APPLICATION OF BOUNDARY ELEMENT METHOD (BEM) TO TWO-DIMENSIONAL POISSON'S EQUATION

G. Karami and M. Hakimhashemi

Department of Mechanical Eng.
School of Eng.
Shiraz University
Shiraz- Iran

Abstract BEM can be used to solve Poisson's equation if the right hand side of the equation $\nabla^2\phi = -\sigma$ is constant because it can easily be transformed to an equivalent Laplace equation. However, if the right hand side is not constant, then such a treatment isimpossible and part of the equation can not be transformed over the boundary, hence, the whole domain has to be discretized. Although this takes away important advantages of BEM over the Finite Element Method (FEM) in which the whole domain also has to be discretized, but the results are more accurate, and a much coarser mesh can be employed to obtain an equivalent accuracy with less effort in data preparation. In this paper the application of BEM to two - dimensional Poisson's equation is described.

A computer program is developed using linear elements to express the geometry and functions. The program, is used to solve the torsion porblem (Poisson's equation) and potential flow around a circle (Laplace's equation), and the results are compared with those of analytical methods and FEM.

چکیده در این مقاله از روش اجزاء مرزی جهت حل معادله پواسان با تابع P غیر ثابت، استفاده شده است. برنامه کامپیوتری به نام BIE نوشته شده و توسط آن دو مسئله مهم، جریان سایل پتانسیل (معادله لاپلاس) و مسئله پیچش مقاطع غیر دایره ای (معادله پواسان)، حل شده اند.

1. Boundary Element Method
2. Finite Element Method

INTRODUCTION

For many years the Finite Difference Method (FDM) was the only numerical tool for calculating the solution of Partial Differential Equations (PDEs). The method is simple and generally applicable, but for each PDE with different boundary conditions a separate computer program must be developed, and its accuracy is limited. The FEM which was developed for structural problems has now expanded to a stage that nearly all sorts of PDEs have been solved by using this method. In this approach the PDE is first written in variational form and the functions are then approximated over each element using usually polynomical interpolation. After implementing the boundary conditions, a system of equations is produced, by which the values of functions are calculated at several points in the domain. Then general computer programs can be developed. BEM is to some extent a new numerical tool in which the governing PDEs are transformed over the boundary, thus producing in a form of integral equation. Only the boundary has to be discretized and the functions are appoximated over the boundary. Ater implementing the boundary conditions a system of equations is produced, by which the values of functions are evaluated at some boundary points. Then

the values of the functions inside the domain are easily calculated.

The BEM has been widely used to solve Laplace equation in recent years [1]. Problems of this sort which include two or three dimensional potential flow [2, 3], elastostatic [4, 5, 6], and many other engineering porblems are solved with relative convenience using BEM. Compared to other numerical methods such as FEM, it displays better accuracy, efficiency and ease of data preparation.

TRANSFORMATION OF POISSON'S EQUATION INTO AN INTEGRAL EQUATION

Consider Poisson's equation for the domain A and boundary S;

$$-\nabla^2 u(\underset{\sim}{x}) = P(\underset{\sim}{x}),\ \underset{\sim}{x} \in A \tag{1}$$

where $\underset{\sim}{x} = (x_1, x_2)$ and the function $P: A \rightarrow IR$, and u are continuous functions. The fundamental solution $G(X; \zeta)$ is defined as [7]:

$$-\nabla^2(\underset{\sim}{x}; \underset{\sim}{\zeta}) = \delta(\underset{\sim}{x}; \underset{\sim}{\zeta}), \quad \underset{\sim}{x}, \underset{\sim}{\zeta} \in IR^2 \tag{2}$$

where $\delta(\underset{\sim}{x}; \underset{\sim}{\zeta})$ is the Dirac delta function and is defined at point ζ (ζ is the source and x is a point), $\delta(x; \zeta) = \delta(\zeta; x)$. For a Laplace operator the fundamental solution is [7]:

$$G(\underset{\sim}{X}; \underset{\sim}{\zeta}) = \frac{-1}{2\pi} 2nr$$
$$r = [(x_1-\zeta_1)^2 + (x_2-\zeta_2)^2]^{1/2} \tag{3}$$

This solution is symmetric, i.e. $G(\underset{\sim}{x}; \underset{\sim}{\zeta}) = G(\underset{\sim}{\zeta}; \underset{\sim}{x})$. If we multiply equation (1) by $G(\underset{\sim}{x}; \underset{\sim}{\zeta})$, and multiply equation (2) by $u(\underset{\sim}{x})$ and then subtract from each other, and by considering the symmetry of the equation; it follows:

$$\nabla^2 u(\underset{\sim}{\zeta}).\ G(\underset{\sim}{x}; \underset{\sim}{\zeta}) + \nabla^2 G(\underset{\sim}{x}; \underset{\sim}{\zeta}).\ u(\underset{\sim}{\zeta})]$$
$$[P(\underset{\sim}{\zeta}).\ G(\underset{\sim}{x}; \underset{\sim}{\zeta}) - \delta(\underset{\sim}{x}; \underset{\sim}{\zeta}).\ u(\underset{\sim}{\zeta})] \tag{4}$$

If we integrate over the domain A from both sides:

$$\int_A [-\nabla^2 u(\underset{\sim}{\zeta}).\ G(\underset{\sim}{x}; \underset{\sim}{\zeta}) + \nabla^2 G(\underset{\sim}{x}; \underset{\sim}{\zeta}).\ u(\underset{\sim}{\zeta})] dA(\underset{\sim}{\zeta})$$
$$\int_A [P(\underset{\sim}{\zeta}).\ G(\underset{\sim}{x}; \underset{\sim}{\zeta}) - \delta(x; \underset{\sim}{\zeta}).\ u(\underset{\sim}{\zeta})] dA(\underset{\sim}{\zeta}) \tag{5}$$

Using Green's theorem [1], and after some rearrangements, the integral form of Poisson's equation is derived as:

$$\oint [-G(\underset{\sim}{x}; \underset{\sim}{\zeta}) \frac{au(\underset{\sim}{\zeta})}{an} + n(\underset{\sim}{\zeta}) \frac{aG(\underset{\sim}{x}; \underset{\sim}{\zeta})}{an}] dS(\underset{\sim}{\zeta}) -$$
$$\int_A P(\underset{\sim}{\zeta})\, G(\underset{\sim}{x}; \underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta}) - \int_A \delta(\underset{\sim}{x}; \underset{\sim}{\zeta})\, u(\underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta}) \tag{6}$$

In this equation the unit vector perpendicular to S is n (n_1, n_2) where;

$$\frac{a(x)}{an} \quad \nabla_{\zeta}(x).\ n \quad (\frac{a}{a\zeta_1} i \quad \frac{a}{a\zeta_2} j).\ n_1 i + n_2 j) \tag{7}$$

But according to the definition of $\delta(x; \zeta)$, we have

$$\int_A \delta(\underset{\sim}{x}, \underset{\sim}{\zeta})\, u(\underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta}) \quad \begin{cases} u(\underset{\sim}{x}). & \underset{\sim}{x} \in A \\ 0 & x \notin \bar{A} \end{cases} \tag{8}$$

where $\bar{A}$ AUS, and also;

$$\frac{aG(\underset{\sim}{x}; \underset{\sim}{\zeta})}{n} \quad F_n(\underset{\sim}{x}; \underset{\sim}{\zeta}) = \frac{1}{2\pi r^2}[(x_1 - \zeta_1)n_1 + (x_2 - \zeta_2)n_2]$$
$$r \quad [(x_1 \quad \zeta_1)^2 + (x_2 - \zeta_2)^2]^{1/2}, \underset{\sim}{n} \quad (n_1, n_2 \tag{9}$$

By substituting $\frac{au(\underset{\sim}{\zeta})}{an}$ $q(\underset{\sim}{\zeta})$ it follows;

$$u(\underset{\sim}{x}) = \oint_S q(\underset{\sim}{\zeta})\, G(x; \underset{\sim}{\zeta})\, dS(\underset{\sim}{\zeta}) - \oint_S u(\underset{\sim}{\zeta})\, F_n(x; \underset{\sim}{\zeta})\, dS(\underset{\sim}{\zeta})$$
$$+ \int_A P(\underset{\sim}{\zeta})\, G(\underset{\sim}{x}; \underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta}) \tag{10}$$

The integral $\int P(\underset{\sim}{\zeta})\, G(x; \underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta})$ is in the form of Cauchy's principal value and can be written as [7];

$$\int P(\underset{\sim}{\zeta})\, G(\underset{\sim}{x}; \underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta}) \quad \int_{A-\{x\}} P(\underset{\sim}{\zeta})\, G(\underset{\sim}{x}; \underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta}) \tag{11}$$

By using equation (10), one can evaluate $\frac{au(\underset{\sim}{x})}{am}$ for a direction $m = (m_1, m_2)$ by differentiating ωrt m direction, thus:

$$\frac{au(\underset{\sim}{x})}{m} = \oint_S q(\underset{\sim}{\zeta})\, F_m(x; \underset{\sim}{\zeta})\, dS(\underset{\sim}{\zeta}) \quad \oint_S u(\underset{\sim}{\zeta})\, H_m(x;$$
$$\underset{\sim}{\zeta}) dS(\underset{\sim}{\zeta})$$
$$+ \int_A P(\underset{\sim}{\zeta})\, F_m(\underset{\sim}{x}; \underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta}) \tag{12}$$

where $H_m(\underset{\sim}{x}; \underset{\sim}{\zeta})$ $\frac{aF_n(\underset{\sim}{x}; \underset{\sim}{\zeta})}{am}$

Now if point $\underset{\sim}{x}$ is moving toward the boundary S, the

equation (10) can be written in as:

$$u(\underset{\sim}{x}) = \oint_S q(\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) - \oint_S u(\underset{\sim}{\zeta_O})\ F_n(\underset{\sim}{x};\ \underset{\sim}{\zeta_O}) dS(\underset{\sim}{\zeta_O})$$

$$+ \int_A P(\underset{\sim}{\zeta})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta})\ dA(\underset{\sim}{\zeta}) \qquad \underset{\sim}{x},\ \underset{\sim}{\zeta_O} \varepsilon S,\ \underset{\sim}{\zeta} \varepsilon A \tag{13}$$

In this equation the integrals $\oint_S u(\underset{\sim}{\zeta_O})\ F_n(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O})$ and $\oint_S (\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O})$ are in the form of Cauchy's principal value, and it can be proved that they can be written as;

$$\oint_S q(\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) = \oint_{S-\{x\}} q(\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O})$$

$$\oint_S u(\underset{\sim}{\zeta_O})\ F_n(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) = -\frac{\theta_e}{2\pi}\ u(\underset{\sim}{x}) + \oint_{S-\{x\}} u(\underset{\sim}{\zeta_O})\ F_n(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O})$$

For simplicity we write S instead of S - {x }, so the equation (14) is written as;

$$\frac{\theta_i}{2\pi}\ u(x) = \oint_S q(\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) - \oint_S u(\underset{\sim}{\zeta_O})\ F_n(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O})$$

$$+ \int_A P(\underset{\sim}{\zeta})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta})\ dA(\underset{\sim}{\zeta})$$

where $\frac{\theta_i}{2\pi} = (1 - \frac{\theta_e}{2\pi})$. ϕ_i and ϕ_e are shown in Figure. 1. When the boundary S is continuous at x then $\theta_i = \pi$, otherwise it is usually equal to the corner angle at x. Having found the final form of the integral equation the boundary conditions must be incorporated.

(I) Dirichlet Boundary Conditions (DBCs)

If we designate $\alpha_i(x)$ to be equal to $\frac{\theta_i(x)}{2\pi}$, for DBSc we have

$u(\underset{\sim}{x}) = \hat{u}(\underset{\sim}{x}),\ \underset{\sim}{x} \varepsilon S$

The form of boundary integral equation is;

$$\oint_S q(\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) = \alpha(\underset{\sim}{x})\ \hat{u}(\underset{\sim}{x}) +$$

$$\oint \hat{u}(\underset{\sim}{\zeta_O})\ F_n(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) - \int_A P)\underset{\sim}{\zeta})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta})\ dA(\underset{\sim}{\zeta}) \tag{16}$$

In equation (16) $q(\underset{\sim}{\zeta_O})$ is unknown.

(II) Newmann Boundary Conditions (NBCs):

For NBCs we have;

$q(\underset{\sim}{x}) = \hat{q}(\underset{\sim}{x}) \qquad \underset{\sim}{x} \varepsilon S$

The form of boundary integral equation is;

$$\alpha(\underset{\sim}{x})\ u(\underset{\sim}{x}) + \oint_S u(\underset{\sim}{\zeta_O})\ F_n(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) =$$

$$\int_S \hat{q}(\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) + \int_A P(\underset{\sim}{\zeta})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta})\ dA(\underset{\sim}{\zeta}) \tag{17}$$

In the above equation $u(\underset{\sim}{\zeta_O})$ is unknown.

(III) Mixed Boundary Conditions (MBCs):

For MBCs we have

$u(\underset{\sim}{x}) = \hat{u}(\underset{\sim}{x}), \qquad \underset{\sim}{x} \varepsilon S_1$

$q(\underset{\sim}{x}) = \hat{q}(\underset{\sim}{x}), \qquad \underset{\sim}{x} \varepsilon S_2$

$S = S_1 \cup S_2$

The form of the boundary integral equation is;

$$\alpha(\underset{\sim}{x})\ u(\underset{\sim}{x})\ |_{x \varepsilon S} + \oint_{S_2} u(\underset{\sim}{\zeta_O})\ Fn(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) - \int_{S_1} q(\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O})$$

$$= \oint_{S_1} \hat{u}(\underset{\sim}{\zeta_O})\ F_n(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O}) + \oint_{S_2} \hat{q}(\underset{\sim}{\zeta_O})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta_O})\ dS(\underset{\sim}{\zeta_O})$$

$$+ \int_A P(\underset{\sim}{\zeta})\ G(\underset{\sim}{x};\ \underset{\sim}{\zeta})\ dA(\underset{\sim}{\zeta})$$

In the above equation $u(\underset{\sim}{\zeta_O})$ and $q(\underset{\sim}{\zeta_O})$ are unknowns at some points on the boundary.

NUMERICAL FORMULATIONS

The boundary $S(S_h)$ is divided into N_e elements, each called a Boundary Element (BE). Every BE has several Nodes (N_p) and its geometrical shape can be either linear or non - linear. For integration over the domain $A(A_h)$, the domain is also discretized into elements (Nein), thus;

$$S_h = \sum_{e=1}^{N_e} S_e,\ A_h = \text{Interior}(S_h)$$

$$A_h = \sum_{e=1}^{Nein} Ae$$

Approximating $u_h(\underset{\sim}{\zeta_O})$ and $q_h(\underset{\sim}{\zeta_O})$ over the new boundary;

$$u_e(\underset{\sim}{\zeta_O}) = \sum_{i=1}^{N_p} u_i^e\ \psi_i^e\ (\underset{\sim}{\zeta_O})$$

$$q_e(\underset{\sim}{\zeta_O}) = \sum_{i=1}^{N_p} q_i^e > _i^e\ (\underset{\sim}{\zeta_O})$$

$\underset{\sim}{\zeta_O} \varepsilon S_h$ (19)

The shape functions $\psi_i^{(e)}$ are Lagrangian type and have the property of;

$$\psi_i^{(e)}(\underset{\sim}{\zeta_j}) = \delta_{ij},\ u_e(\underset{\sim}{\zeta_j}) = u_i \tag{20}$$

$i,j = 1, 2, \ldots, N_p$

The numerical formulation for MBCs is described here, as for the other two types of BCs follow the same suite. The boundary conditions for MBCs are given as:

$$u_h(\underset{\sim}{x}) = \hat{u}(\underset{\sim}{x}), \qquad \underset{\sim}{x} \varepsilon S_{1h}$$
$$q_h(\underset{\sim}{x}) = \hat{q}(\underset{\sim}{x}), \qquad \underset{\sim}{x} \varepsilon S_{2h}$$
$$S_h = S_{1h} \cup S_{2h}, \; S_{1h} = \bigcup_{e=1}^{N_{1e}} S_1 e, \qquad S_{2h} = \bigcup_{e=1}^{N_{2e}} S_{2e}$$

where N_{1e} is the number of elements with DBCs, and N_{2e} is the number of elements with NBCs. The form of the equation for every node is:

$$\alpha(\underset{\sim}{x}_i)\, u_h(\underset{\sim}{x}_i) + \sum_{e=1}^{N_{2e}} \int_{S2e} \Big[\sum_{j=1}^{N_p} u_j^e(\underset{\sim}{\zeta}_O)\Big] F_n(\underset{\sim}{x}_i; \underset{\sim}{\zeta}_O)\, dS(\underset{\sim}{\zeta}_O) + \sum_{e=1}^{N_{2e}} \int_{S1e} \Big[\sum_{j=1}^{N_p} q_j^e \psi_j^e(\underset{\sim}{\zeta}_O)\Big] G(\underset{\sim}{x}_i; \underset{\sim}{\zeta}_O)\, dS(\underset{\sim}{\zeta}_O) = -\sum_{e=1}^{N_{1e}} \int_{S1e} \Big[\sum_{j=1}^{N_p} \hat{u}_j^e \psi_j^e(\underset{\sim}{\zeta}_O)\Big] F_n(\underset{\sim}{x}_i; \underset{\sim}{\zeta}_O)\, dS(\underset{\sim}{\zeta}_O) \qquad (21)$$
$$+ \sum_{e=1}^{N_{2e}} \int_{S2e} \Big[\sum_{j=1}^{N_p} \hat{q}_j^e \psi_j^e(\underset{\sim}{\zeta}_O)\Big] G(\underset{\sim}{x}; \underset{\sim}{\zeta}_O)\, dS(\underset{\sim}{\zeta}_O) + \int_{Ah} P(\underset{\sim}{\zeta})\, G(\underset{\sim}{x}_i; \underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta})$$

The compact form of the equation (21) becomes;

$$\alpha_i u_i + \sum_{j=1}^{N_2} b_{ij} u_j - \sum_{j=1}^{N_1} a_{ij} q_j = \sum_{j=1}^{N_2} q_{ij} \hat{q}_j - \sum_{j=1}^{N_1} b_{ij} \hat{u}_j + d_i$$
$$i = 1,2,\ldots,N \qquad (22)$$

where N_1, and N_2 are the number of nodes with DBCs and NBCs. Also,

$$\alpha_i = \alpha(\underset{\sim}{x}_i) = \frac{\theta_i}{2\pi},$$

θ_i = corner angle at node

$$\hat{u}_i = \hat{u}_h(\underset{\sim}{x}_i), \; q_i = q_h(\underset{\sim}{x}_i)$$
$$a_{ij} = \sum_{e=1}^{N_e} a_{ij}^e,$$
$$a_{ij}^e = \int_{Se} G(\underset{\sim}{x}_j; \underset{\sim}{\zeta}_O)\, \psi_i^e(\underset{\sim}{\zeta}_O)\, dS(\underset{\sim}{\zeta}_O)$$
$$b_{ij} = \sum_{e=1}^{N_e} b_{ij}^e,$$
$$b_{ij}^e = \int_{Se} F_n(\underset{\sim}{x}; \underset{\sim}{\zeta}_O)\, \psi_j^e(\underset{\sim}{\zeta}_O)\, dS(\underset{\sim}{\zeta}_O)$$
$$d_i = \int_{Ah} P(\underset{\sim}{\zeta})\, G(\underset{\sim}{x}; \underset{\sim}{\zeta})\, dA(\underset{\sim}{\zeta})$$

For the computer program written, linear shape functions have been assumed for boundary elements as well as interior elements. Thus $\psi_{i(e)}$ for the two nodes of each boundary element as a function of intrinsic coordinates ξ, is written as;

$$\psi_1(\xi) = 0.5\,(1 - \xi)$$
$$\psi_2(\xi) = 0.5\,(1 + \xi). \qquad -1 \le \xi \le 1$$

Each interior elements has four nodes, for which the shape function is written as a function of ξ. η the intrinsic coordinates;

$$\psi_1(\xi, \eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$\psi_2(\xi, \eta) = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$\psi_3(\xi, \eta) = \tfrac{1}{4}(1+\xi)(1+\eta) \qquad -1 \le \xi, \eta \le 1$$
$$\psi_4(\xi, \eta) = \tfrac{1}{4}(1-\xi)(1+\eta)$$

The integrations are performed using Gaussian quadrature. For this purpose the integrands are first transformed into ξ, η local coordinates by use of the Jacobian of transformation. Different strategies have been employed for singular or nonsingular integrands.

The matrix form of the equation (22) is written as;

$$[B]\{\bar{U}\} = [A]\{\bar{Q}\} + \{D\}$$

where $\{\bar{U}\} = u_i$ and $\{\bar{Q}\} = \bar{q}_i$
$\bar{u}_i = \hat{u}_i$ and $\bar{q}_i = q_i$ for DBCs and $\bar{u}_i = u_i$, and $\bar{q}_i = \hat{q}_i$ for NBCs. Also

$$[A] = [a_{ij}], \; [B] = [\bar{b}_{ij}], \; \{D\} = \{d_i\}, \; b_{ij} = \bar{b}_{ij} + \alpha_i \delta_{ij}$$

After rearrangement the form of the equation (23) becomes;

$$[K]\{V\} = [M]\{\omega\} + \{D\}$$

where,

$$[K] = [K_{ij}], \begin{cases} v_j = u_j \, K_{ij} = b_{ij} \\ v_j = q_j \, k_{ij} = -a_{ij} \end{cases}$$
$$[M] = [m_{ij}] \begin{cases} w_j = \hat{u}_j \, m_{ij} = -b_{ij} \\ w_j = \hat{q}_j \, m_{ij} = a_{ij} \end{cases}$$

and finally;

$$[K]\{V\} = \{R\}$$

where;

$$\{R\} = \{r_i\}, \; r_i = \sum_{i=1}^{N} m_{ij}\omega_j + d_i \qquad (25)$$

A convinient way to calculate α_i is the use of rigid body motion property of the Poisson's equation. Suppose we have;

$$\nabla^2 u(\underset{\sim}{x}) = 0, \quad x \varepsilon A$$

$$\frac{\partial u(\underset{\sim}{x})}{\partial \underset{\sim}{n}} = 0, \quad x \varepsilon S$$

the solution to this equation is $u(x) = u_o = \text{const}$. Now if we formulate the equation ωrt the given boundary condition;

$$[B]\{U\} = [A]\{Q\}$$

Since $q = 0$, then $\{Q\} = 0$ and thus $[B]\{u\} = \{0\}$, but we know that $u_i = u_o$, thus $\{U\} = [U_o, U_o, \ldots, U_o]^T$. Hence;

$$u_o \sum_{i=1}^{N} \bar{b}_{ij} = 0, \quad i = 1,2,\ldots,N$$

since the Lapalace equation is true for every u_o, thus;

$$\sum_{j=1}^{N} b_{ij} = 0 \rightarrow \alpha_i + \sum_{i=1}^{N} b_{ij} = 0$$

therefore;

$$\alpha_i = -\sum_{j=1}^{N} b_{ij} \qquad (30)$$

The matrix [K] in equation (25) is a fully populated matrix, and if a simple elimination routine is employed, the round - off - errors could cause serious inaccuracy in the solutions. Thus a Gaussian elimination method with total pivoting and scaling is employed.

APPLICATIONS

In order to check the accuracy and efficiency of the method two problems have been solved here.

1. Torsion Problem:

The governing equation for torsion of noncircular members is a kind of Poisson's equation. The equation with the boundary conditions is given [8];

$$\nabla^2 \phi(\underset{\sim}{x}) = 2G\theta, \quad x \varepsilon A$$
$$\phi(\underset{\sim}{x}) = 0, \quad x \varepsilon A$$
$$A = \{\underset{\sim}{x} = (x, y) \quad -b < x < b, -a < y < a\}$$
$$S = \{\underset{\sim}{x} = (x, y) \quad -b \leq x \leq b, -a \leq y \leq a\} - A$$

The function ϕ is a kind of stress function by which τ_{yz} and τ_{xz} are derived according to;

$$\tau_{yz} = \frac{-a\phi}{ax}, \quad \tau_{xz} = \frac{a\phi}{ay}$$

The analytical solution to the equation is found by use of series functions [8]. For BE discretization a square shape with $a = b = 0.5$ is assummed. It has 40 boundary elements and 100 interior elements. The number of nodes is 40 on the boundary and [2] on the interior. With $G\theta = 10 \times 10^{-2}$, the problem is solved. For comparison the values of ϕ, τ_{yz} and τ_{xz} are calculated at some nodal points on the boundary and at the interior of the cross section are tabulated in Table 1. As it is shown the BEM would yield very accurate values for τ_{yz} and τ_{xz} but FEM [10] has given better values for ϕ. For FEM discretiztion the same number of elements (100 elements with 121 nodes) has been used.

2. Potential Flow Around a Circle:

BEM has been applied to solve a Laplace equation, here the potential flow around a circle. The problem and its boundary conditions are expressed as [9];

$$\nabla^2 \psi(\underset{\sim}{x}) = 0, \quad \underset{\sim}{x} \varepsilon 1R^2 - (AUS), \quad \underset{\sim}{x} = (x, y)$$
$$\psi(\underset{\sim}{x}) = 0, \quad \underset{\sim}{x} \varepsilon S$$
$$\psi(\underset{\sim}{x}) = V_\infty y, \quad |\underset{\sim}{x}| \rightarrow \infty \quad |\underset{\sim}{x}| = \sqrt{x^2 + y^2}$$
$$A = \{\underset{\sim}{x} \mid x^2 + y^2 < a^2\}, \quad S = \{\underset{\sim}{x} \mid x^2 + y^2 = a^2\}$$

The analytical solution has been derived for an infinite domain [9]. Although indirect BEM can be formulated for infinite domain but here a direct BEM has been applied to a finite domain with the following form of discretization;

$$\nabla^2 \psi(\underset{\sim}{x}) = 0, \quad \underset{\sim}{x} \varepsilon A, \quad \underset{\sim}{x} = (x, y)$$
$$\psi(\underset{\sim}{x}) = 0, \quad \underset{\sim}{x} \varepsilon S$$
$$\psi(\underset{\sim}{x}) = V_\infty y, \quad \underset{\sim}{x} \varepsilon S_\infty$$
$$A = \{\underset{\sim}{x} \mid a^2 < x^2 + y^2 < (100a)^2\}$$
$$S = \{\underset{\sim}{x} \mid x^2 + y^2 = a^2\}, \quad S = \{\underset{\sim}{x} \mid x^2 + y^2 = (100)^2\}$$

The domain is finite and doubly connected. The domain S and S are divided into 64 boundary elements (32 for S and 32 for S_∞). The values of ψ, $\frac{a\psi}{aX}$, $\frac{a\psi}{ay}$ are

tabulated in Table 2. The results match very well with the analytical solution, although a finite domain is assumed.

CONCLUSION:

A BEM program is presented for Poisson's equation. The BEM requires the formulations of the functions in terms of integral over the boundary. This would cause very much simplification as far as the data preparation and the input data is concerned. The method has also its own drawback, as in case of Poisson's equation part of the integral equations can not be transferred over the boundary.

With respect to Poisson's and Laplace' equations the following conclusions may be made.

In case of Laplace equation:

1. In BEM ψ is approximated over the boundary S, while in FEM it is approximated over the domain A, and hence increases the errors
2. The size of the domain doesnot cause any increase in calculations in case of BEM, But in FEM by increasing the size, the number of elements must be increased.
3. In BEM the functions and their derivatives are easily and accurately calculated at internal points, while for FEM the points are usually the nodes and the values of derivatives are not as much accurate.

In Case of Poisson's equations:

1. The volume of calculations due to an integral over the domain has been consideribly increased for BEM.
2. A much coarser interior mesh generation can be done for BEM in comparison with FEM at the same accuracy.
3. The computation time for BEM is bigger than FEM because of a fully populated matrix of coefficients.
4. Those in 2, and 3 in case of Laplace equation are true for the case of Poisson's equation.

ACKNOWLEDGMENT

The authors would like to express their gratitude to Mrs. M. Shaterpouri for typing the manuscript. The numerical computation was carried out at the computer center of Shiraz University.

REFERENCES

1. Banerjee, P. K., and Butterfield, Boundary Element Methods in Engineering, Butterworths (1980).
2. Jaswon, M. A., «Integral Equation Methods in Potential Theory I», Proc. R. Soc. London, A275, 23-32, (1977).
3. Symm, G. T., «Integral Equation Methods in Potential Theory II», Proc. R. Soc., London A275, 33-46, (1977).
4. Cruse, T. A., «Mathematical Foundation of the Boundary Integral Equation Method in Solid Mechanics, Report No. AFOSR-TR- 1002, Pratt & Whitney Aircraft Group (1977).
5. Karami, G. and Fenner, R. T., «Analysis of Mixed Mode Fracture and Crack Closure Using Boundary Integral Equation Methods, Int. J. Fracture, Vol. 30, No. 1, (1986).
6. Karami, G., «Boundary Element Method for Elastic Thermo-elastic Body Force Problems Using Pseudo Body Force Approach», Computational Mechanics '88, Theory and Applications, Vol. 1, Editors (S. N. Atluri, G. Yagawa), Springer - Verlag, Berlin (1988).
7. Sokolnikoff, I. S. and Redheffer, R. M., Mathematics of Physics and Modern Engineering, McGraw - Hill (1966).
8. Timoshenko, S. P., and Goodier, J. N., «Theory of Elasticity», McGraw-Hill (1983).
9. Kuethe, A. M. and Chow, Y. C., «Foundation of Aerodynamics», John Wiley & Sons (1976).
10. Karami, G. and Hakimhashemi, M., «CODE2-A Finite Element Program for Solving Two- Dimensional Poisson's Equation», Department of Mechanical Engineering, Shiraz University (Internal Report), June 1986.
11. Hakimhashemi, M., «Application of Boundary Element Method to Two- Dimensional Poisson's Equation», M. Sc Thesis, Department of Mechanical Engineering, Shiraz University, 1987.

Table 1(a): Values of ϕ (Analytic V. S. BEM)

Point #	Analytic value	BEM value	e% BEM
1	0.14734257E-2	0.14489902E-2	1.6584
	0.14230610E-2	0.13986203E-2	1.7174
	0.12675927E-2	0.12430740E-2	1.9342
	0.99396109E-3	0.96910693E-3	2.5005
	0.58083665E-3	0.55510297E-3	4.4304
	0.14230610E-2	0.13986203E-2	1.7174
	0.12675927E-2	0.12430740E-2	1.9342
8	0.99396109E-3	0.96910693E-3	2.5005
9	0.58083665E-3	0.55510297E-3	4.4304
10	0.13748848E-2	0.13504703E-2	1.7757
	0.10968195E-2	0.10727214E-2	2.1971
12	0.69293764E-3	0.67026314E-3	3.2722
13	0.26142494E-3	0.24326972E-3	6.9447

Table 1(b): Values of ϕ (Analytic V.S. FEM)

Point #	Analytic value	FEM2 value	e% FEM2
1	0.14734257E-2	0.14733923E-2	0.0022
2	0.14230610E-2	0.14229543E-2	0.0075
	0.12675927E-2	0.12675525E-2	0.0031
4	0.99396109E-3	0.99386760E-3	0.0094
	0.58083665E-3	0.58079151E-3	0.0077
6	0.14230610E-2	0.14229543E-2	0.0075
7	0.12675927E-2	0.12675525E-2	0.0031
8	0.99396109E-3	0.99386760E-3	0.0094
9	0.58083665E-3	0.58079151E-3	0.0077
10	0.13748848E-2	0.13747074E-2	0.0129
11	0.10968195E-2	0.10967966E-2	0.0021
12	0.69293764E-3	0.69288640E-3	0.0074
13	0.26142494E-3	0.26214551E-3	0.2756

Table 1(c): Values of $d\phi/dx$ (Analytic V.S. BEM)

Point #	Analytic value	BEM value	e% BEM
	0.0	0.0	0.0
2	0.10145858E-2	0.10147975E-2	0.0208
3	0.21165837E-2	0.21182574E-2	0.0790
4	0.33917714E-2	0.33973342E-2	0.1640
5	0.49181034E-2	0.49308390E-2	0.2589
6	0.0	0.0	0.0
7	0.0	0.0	0.0
8	0.0	0.0	0.0
9	0.0	0.0	0.0
10	0.97081015E-3	0.97039424E-3	0.028
11	0.17655405E-2	0.17621529E-2	0.1918
12	0.21940042E-2	0.21818066E-2	0.5559
13	0.19776714E-2	0.19386562E-2	1.9727

Table 1(d): Values of $d\phi/ds$ (Analytic V.S. FEM)

Point #	Analytic value	FEM2 value	e% FEM2
1	0.0	0.0	0.0
2	0.10145858E-2		
3	0.21165837E-2	0.21454334E-2	1.3630
4	0.33917714E-2		
5	0.49181034E-2	0.49693380E-2	
6	0.0		
7	0.0	0.0	0.0
8	0.0		
9	0.0	0.0	0.0
10	0.97081015E-3		
11	0.17655405E-2	0.17957373E-2	
12	0.21940042E-2		
13	0.19776714E-2	0.20845348E-2	

Table 1(e): Values of $d\phi/dy$ (Analytic V.S. BEM)

Point #	Analytic value	BEM value	e% BEM
1	0.0	0.0	0.0
2	0.0	0.0	0.0
3	0.0	0.0	0.0
4	0.0	0.0	0.0
5	0.0	0.0	0.0
6	-0.10145858E-2	-0.10147975E-2	0.0208
7	-0.21165837E-2	-0.21182574E-2	0.0790
8	-0.33917714E-2	-0.33973342E-2	0.1640
9	-0.49181034E-2	-0.49308390E-2	0.2589
10	-0.97081015E-3	-0.97039424E-3	0.0428
11	-0.17655405E-2	-0.17621529E-2	0.1918
12	-0.21940042E-2	-0.21818066E-2	0.5559
13	-0.19776714E-2	0.19386562E-2	1.9727

Table 1(f): Values of $\frac{d\phi}{dy}$ (Analytic V.S. FEM)

Point #	Analytic value	FEM2 value	e% FEM2
1	0.0	0.0	0.0
2	0.0		
3	0.0	0.0	0.0
4	0.0		
5	0.0	0.0	0.0
6	-0.10145858E-2		
7	0.21165837E-2		1.3630
8	-0.33917714E-2		
9	-0.49181034E-2	-0.49693380E-2	
10	-0.97081015E-3		
11	-0.17655405E-2	-0.17957373E-2	1.7103
12	-0.21940042E-2		
13	-0.19776714E-2	-0.21454334E-2	