simply supported beam was assumed. It is interesting to note that by dropping the second term on the L. H. S. of Equation 21, i.e., the coupling term which is due to the inertial effects of the moving mass, it reduces to Equation 4, with $M_p g = P$. Therefore, the moving force solution may be considered as a special form of a more general case of beam vibration when carrying a moving mass.

Three different numerical techniques, namely, finite element method, Runge-Kutta method of fourth order, and central difference expansions for derivatives, were used to solve the problem. These methods are applicable for arbitrary number of mode shapes and were employed with the following numerical data.

$$E = 2.07\times10^{11}\ N/m^2;\ I = 1.04\times10^{-6}\ m^4;\ V = 12\ Km/h$$

$$L = 10\ m\ ;\ a = 174.9\ m^2/s\ ;\ M_p = 70\ Kg$$

$$g = 9.81\ m/s^2\ ;\ m = 7.04\ Kg/m \tag{22}$$

1- Finite Element Method

The version 7.1 of the SUPERSAP software package, [10] which has been employed in the present analysis, is capable of accepting finite element models that are drawn in AutoCAD. Alternatively, the models can be generated through the package itself. It is assumed, first, that the material behaves elastically, with no structural damping. Next, the data for material properties and boundary conditions are fed into the machine. Finally, this model is subjected to the assumed loading. In order to simulate the motion of the load, an incremental procedure for imposing the load has been considered.

In this example, the beam was modeled in two different manners, i.e., using 20 and 60 elements, and the solution for both cases were determined. It has been found that the first three computed natural frequencies for beam vibration, were very close to those obtained analytically, as shown in Table 1.

TABLE 1. Comparison between the first three natural frequencies, (rad/s).

Mode Number	Analytical solution	F.E.M.Solution
1	17.26	17.25
2	69.04	68.99
3	155.33	155.22

Using the calculated modal shapes and natural frequencies, the software evaluates the deflections at nodes, for time increments equal to 0.05 s. The results obtained for the two finite element models have a maximum discrepancy of 10%, when compared with those corresponding to the analytical solution, determined from Equation 5, for the first three modes. It is believed that the deviation and alternating behavior of the F.E.M. solution are mainly due to the incremental method used for modelling the load movement.

2- Fourth Order Runge-Kutta Method

In order to apply the fourth order Runge-Kutta method to solve Equation 21, this system of equations should first be properly arranged, [11]. In fact, it should be rewritten in such a way that derivatives of all the functions are expressed in terms of the other functions (not their derivatives). Considering Equation 21 and applying Kramer's rule for solution of a set of linear algebraic equations, one obtains

$$\ddot{\phi}_1(t) = \frac{\begin{vmatrix} -M_p g X_1 - EI\beta_1^4\phi_1 & M_pX_1X_2 & M_pX_1X_3 & \cdots & M_pX_1X_N \\ -M_p g X_2 - EI\beta_2^4\phi_2 & \rho A + M_pX_2^2 & M_pX_2X_3 & \cdots & M_pX_2X_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -M_p g X_N - EI\beta_N^4\phi_N & M_pX_2X_N & M_pX_3X_N & \cdots & \rho A + M_pX_N^2 \end{vmatrix}}{\begin{vmatrix} \rho A + M_pX_1^2 & M_pX_1X_2 & M_pX_1X_3 & \cdots & M_pX_1X_N \\ M_pX_1X_2 & \rho A + M_pX_2^2 & M_pX_2X_3 & \cdots & M_pX_2X_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M_pX_1X_N & M_pX_2X_N & M_pX_3X_N & \cdots & \rho A + M_pX_N^2 \end{vmatrix}} \quad (23)$$

Similar expressions for other derivatives can be written, as well.

Using Equation 23 and other equations for derivatives, the problem can be solved for any finite number of mode shapes. Clearly it is necessary to apply also an algorithm for calculation of determinats as given by Equation 23. Using the described method for two modes, it was observed that neglecting inertial effects, the results of this numerical method with time interval of 0.01 s, have been nearly the same (within 10^{-2}% deviations) as those found analytically. However, considering these effects, the maximum difference reaches to about 17% (at t=0.8 s and t=1.9 s). This value is an estimation for the effect of mass inertia in the present example. The graph corresponding to application of this method for data given by Equation 22 has been plotted in Figure 2.

3- Central Difference Expansions for Derivatives

Using approximate central differnce formulas, [11], the derivatives in Equation 21 can be substituted with corresponding difference expressions. With this tech-

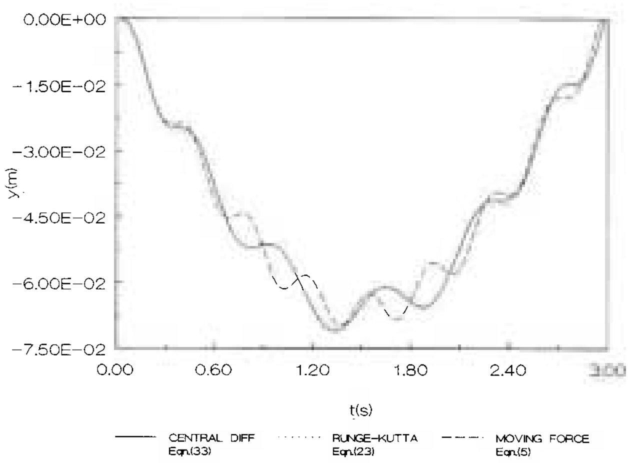

Figure 2. Time history diagram of center of the beam.

nique, and for N modal shapes, Equation 21 would then be transformed to a system of N linear algebraic equations, which must be solved for each time interval. Regarding the approximations involved, the time step should be chosen to be sufficiently small.

In order to present the method, Equation 21 is first rearranged in the following form,

$$[H(t)]\,(\ddot{\phi}\,(t)) = -(\alpha\phi\,(t)) + (C\,(t)) \tag{24}$$

where,

$$[H\,(t)]=\begin{bmatrix} \rho A+M_pX_1^2(Vt) & M_pX_1X_2 & \cdots & M_pX_1X_N \\ M_pX_2X_1 & \rho A+M_pX_2^2(Vt) & \cdots & M_pX_2X_N \\ \vdots & \vdots & \vdots & \vdots \\ M_pX_NX_1 & M_pX_NX_2 & \cdots & \rho A+M_pX_N^2(Vt) \end{bmatrix}$$

$$(\phi\,(t))=\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \vdots \\ \phi_N(t) \end{pmatrix} \tag{25}$$

and,

$$\alpha_i = EI\beta_i^4 \quad , \quad C_i = -M_p g X_i\,(Vt) \tag{26}$$

Assuming rest boundary conditions,

$$(\phi\,(0))=0 \quad , \quad \left(\frac{d\phi}{dt}(0)\right)=0 \tag{27}$$

and considering the following forward difference equation,

$$\frac{d\phi}{dt}(0)=\frac{\phi\,(h)-\phi\,(0)}{h} \tag{28}$$

Equation 27 can be rewritten as,

$$(\phi\,(0)) = 0 \quad , \quad (\phi\,(h)) = 0 \tag{29}$$

where h is the time step size. Hence, the forward difference procedure was implemented for proper presentation of initial conditions. At this stage, the central difference formulas can be used for completeing the solution. In fact,

$$\ddot{\phi}_i\,(t_k)=\frac{\phi_i\,(t_{k+1})-2\,\phi_i\,(t_k)+\phi_i\,(t_{k-1})}{h^2} \tag{30}$$

where,

$$t_k = kh \;,\; t_{k+1} = (k+1)\,h \quad , \quad t_{k-1} = (k-1)\,h \tag{31}$$

Considering the following definitions,

$$\left(\phi_{t_k}\right)=\left(\phi\,(t_k)\right) \;,\; \left[H_{t_k}\right]=\left[H\,(t_k)\right] \tag{32}$$

Equation 24 can now be rewritten in the folowing form,

$$\frac{1}{h^2}\left[H_{t_k}\right]\left(\phi_{t_{k+1}}-2\,\phi_{t_k}+\phi_{t_{k-1}}\right)=-\left(\alpha\,\phi_{t_k}\right)+\left(C_{t_k}\right) \tag{33}$$

which can be solved using standard algorithms for solving a set of linear algebraic equations, such as the Gaussian elimination.

The foregoing algorithm, adapting 0.01 s time interval and three mode numbers, was implemented for solving the problem introduced by Equation 22. It was noted that neglecting inertial effects, the numerical solution is within 0.5% deviation from those found analytically. On the other hand, if the inertial effects are included in the solution, the maximum difference reaches to about 18%. This figure is nearly the same as the one obtained previously, using the Runge-Kutta method. The results obtained are illustrated in Figure 2.

MAXIMUM SHEAR FORCE AND BENDING MOMENT

So far, only the dynamic values of the beam deflection were analyzed. Although these values are of great practical importance, there are also other parameters, such as shear force and bending moment which should be studied.

It is known that neglecting shear deformation, the bending moment function can be related to the dynamic deflection function as follows,

$$EI\frac{\partial^2 y}{\partial x^2}=M(x,t) \tag{34}$$

Substituting Equation 2 in Equation 34, one obtains

$$EI\sum_{i=1}^{\infty}X_i''(x)\,\phi_i(t)=M(x,t) \tag{35}$$

And similarly for the shear force,

$$EI\sum_{i=1}^{\infty}X_i'''(x)\,\phi_i(t)=S(x,t) \tag{36}$$

That is, bending moment and shear force can be calculated at any instant of time and for any point of the beam. Therefore, the critical values of these parameters for a set of assumed data can be determined, and hence, dynamic collapse of the structure prohibited.

Graphs showing variation of maximum bending moment and shear force along the beam with time are also very helpful. Figures 3 and 4 are such graphs

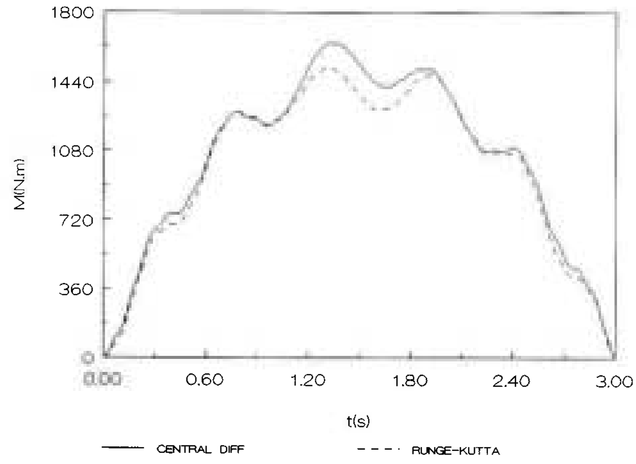

Figure 3. Maximum bending moment variation (positive or negative).

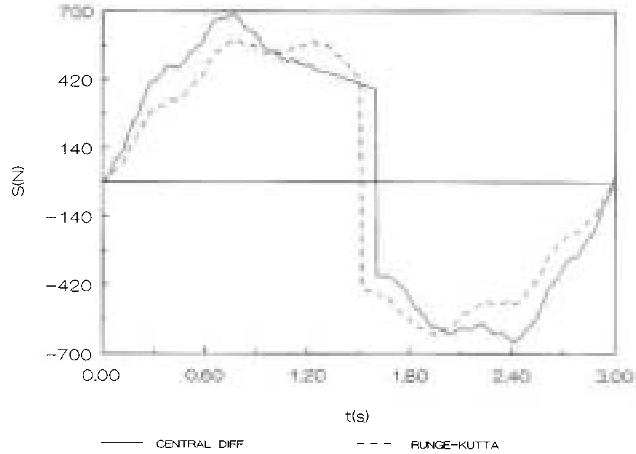

Figure 4. Maximum shear force variation (positive or negative).

which represent the desired values using both the Runge-Kutta and central difference algorithms.

CONCLUSIONS

Dynamic behavior of a beam carrying a moving mass, when including or excluding the inertial effects, was fully analyzed. The procedure imposes no restricting assumptions on the boundary conditions of the beam. It was noted that for a simply supported beam, the governing equation in the case of moving mass would reduce to that of the moving load, provided the coupling term resulting from inertial effects is dropped out. Three numerical techniques, namely, finite element method, fourth order Runge-Kutta and, central difference expansion method were employed to solve the equations. It is found that when neglecting inertial effects, these methods produce nearly the same results, which are identical to the analytical solution of the problem. On the other hand, considering the inertial effects, their differences could reach to a maximum value of about 18%. Therefore, it is concluded that the consideration of the load inertial effects is absolutely essential for modeling such dynamic systems. Finally, using the obtained dynamic deflection functions, graphs representing variations of maximum bending moment and shear force were also plotted. In practice, these graphs can be used in order to design safe structures against dynamic collapse.

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